The First Measurable is a Lower Bound for the Hanf Number for Joint Embedding.

From [1], if $\mu$ is a strongly compact cardinal, $K$ is an Abstract Elementary Class (AEC) with $\text{LS}(K) < \mu$, and $K$ satisfies joint embedding cofinally below $\mu$, then $K$ satisfies joint embedding $\geq \mu$. The question was raised if the strongly compact upper bound was optimal.

The following theorem provides a lower bound.

**Theorem** [2] There exists an AEC $K$ axiomatized by an $L_{\omega_1,\omega}$-sentence, so that if $\mu$ is the first measurable cardinal, then joint embedding holds/fails cofinally below $\mu$, and everywhere above $\mu$.

This proves that the Hanf number for joint embedding is contained between the first measurable and the first strongly compact. Since these two cardinals can consistently coincide, the upper bound from [1] is consistently optimal.

Moreover, it is consistent that for any club $C$ on the first measurable $\mu$, JEP holds exactly on $\text{lim} \ C$ and everywhere above $\mu$.

**References**

Hanf numbers and presentation theorems in aecs.

A Lower Bound for the Hanf Number for Joint Embedding
pre-print: https://arxiv.org/abs/1808.03017 (Received September 13, 2018)