Random Metrics

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Contents

CHAPTER 1

Laplace Beltrami Operator

We work on a compact Riemannian manifold (M, g) of dimension n.
We define

$$g_{ij} := \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle, \qquad G := (g_{ij})_{ij},$$
$$g := \det(G), \qquad G^{-1} := (g^{ij})_{ij},$$

Let $\mathfrak{X}(M)$ denote the space of vector fields $X : M \to TM$. Fix $f \in C^k(M)$ for $k \geq 1$. Consider the auxiliar map $\psi : \mathfrak{X}(M) \to \mathbb{R}^M$ defined by $\psi(X) = X(f)$ where $X(f) : M \to \mathbb{R}$ satisfies $X(f)(p) = X(p)(f)$. Since ψ is linear we can introduce the following definition.

Definition 1.0.1 (Gradient)
Given $f \in C^k(M)$ with $k \geq 1$, we define the *gradient* of f, ∇f, to be the vector field on M satisfying that for all $X \in \mathfrak{X}(M)$

$$\langle \nabla f, X \rangle = X(f).$$

In local coordinates,

$$\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}.$$

This implies that if $f \in C^k(M)$ then ∇f is a C^{k-1} vector field.
We also denote by ∇ the Riemannian connection associated to g,
$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. Note that for a fixed $p \in M$ it makes sense to consider $\nabla : T_pM \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. Although we are using the same notation, the reader should not confuse the gradient operator with the connection.

Definition 1.0.2 (Divergence)
Given X a C^k vector field on M with $k \geq 1$, we define the real valued function on M

$$div\, X : M \to \mathbb{R},$$

$$div\, X(p) = trace(Y \to \nabla_Y X).$$

In local coordinates, if $X = \sum_j a_j \frac{\partial}{\partial x_j}$,

$$div\, X = \frac{1}{\sqrt{g}} \sum_j \frac{\partial (a_j \sqrt{g})}{\partial x_j}.$$

Definition 1.0.3 (Laplacian)
Given $f \in C^k(M)$ with $k \geq 1$, we define the *Laplacian* of f, Δf, by

$$\Delta f := div(\nabla f).$$

It is easy to see that the Laplacian is a linear operator on $C^k(M)$. In local coordinates,

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{g} \frac{\partial f}{\partial x_j} \right).$$

In particular, if $f \in C^k(M)$ then $\Delta f \in C^{k-2}(M)$.
The Laplacian on M is self-adjoint, nonnegative-definite and has discrete spectrum. Its eigenfunctions play a role similar to trigonometric polynomials on the circle:

Theorem 1.0.4
For compact and connected M, there exists a complete orthonormal basis $\{\phi_0, \phi_1, \dots\}$ of $L^2(M)$ consisting of eigenfunctions of Δ with ϕ_j having eigenvalue λ_j satisfying

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \to \infty.$$

Let ω_n be the volume of the unite ball in $\mathbb{R}^n$,

$$\omega_n := \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

Then the following theorem holds for the eigenvalues of Δ:

Theorem 1.0.5 (Weyl's asymptotic formula)
Let M be a compact Riemannian manifold with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \dots$, each distinct eigenvalue repeated according to its multiplicity.
Then for $N(\lambda) := \#\{j : \lambda_j \leq \lambda\}$, we have

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} Vol(M) \lambda^{n/2}, \qquad \lambda \to \infty.$$

In particular,

$$\lambda_j \sim \frac{\sqrt{2\pi}}{(\omega_n Vol(M))^{2/n}} j^{2/n}, \qquad j \to \infty.$$

For future reference, we define the heat kernel on M and summarize its main properties. Set $\mathcal{C} := \{f \in C^0(M \times (0,\infty)) : f(\cdot,t) \in C^2(M) \text{ and } f(x,\cdot) \in C^1((0,\infty))\}$. The heat operator $L : \mathcal{C} \to \mathcal{C}$ is defined by $L := \Delta - \partial/\partial t$. In this setting the *Heat equation* is given by

$$Lu = 0.$$

We say that a *fundamental solution* of the heat equation is a continuous function $p : M \times M \times (0,\infty) \to \mathbb{R}$ which is C^2 with respect to x, C^1 with respect to t and such that

$$L_x p = 0 \qquad \text{and} \qquad \lim_{t\to 0} p(\cdot, y, t) = \delta_y.$$

It can be shown that

$$p(x,y,t) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \phi_j(x)\phi_j(y).$$

The following asymptotic expansion for the heat kernel is standard [**?**]:

Theorem 1.0.6

$$p(x,x,t) \sim_{t\to 0^+} \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) t^{j-n/2};$$

here $a_j(x)$ *is the* j*-th* heat invariant*, where*

$$a_0(x) = 1, \;\; a_1(x) = R(x)/6.$$

In particular,

$$\lim_{t\to 0^+} e(x,x,t)t^{n/2} = \frac{1}{(4\pi)^{n/2}}.$$

CHAPTER 2

Sobolev Spaces

This Chapter follows the exposition given in "Non-perturbative conformal quantum gravity" , [?].

