The connection between Hochschild and cyclic cohomologies with generalized De Rham homology and index theories for arbitrary algebras has long been established by the work of Connes, Karoubi, Loday, Feigin, Tsygan, et al. Here we generalize these cohomology theories even further, essentially creating a theory that establishes a step-wise bridge between the two, which we call "Bridge Cohomology". Motivation for this construction comes from trying to generalize the Hochschild-Kostant-Rosenberg-Connes theorem to manifolds with boundary, and applications in tracial constructions in certain classes of pseudodifferential operators. We end with a geometric example and extend the theorems by Hochschild–Kostant–Rosenberg and Connes to manifolds with boundary. Further geometric and topological interests of this theory include extending Chern-Weil theory to manifolds with boundary via pairings between bridge cohomology and higher K-theories. (Received September 07, 2019)