Points on Quadratic Twists of $X_0(N)$.

Let $X^d(N)$ be the modular curve described as quadratic twist of $X_0(N)$ by a quadratic field $K = \mathbb{Q}(\sqrt{d})$ and $w_N$. Rational points on this twist are $K$-rational points of $X_0(N)$ that are fixed by $\sigma$ composed with $w_N$ where $\sigma$ is the generator of $\text{Gal}(K/\mathbb{Q})$. Unlike $X_0(N)$, it’s not immediate to say that there are points (global or local) on $X^d(N)$. Given $(N, d, p)$ we give necessary and sufficient conditions for existence of a $\mathbb{Q}_p$-rational point on $X^d(N)$, answering the following question of Ellenberg:

For which $d$ and $N$ there exists points on $X^d(N)$ for every completion of $\mathbb{Q}$? (Received September 22, 2009)