2.1 Sobolev spaces as subsets of $\mathbb{R}^\infty$

Our aim in this section is to show that the sobolev space $H^r \subset \mathbb{R}^\infty$ after some identifications. Indeed, we will show that

$$H^r = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \left\{ (u_j)_j \in R^\infty : \sum_{j=0}^{m} (\lambda_j + 1)^r u_j^2 < n \right\}. \tag{2.1}$$

Let $\phi_0, \phi_1, \ldots$ be an orthornormal basis of $L^2(M)$ of eigenfunctions of the Laplacian with corresponding eigenvalues $0 = \lambda_0 < \lambda_1 < \ldots$, that is, $\Delta\phi_j + \lambda_j \phi_j$.

Definition 2.1.1 (Sobolev Spaces)
For each real number r, the *Sobolev space* $H^r(M)$ is defined to be the completion of $C^\infty(M)$ relative to the inner product

$$\langle f, h \rangle_r := \langle (I - \Delta)^r f, h \rangle_{L^2(M)} \quad ,$$

where I is the identity operator.

Notice that $\langle (I - \Delta)^r f, h \rangle = \sum (\lambda_j + 1)^r f_j h_j$, where f_j, h_j are the Fourier coefficients of f and h repectively ($f_j = \langle f, \phi_j \rangle$).

Remark 2.1.2
If we set $e_k := (\lambda_k+1)^{-r/2}\phi_k$ then $e_0, e_1, \ldots.$. is an orthonormal basis of $H^r(M)$. Indeed,

$$\|e_k\|_r^2 = \sum_j (\lambda_j + 1)^r \langle e_k, \phi_j\rangle^2 = 1.$$

We will denote by $C^k(M)$ the Banach space of C^k functions on M equiped with the norm

$$\|f\|_k := \max_{x\in M}\{|\nabla^m f(x)| : \ 0 \le m \le k\},$$

where $\nabla^m f$ is the covariant tensor of degree m obtained by applying m times the Levi-Civita covariant derivative to f.

The norms of $C^k(M)$ and $H^r(M)$ can be compared by means of the Sobolev inequalities:

Theorem 2.1.3 (Sobolev embedding Theorem)
For $0 \le k < r - n/2$ there exists a constant $C_{k,r}$ such that for all $f \in C^k(M)$

$$\|f\|_k \le C_{k,r}\|f\|_r.$$

Definition 2.1.4 (Space of Distributions)
We define the *Space of Distributions* $D(M)$ by

$$D(M) := \{u : C^\infty(M) \to \mathbb{R}, \text{ linear and continuous}\}.$$

It can be shown that

$$u \in D(M) \longleftrightarrow \exists r, K_r \ \text{ such that } |u(f)| \le K_r\|f\|_r \ \ \forall f \in C^\infty(M).$$

For every $u \in D(M)$ and j set $u_j := u(\phi_j)$. Notice that with this notation $\exists r, K_r$ such that

$$|u_j| \le K_r\|\phi_j\|_r = K_r(c_j + 1)^{r/2}.$$

Since $(I - \Delta)^s : C^\infty(M) \to C^\infty(M)$ extends to an isometry $(I - \Delta)^s : H^r(M) \to H^{r-2s}(M)$, there is a natural pairing $H^{-r} \times H^r \to \mathbb{R}$ given by

$$(f, h) := \left\langle (I - \Delta)^{-r/2} f, (I - \Delta)^{r/2} h \right\rangle_0.$$

In this situation we have $|(f, h)| \le \|f\|_{-r}\|h\|_r$. Therefore, we may identify $H^r M)$ with the dual space of $H^{-r}(M)$ under $h \to (\cdot, h)$. Moreover, the functional $(\cdot, h)$ is continuous on $H^{-r}(M)$ and so is its restriction to $C^\infty(M)$. As a consequence, from now on we will identify

$$H^r(M) \longleftrightarrow \{u \in D(M) : \ u \text{ is continuos w.r.t. } \|\cdot\|_r\}$$

via

$$h \in H^r(M) \longrightarrow (\cdot, h)\ |_{C^\infty(M)} \in D(M).$$

For $h \in H^r(M) \subset D(M)$, we have that $h_j = h(\phi_j) = (\phi_j, h)$ and

$$\|h\|_r^2 = \langle h, h \rangle_r = \langle (I - \Delta)^r h, h \rangle = \sum (\lambda_j + 1)^r h_j^2.$$

Thus, we have the following identification:

$$H^r(M) \longleftrightarrow \left\{ u \in D(M) : \sum (\lambda_j + 1)^r u_j^2 < \infty \right\}.$$

Finally we will identify a distribution $u \in D(M)$ with its sequence of Fourier coefficients $\{u_j\}_j \in \mathbb{R}^\infty$. So

$$H^r(M) \longleftrightarrow \left\{ \{u_j\}_j \in \mathbb{R}^\infty : \sum (\lambda_j + 1)^r u_j^2 < \infty \right\}, \tag{2.2}$$

or what is the same, (2.1) holds.

2.2 Probablity Measures on Sobolev Spaces

Given a sequence of positive numbers $\{s_0, s_1, \dots\}$ we can define a measure ν on $\mathbb{R}^\infty$ with sigma-field generated by cylinder sets of the form

$$E(i_1, \dots, i_n; A) := \{x \in \mathbb{R}^\infty : (x_{i_1}, \dots, x_{i_N}) \in A\}$$

where A is a Borell subset of $\mathbb{R}^n$. We will define ν on the cylinder sets and by the Kolmogorov construction we will extend ν to a countably additive measure on the entire sigma-field. We define ν as follows:

$$\nu(E(i_1, \dots, i_N; A)) := \prod_{j=1}^{N} (2\pi s_{n_j})^{-1/2} \int_A e^{-\frac{1}{2} \sum_{i=1}^{N} x_{i_j}^2 / s_{n_j}} dx_{i_1} \dots dx_{i_N}. \tag{2.3}$$

Here if $s = 0$ then $(2\pi s)^{-1/2} e^{-x^2/2s} = \delta(x)$.

