An $L$-space is a rational homology sphere $Y$ whose Heegaard Floer homology is as small as possible: $\widehat{\mathcal{HF}}(Y) \cong \mathbb{Z}^{|H_1(Y;\mathbb{Z})|}$. Boyer, Gordon, and Watson have conjectured that $Y$ is an $L$-space if and only if the fundamental group of $Y$ is non-left-orderable, a conjecture that is known to hold for all non-hyperbolic geometric manifolds. We show that if an $L$-space $Y$ admits a Heegaard diagram whose Heegaard Floer complex has exactly $|H_1(Y;\mathbb{Z})|$ generators and thus has vanishing differential, then $\pi_1(Y)$ is non-left-orderable. We call such manifolds strong $L$-spaces. Examples include double branched covers of alternating links; on the other hand, the Poincaré homology sphere is an $L$-space but not a strong $L$-space. (Received September 21, 2011)