Fixing $t \in \mathbb{R}$ and a finite field $\mathbb{F}_q$ of odd characteristic, we give an upper bound on the proportion of genus $g$ hyperelliptic curves over $\mathbb{F}_q$ whose zeta function vanishes at $\frac{1}{2} + it$. Our upper bound is independent of $g$ and tends to 0 as $q$ grows. This result is obtained by studying rational points on twisted Hurwitz spaces over finite fields. (Received September 03, 2019)