Remark 2.2.1
With this construction ν turns to be a mean zero gaussian probability measure on $\mathbb{R}^\infty$.

Via identification (2.1) the Sobolev spaces $H^r(M) \subset \mathbb{R}^\infty$ are ν-measurable.

Theorem 2.2.2 (Mourier)
If ν is a mean zero gaussian porbability measure on a Hilbert space H, there is a positive trace-class operator S such that

$$\langle Sx, y \rangle = \int_H \langle x, z \rangle \langle z, y \rangle \; d\nu(z).$$

Using Mourier's Theorem we aim to prove the next useful result [?, Proposition 1].

Theorem 2.2.3 (Bleecker)
Let ν be the probability measure on $\mathbb{R}^\infty$ defined by (3.8)*. Then,*

$$\nu(H^r(M)) = 1 \qquad \textit{if and only if} \qquad \sum_k (\lambda_k + 1)^r s_k < \infty.$$

Proof. If $\nu(H^r(M)) = 1$ by Theorem 2.2.2 there exists a positive trace-class operator S_r such that

$$\langle S_r x, y\rangle_r = \int_{H^r(M)} \langle x, z\rangle_r \langle z, y\rangle_r \; d\nu(z).$$

Set $x = y = e_k$ with e_k defined as in Remark 2.1.2 and observe that $\langle e_k, z\rangle_r = (\lambda_j + 1)^{r/2} z_j$ to obtain

$$\langle S_r e_k, e_k\rangle_r = \int_{H^r(M)} (\lambda_k + 1)^r z_k^2 \; d\nu(z).$$

Let $X_k \sim \mathcal{N}(0, s_k)$. Using that

$$s_k = Var(X_k) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s_k}} e^{-z_k^2/(2s_k)} z_k^2 \; dz_k = \int_{\mathbb{R}^\infty} z_j^2 \; d\nu(z) = \int_{H^r} z_j^2 \; d\nu(z)$$

and summing in k we conclude that

$$tr(S_r) = \sum_k (\lambda_k + 1)^r s_k.$$

Since $tr(S_r) < \infty$ we must have $\sum_k (\lambda_k + 1)^r s_k < \infty$.

Conversely, suppose $\sum_k (\lambda_k + 1)^r s_k < \infty$ holds.
Recall that $\sum_k (\lambda_k + 1)^r s_k = \sum_k \int_{\mathbb{R}^\infty} \langle e_k, z\rangle_r \; d\nu(z)$ and apply the Monotone Convergence theorem to the right hand side of previous equality to obtain

$$\sum_k (\lambda_k + 1)^r s_k = \int_{\mathbb{R}^\infty} \sum_k (\lambda_k + 1)^r z_k^2 \; d\nu(z) = \int_{\mathbb{R}^\infty} \|z\|_r^2 \; d\nu(z).$$

We conclude that we must have $\int_{\mathbb{R}^\infty} \|z\|_r^2 \; d\nu(z) < \infty$. Therefore, $\|z\|_r < \infty$ a.s. or equivalently, $z \in H^r$ a.s.

$\square$

CHAPTER 3

Random Fields

We present an introduction to Random Fields and Borell-TIS inequality that follows the exposition given in the book "Random Fields and Geometry" , [?].

Definition 3.0.4 (Random field)
Let $(\Omega, \mathcal{M}, \mathbb{P})$ be a complete probability space and M a manifold. Then a measurable mapping $f : \Omega \to \mathbb{R}^M$ is called a real-valued random field.

Therefore, $f(\omega) : M \to \mathbb{R}$ is a function and $f(\omega)(x)$ is its value at $x \in M$. Usually we will adopt the notation $f_x = f(x) = f(x, \omega) = f(\omega)(x)$.

Definition 3.0.5 (Gaussian Random Variable)
A random variable $X : \Omega \to \mathbb{R}$ is said to be *Gaussian* if for some $a \in \mathbb{R}$ and $\sigma > 0$ its density function is

$$\varphi(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(t-a)^2/(2\sigma^2)}, \qquad t \in \mathbb{R}.$$

We abbreviate this by writing $X \sim \mathcal{N}(a, \sigma^2)$. It is easy to check that $\mathbb{E}\{X\} = a$ and $Var(X) = \sigma^2$. When $X \sim \mathcal{N}(0, 1)$ we say that X has a *standard* normal distribution. We will also adopt the notation

$$\Psi(x) := 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \varphi(t)\, dt. \tag{3.1}$$

Notice that $\mathbb{P}\{X > u\} = \Psi(u)$. Altough Ψ doesn't have a nice explicit formula, there are bounds that hold for every $x > 0$:

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) \varphi(x) < \Psi(x) < \frac{1}{x} \varphi(x). \tag{3.2}$$

In what follows, we will always work with *centered Gaussian random fields.* This means that f will be a random field on M such that for every $x \in M$, $f(x)$ is a centered Gaussian Random Variable.

