Decay of spherical means of Fourier transforms and distance sets of measures.

Suppose $\mu \in M(\mathbb{R}^n)$ is a measure with $\|\mu\| > 0$, $\sigma$ is surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and $\phi \in L^2(S^{n-1})$ is a function with $\|\phi\|_{L^2(S^{n-1})} > 0$. If

$$
\int_0^\infty \int_{S^{n-1}} |\hat{\mu}(r\theta)|^2 d\sigma(\theta) r^{n-1} dr = \int_{\mathbb{R}^n} |\hat{\mu}(\xi)|^2 d\xi < \infty,
$$

then, as is well-known, $d\mu \ll dx$, and since $\|\mu\| > 0$, it follows that $|\text{supp } \mu| > 0$. Now by the Cauchy-Schwarz inequality,

$$
\left| \int_{S^{n-1}} \hat{\mu}(r\theta) \phi(\theta) d\sigma(\theta) \right|^2 \leq \|\phi\|^2_{L^2(S^{n-1})} \int_{S^{n-1}} |\hat{\mu}(r\theta)|^2 d\sigma(\theta),
$$

so it is natural to ask the question, what can we say about $\text{supp } \mu$ under the weaker assumption

$$
\int_0^\infty \left| \int_{S^{n-1}} \hat{\mu}(r\theta) \phi(\theta) d\sigma(\theta) \right|^2 r^{n-1} dr < \infty?
$$

We give an answer to this question in the case $\phi \in C^\infty(S^{n-1})$. We also give an application of our result to Falconer’s distance set problem. (Received July 28, 2011)