3.1 Borell-TIS inequality

In this section we introduce an inequality that gives a universal bound for the excursion probability

$$\mathbb{P}\left\{\sup_{x\in M} f(x) \geq u\right\}, \qquad u > 0,$$

for any centered continuous Gaussian field. This inequality was discovered independently, by Borell and Tsirelson, Ibragimov, and Sudakov. We shall call it the Borell-TIS inequality. Before stating the inequality we need to introduce some notation. From now on we will write

$$\|f\|_M := \sup_{x\in M} f(x).$$

Notice that despite the norm notation, $\|\cdot\|_M$ is not a norm. Observe also that if we need to bound the tail of $\sup_x |f(x)|$, the symmetry of the Gaussians gives

$$\mathbb{P}\left\{\sup_{x\in M} |f(x)| \geq u\right\} \leq 2\mathbb{P}\left\{\sup_{x\in M} f(x) \geq u\right\}. \tag{3.3}$$

Indeed, $\{\sup_{x\in M} |f(x)| \geq u\} \subset \{\exists x \in M : \ f(x) > u\} \cup \{\exists x \in M : \ f(x) < -u\}$, and $\mathbb{P}\{\exists x \in M : \ f(x) > u\} = \mathbb{P}\{\exists x \in M : \ f(x) < -u\}$ because $f(x)$ is centered for every $x \in M$.

We shall also define

$$\sigma_M^2 := \sup_{x\in M} \mathbb{E}\left\{f_x^2\right\}.$$

Notice that if f is centered then $\sigma_M^2 := \sup_{x\in M} Var\{f_x\}$.

Theorem 3.1.1 (Borell-TIS)
Let f_x be a centered gaussian field a.s. bounded on M. Then $\mathbb{E}\{\|f\|_M\} < \infty$ and for all $u > 0$,

$$\mathbb{P}\left\{\|f\|_M - \mathbb{E}\{\|f\|_M\} > u\right\} \leq e^{-u^2/(2\sigma_M^2)}.$$

Corollary 3.1.2
There exists a contant $C > 0$ depending only on $\mathbb{E}\{\|f\|_M\}$ for which

$$\mathbb{P}\{\|f\|_M > u\} \leq e^{Cu-u^2/(2\sigma_M^2)}$$

provided $u > \mathbb{E}\{\|f\|_M\}$.

3.2 Parametrizing random metrics in a conformal class

This Section follows the exposition given in "Non-perturbative conformal quantum gravity" , [?].

We denote by (M, g) the Riemannian manifold with metric tensor g. If the angles between two vectors with respect to g_0 and g_1 are always equal at each point of the manifold, the Riemannian metrics g_0 and g_1 on M are said to be conformally related, or to be *conformal to each other*. It is known that the necessary and sufficient condition for g_0 and g_1 of M to be conformal to each other is that there exists a function f on M such that $g_1 = e^f g_0$. We call such a change of metric $g_0 \to g_1$ a conformal change of Riemannian metric.
We consider a conformal class of metrics on a Riemannian manifold M of the form

$$g_1 = e^{af} g_0, \tag{3.4}$$

where g_0 is a *reference* Riemannian metric on M, a is a constant, and $f = f(x)$ is a C^2 function on M.
Given a metric g_0 on M and the corresponding Laplacian Δ_0, let $\{\lambda_j, \phi_j\}$ denote an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $-\Delta_0$; we let $\lambda_0 = 0, \phi_0 = 1$. We define a random conformal multiple $f(x)$ by

$$f(x) = -\sum_{j=1}^{\infty} a_j c_j \phi_j(x), \tag{3.5}$$

where $a_j \sim \mathcal{N}(0, 1)$ are i.i.d standard Gaussians, and c_j are positive real numbers. We write the minus sign for convenience purposes only.

We assume that $c_j = F(\lambda_j)$, where $F(t)$ is an eventually monotone decreasing function of t, $F(t) \to 0$ as $t \to \infty$. For example, we may take $c_j = e^{-\tau\lambda_j}$ or $c_j = \lambda_j^s$.

3.2.1 Regularity of the conformal factor

We can think of $L^2(M)$ as a subset of $\mathbb{R}^\infty$ by associating to every function $f \in L^2$ its sequence of Fourier coefficients $(f_j)_j$. This way

$$L^2(M) \longleftrightarrow \left\{ (f_j)_j \in \mathbb{R}^\infty : \sum f_j^2 < \infty \right\}. \tag{3.6}$$

Recall also from Chapter 2 that for every r we can identify H^r with a subset of $\mathbb{R}^\infty$ as follows from (2.2)

$$H^r(M) \longleftrightarrow \left\{ (f_j)_j \in \mathbb{R}^\infty : \sum (\lambda_j + 1)^r f_j^2 < \infty \right\}. \tag{3.7}$$

As we did in Section 2, we equip the space of functions $L^2(M) \subset \mathbb{R}^\infty$ with a probability measure. We define it using the sequence $s_j = c_j^2$ for c_j as in (3.5). Thus, $\nu = \nu_{\{c_n^2\}_{n=1}^\infty}$

is generated by the densities on the finite cylinder sets

$$d\nu_{(n_1,n_2,\dots n_l)}(f) = \frac{1}{\prod_{j=1}^{l}(2\pi c_{n_j}^2)^{1/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{l}\frac{f_{n_j}^2}{c_{n_j}^2}\right) df_{n_1}\dots df_{n_l}, \tag{3.8}$$

where $f_n = \langle f, \phi_j\rangle_{L^2(M)}$ are the Fourier coefficients, see Section 2.2.

We note that the Fourier coefficients f_j of the conformal factor f defined as in equation (3.5) are centered Gaussians of variance c_j^2, that is, $f_j \sim \mathcal{N}(0, c_j^2)$. Moreover, since the a_j's are i.i.d. the f_j are independent. Since the density of the joint distribution is the product of the individual densities when the variables are independent, this implies that equation (3.8) describes the density function of the joint distribution of $(f_{n_1}, \dots, f_{n_l})$. In particular, $\mathbb{P}\{(f_{n_1}, \dots, f_{n_l}) \in A\} = \nu_{(n_1,n_2,\dots n_l)}(A)$. The latter observation translates, via Kolmogorov's construction, to

$$\mathbb{P}\{(f_j)_j \in A\} = \nu(A).$$

The preceding observation lead us to conclude that

$$\mathbb{P}\{f \in H^r(M)\} = \nu(H^r(M)).$$

It follows that $f \in H^r(M)$ a.s. is and only if $\nu(H^r(M)) = 1$. Therefore, the smoothness of the Gaussian random field f can be derived from a restatement of Theorem 2.2.3:

Proposition 3.2.1 (Regularity on Sobolev space)

$$f \in H^r(M) \qquad \textit{if and only if} \qquad \sum_j (\lambda_j + 1)^r c_j^2 < \infty.$$

Choosing $c_j = \lambda_j^{-s}$, $\sum_{j=1}^{\infty}(\lambda_j+1)^r c_j^2 < \infty$ translates to $\sum_{j=1}^{\infty}\lambda_j^{r-2s} < \infty$.
It follows from Weyl's law that $\lambda_j \sim j^{2/n}$ as $j \to \infty$; so

$$\sum_{j=1}^{\infty}\lambda_j^{r-2s} < \infty \longleftrightarrow \sum_{j=1}^{\infty} j^{\frac{2(r-2s)}{n}} < \infty.$$

We find that

$$\text{If} \quad \frac{2(r-2s)}{n} < -1, \text{ then } f(x) \in H^r(M) \textit{ a.s.}$$

Equivalently,

$$\text{If } \; r < 2s - \frac{n}{2}, \text{ then } f(x) \in H^r(M) \textit{ a.s.}$$

By the Sobolev embedding theorem, $H^r(M) \subset C^k(M)$ for $k + \frac{n}{2} < r$. Substituting into the formula above, we find that

$$f \in C^k(M) \quad \text{if} \quad k + \frac{n}{2} < r < 2s - \frac{n}{2}.$$

Rephrasing,

$$If\ c_j = O(\lambda_j^{-s}),\ \text{and}\ s > \frac{n+k}{2}, \quad then\ f(x) \in C^k(M)\ a.s. \tag{3.9}$$

We will be mainly interested in $k = 0$ and $k = 2$. Accordingly, we conclude the following

Proposition 3.2.2

- *If* $c_j = O(\lambda_j^{-s})$ *and* $s > n/2$, *then* $f \in C^0(M) a.s.$
- *If* $c_j = O(\lambda_j^{-s})$ *and* $s > n/2 + 1$, *then* $f \in C^2(M) a.s.$
- *If* $c_j = O(\lambda_j^{-s})$ *and* $s > n/2 + 1$, *then* $\Delta_0 f \in C^0(M) a.s.$
- *If* $c_j = O(\lambda_j^{-s})$ *and* $s > n/2 + 2$, *then* $\Delta_0 f \in C^2(M) a.s.$

CHAPTER 4

Random Scalar Curvature

This Chapter presents a selection of the results proven in "Scalar curvature and Q-curvature of random metrics"*, [?].*

Since the 19th century, many results have been established comparing geometric and topological properties of manifolds where the (sectional or Ricci) curvature is bounded from above or from below, with similar properties of manifolds of constant curvature. Examples include Toponogov theorem (comparing triangles); sphere theorems of Myers and Berger-Klingenberg for positively-curved manifolds; volume of the ball comparison theorems of Gromov and Bishop; splitting theorem of Cheeger and Gromoll; Gromov's pre-compactness theorem; theorems about geodesic flows and properties of fundamental group for negatively-curved manifolds; and numerous other results.

When studying such questions for random Riemannian metrics, a natural question is to estimate the *probability* of the metric satisfying certain curvature bounds, in a suitable regime.

In the present chapter, we consider a random conformal perturbation g_1 of the reference metric g_0 , and study the following questions about the scalar curvature R_1 of the new metric:

i) Assuming $R_0 \neq 0$, estimate the probability that R_1 changes sign;

ii) Estimate the probability that $||R_1 - R_0||_\infty > u$, where $u > 0$ is a parameter.

4.1 Random metrics in a conformal class

As explained in Section 3.2 we consider a conformal class of metrics on a Riemannian compact manifold M of the form

$$g_1(a) := e^{af} g_0, \tag{4.1}$$

where g_0 is a "reference" Riemannian metric on M, a is a constant, and $f = f(x)$ is a C^2 function on M.
Given a metric g_0 on M and the corresponding Laplacian Δ_0, let $\{\lambda_j, \phi_j\}$ denote, as usual, an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of $-\Delta_0$; we let $\lambda_0 = 0, \phi_0 = 1$. We define a random conformal multiple $f(x)$ by

$$f(x) = -\sum_{j=1}^{\infty} a_j c_j \phi_j(x), \tag{4.2}$$

where $a_j \sim \mathcal{N}(0,1)$ are i.i.d standard Gaussians, and c_j are positive real numbers, and we use the minus sign for convenience purposes only.

Remark 4.1.1 (Smoothness)
By Proposition 3.2.2 the smoothness of the metric $g_1 = e^{af} g_0$ is almost surely determined by the coefficients c_j. In some sense, a can be regarded as the radius of a sphere (in an appropriate space of Riemannian metrics on M) centered at g_0. Most of the results in this paper hold in the limit $a \to 0$; thus, we are studying *local* geometry of the space of Riemannian metrics on M.

4.1.1 Volume

We next consider the volume of the random metric in (4.1).
Since the volume form associated to the metric g_1 is given in local coordinates by $dV_1 = \sqrt{|g_1|}\, dx_1 \wedge \cdots \wedge dx_n$, where $|g_1|$ is the absolute value of the determinant of g_1, the volume element dV_1 corresponding to g_1 is given by

$$dV_1 = e^{naf/2} dV_0, \tag{4.3}$$

where dV_0 denotes the volume element corresponding to g_0.
We consider the random variable $V_1 = \mathrm{vol}(M, g_1)$. We shall prove the following

Proposition 4.1.2
With the same notation as above,

$$\lim_{a \to 0} \mathbb{E}\{V_1(a)\} = V_0,$$

where V_0 denotes the volume of (M, g_0).

Proof. Recall that $f(x)$ defined by (4.2) is a mean zero Gaussian with variance $\sigma(x)^2 = r_f(x,x)$. One may compute explicitly $\mathbb{E}\left\{e^{naf(x)/2}\right\}$. Suppose $X \sim \mathcal{N}(0,1)$, then for every t

$$\mathbb{E}\left\{e^{tX}\right\} = \frac{1}{\sqrt{2\pi}} \int e^{tx} e^{-\frac{x^2}{2}}\, dx = e^{t^2} \frac{1}{\sqrt{2\pi}} \int e^{-\frac{(x-t)^2}{2}}\, dx = e^{t^2}.$$

In particular, for $Y \sim \mathcal{N}(0,\sigma^2)$, say $Y = \sigma X$ with $X \sim \mathcal{N}(0,1)$, we have $\mathbb{E}\left\{e^{tY}\right\} = e^{\frac{(t\sigma)^2}{2}}$. In our situation this translates to

$$\mathbb{E}\left\{e^{naf(x)/2}\right\} = e^{\frac{1}{8}n^2a^2 r_f(x,x)}.$$

Hence, using Fubini's theorem we obtain

$$\begin{aligned}\mathbb{E}\left\{V_1(a)\right\} &= \mathbb{E}\left\{\int_M dV_1\right\} = \int_\Omega \int_M dV_1\, d\mathbb{P} = \int_\Omega \int_M e^{\frac{naf}{2}}\, dV_1\, d\mathbb{P} = \int_M \int_\Omega e^{\frac{naf}{2}}\, d\mathbb{P} dV_0 \\ &= \int_M \mathbb{E}\left\{e^{\frac{naf}{2}}\right\} dV_0 = \int\limits_M e^{\frac{1}{8}n^2a^2 r_f(x,x)} dV_0.\end{aligned}$$

Since $r_f(x,x)$ is continuous, as $a \to 0$, the latter converge to V_0 by the dominated convergence theorem. □

4.2 Scalar curvature

The *scalar curvature* assigns to each point on a Riemannian manifold a real number determined by the geometry of the manifold near that point. In two dimensions, the scalar curvature is twice the Gaussian curvature, and completely characterizes the curvature of a surface.

Let (M, g) be an n-dimensional compact manifold, $n \geq 2$. Recall that the *Riemann curvature tensor* is defined by $R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, where ∇ denotes the Levi-Civita connection.

Definition 4.2.1 (Scalar curvature)
For $x \in M$, let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of T_xM. The *Scalar curvature* at x, $R(x)$, is then defined by

$$R(x) = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \langle R(v_i, v_j)v_i, v_j \rangle .$$

Geometrically, the scalar curvature $R(x)$ represents the amount by which the volume of a geodesic ball in a curved Riemannian manifold deviates from that of the standard ball in Euclidean space. Indeed,

$$\mathrm{vol}(B_M(x,r)) = \mathrm{vol}(B_{\mathbf{R}^n}(r)) \left[1 - \frac{R(x)r^2}{6(n+2)} + O(r^4)\right].$$

We will study the behavior of scalar curvature for random Riemannian metrics in a fixed conformal class, where the conformal factor is a random function possessing certain smoothness. We consider random metrics that are close to a "reference" metric that we denote g_0.
Our techniques are inspired by [**?**, **?**, **?**, **?**].

4.2.1 Scalar curvature in a conformal class

It is well-known that the scalar curvature R_1 of the metric g_1 in (4.1) is related to the scalar curvature R_0 of the metric g_0 by the following formula ([**?**, §5.2, p. 146])

$$\boxed{R_1 = e^{-af}\left[R_0 - a(n-1)\Delta_0 f - a^2(n-1)(n-2)|\nabla_0 f|^2/4\right],} \tag{4.4}$$

where Δ_0 is the negative definite Laplacian for g_0 ($\Delta_0\phi_j = \lambda_j\phi_j$), and ∇_0 is the gradient corresponding to g_0. We observe that the last term vanishes when $n = 2$:

$$\boxed{R_1 = e^{-af}(R_0 - a\Delta_0 f).} \tag{4.5}$$

Substituting (4.2), we find that

$$R_1(x)e^{af(x)} = R_0(x) - a\sum_{j=1}^{\infty}\lambda_j a_j c_j \phi_j(x). \tag{4.6}$$

The smoothness of the scalar curvature for the metric g_1 is determined by the random field $a(n-1)\Delta f + a^2(n-1)(n-2)|\nabla f|^2/4$.
We remark that it follows easily from (4.4) and Corollary 3.2.2 that

Proposition 4.2.2

If $R_0 \in C^0(M)$ *and* $c_j = O(\lambda_j^{-s}), s > n/2+1$ *then* $R_1 \in C^0(M)$ *a.s.*

If $R_0 \in C^2(M)$ *and* $c_j = O(\lambda_j^{-s}), s > n/2+2$ *then* $R_1 \in C^2(M)$ *a.s.*

For $n = 2$, the expression (4.4) for the curvature is of a particular simple shape, studying which it is convenient to work with the random centered Gaussian field

$$h(x) := \Delta_0 f(x) = \sum_{j=1}^{\infty} a_j c_j \lambda_j \phi_j(x) \tag{4.7}$$

In principle, one may derive any property of h in terms of the function r_h and its derivatives by the Kolmogorov theorem.

4.3 Sign of R_1 on Surfaces

In this section, we shall use Borell-TIS inequality to estimate the probability that the curvature R_1 of a random metric on a compact orientable surface S_γ of genus $\gamma \neq 1$ changes sign assuming that the curvature R_0 for reference metric has constant sign.

We remark that by Gauss-Bonnet theorem, when S_γ is a compact surface without boundary of genus γ, $\int_{S_\gamma} R = 4\pi(1-\gamma)$. In particular for $S = \mathbf{T}^2$ we have that $\int_{\mathbf{T}^2} R = 0$ so the curvature R_0 has to change sign on $\mathbf{T}^2$.
We denote by $M = S_\gamma$ a compact surface of genus $\gamma \neq 1$. We choose a reference metric g_0 so that R_0 has constant sign (positive if $S_\gamma = S^2$, and negative if S_γ has genus ≥ 2). We remark that by uniformization theorem, such metrics exist in every conformal class. In fact, every metric on M is conformally equivalent to a metric with $R_0 \equiv const$.
Define the random metric on S_γ by $g_1(a) = e^{af} g_0$, (as in (4.1)) and f is given by (4.2), as usually.
In this section we shall estimate the probability $\mathbf{P}(a)$ defined by

$$\mathbf{P}(a) := \mathbb{P}\Big\{\exists x \in M : \operatorname{sgn}(R_1(a)(x)) \neq \operatorname{sgn}(R_0)\Big\}, \tag{4.8}$$

i.e. $\mathbf{P}(a)$ is the probability that the curvature $R_1(a)$ of the random metric $g_1(a)$ *changes sign* somewhere on S.

We shall estimate $\mathbf{P}(a)$ in the limit $a \to 0$. Geometrically, since $g_1(a) \to g_0$, $\mathbf{P}(a)$ should go to zero as $a \to 0$; below, we shall estimate the *rate*. To do that, we shall use the strong version of the Borell-TIS inequality 3.1.2. We prove the following

Theorem 4.3.1
Assume that $R_0 \in C^0(M)$ has constant sign and that $c_j = O(\lambda_j^{-s}), s > 2$. Then there exist constants $C_1, C_2 > 0$ such that the probability $\mathbf{P}(a)$ satisfies

$$C_1(a - a^3)e^{-1/(2a^2\sigma_v^2)} \leq \mathbf{P}(a) \leq e^{C_2/a - 1/(2a^2\sigma_v^2)},$$

as $a \to 0$. In particular

$$\lim_{a\to 0} a^2 \ln \mathbf{P}(a) = \frac{-1}{2\sigma_v^2}.$$

Proof.
Recall that according to equation (4.5), in dimension two, $R_1 = e^{-af}(R_0 - ah)$ where $h = \Delta_0 f$ was defined earlier in (4.7). Therefore, using that $R_0 \neq 0$, we get $\operatorname{sgn}(R_1) = \operatorname{sgn}(R_0)\operatorname{sgn}(1 - ah/R_0)$. It is then convenient to introduce a random field v defined by

$$v(x) = h(x)/R_0(x) \tag{4.9}$$

We remark that $r_v(x,x) = r_h(x,x)/[R_0(x)]^2$, and we let $\sigma_v^2 = \sup_{x\in M} r_v(x,x)$.
We denote by $||v||_M := \sup_{x\in M} v(x)$. With this notation $\operatorname{sgn}(R_1(a)(x)) \neq \operatorname{sgn}(R_0)$ if and only if $\operatorname{sgn}(1 - av(x)) = -1$. It follows that

$$\mathbf{P}(a) = \mathbb{P}\{||v||_M > 1/a\}. \tag{4.10}$$

Since $c_j = O(\lambda_j^{-s}), s > 2$, Proposition 4.2.2 implies that h and R_1 are a.s. C^0 and hence bounded, since M is compact.

Assume that the supremum of σ_v^2 is attained at $x = x_0$. We shall use (4.10) to estimate $\mathbf{P}(a)$ from above and below. To get a lower bound for $\mathbb{P}\{\|v\|_M > 1/a\}$, choose $x = x_0$. Since the random variable $v(x_0)$ is Gaussian with mean 0 and variance σ_v^2 we obtain

$$\mathbb{P}\{\|v\|_M > 1/a\} \geq \mathbb{P}\{v(x_0) > 1/a\} = \Psi\left(\frac{1}{a}\right).$$

where the error function Ψ is given by formula (3.1). We obtain the lower bound using inequality (3.2), $\Psi\left(\frac{1}{a}\right) > \frac{1}{\sqrt{2\pi}\sigma_v} e^{-1/(2a^2\sigma_v^2)}(a - a^3)$.

We obtain an upper bound by a straightforward application of Theorem 3.1.2 on our problem. Since v is C^0 a.s. (because h and R_0 are), there exist a constant $C_2 > 0$ so that

$$\mathbb{P}\{\|v\|_M > 1/a\} \leq e^{C_2/a - 1/(2a^2\sigma_v^2)}.$$

□

4.3.1 Using results of Adler-Taylor on the 2-Sphere

The sphere is special in that the curvature perturbation is *isotropic*, so that in particular the variance is constant. In this case a special theorem due to Adler-Taylor gives a precise asymptotics for the excursion probability.

For an integer m let $\mathcal{E}_m$ be the space of spherical harmonics of degree m of dimension $N_m = 2m + 1$ associated to the eigenvalue $E_m = m(m+1)$, and for every m fix an L^2 orthonormal basis $B_m = \{\eta_{m,k}\}_{k=1}^{N_m}$ of $\mathcal{E}_m$.

To treat the spectrum degeneracy it will be convenient to use a slightly different parametrization of the conformal factor than the usual one (4.2)

$$f(x) = -\sqrt{|\mathcal{S}^2|} \sum_{m \geq 1,\, k} \frac{\sqrt{c_m}}{E_m \sqrt{N_m}} a_{m,k} \eta_{m,k}(x), \tag{4.11}$$

where $a_{m,k}$ are standard Gaussian i.i.d. and $c_m > 0$ are some (suitably decaying) constants. For extra convenience we will assume in addition that

$$\sum_{m=1}^{\infty} c_m = 1, \tag{4.12}$$

which has an advantage that a random field $\Delta_0 f(x)$ defined below is of unit variance. We stress that for convenience, in the present section, the random field f defined differently than in the rest of the paper. The reason for the new definitions is spectral degeneracy on S^2.

Lemma 4.3.2
Given a sequence c_m satisfying (4.12), we have

$$f(x) \in H_r(\mathcal{S}^2) \text{ a.s.} \qquad \text{if and only if} \qquad \sum_{m=1}^{\infty} m^{2r-4} c_m < \infty.$$

In what follows we will always assume that

$$c_m = O\left(\frac{1}{m^s}\right). \tag{4.13}$$

Thus $f(x) \in H_r(\mathcal{S}^2)$ precisely for $r < \frac{s}{2} + \frac{3}{2}$.

Theorem 4.3.3
Let $s > 7$, and the metric g_1 on $\mathcal{S}^2$ be given by

$$g_1 = e^{af} g_0$$

where f is given by (4.11)*. Also, let $c_m \neq 0$ for at least one* odd m*. Then as $a \to 0$, the probability that the curvature is everywhere positive is given by*

$$\mathbb{P}\left\{R_1(x) > 0, \forall x \in \mathcal{S}^2\right\} \sim 1 - \left(\frac{C_1}{\sqrt{2\pi}} - \frac{C_2}{a}\right) e^{-\frac{1}{2a^2}},$$

where $C_1 = 2$, $C_2 = \frac{1}{\sqrt{2\pi}} \sum_{m \geq 1} c_m E_m$ and $\alpha > 1$.

4.4 Random real-analytic metrics and comparison results

In this section, we let M be a compact orientable surface, $M \ncong \mathbf{T}^2$. We shall consider random *real-analytic* conformal deformations; this corresponds to the case when the coefficients c_j in (4.2) decay *exponentially.* We shall use standard estimates for the heat kernel to estimate the probabilities computed in the previous section 4.3.
We fix a real parameter $T > 0$ and choose the coefficients c_j in (4.2) to be equal to

$$c_j = e^{-\lambda_j T/2} / \lambda_j. \tag{4.14}$$

Then it follows that

$$r_h(x,x) = p^*(x,x,T) = \sum_{j:\lambda_j > 0} e^{-\lambda_j T} \phi_j(x)^2,$$

where $p^*(x,x,T)$ denotes the *heat kernel* on M <u>*without*</u> the constant term, evaluated at x at time T.
The heat kernel $p(x,y,t) = \sum_j e^{-\lambda_j t} \phi_j(x)\phi_j(y)$ defines a fundamental solution of the heat equation on M. It is well-known that $p(x,y,t)$ is smooth in x, y, t for $t > 0$, and that $p^*(x,y,t)$ decays exponentially in t, [**?**, **?**].

Notice that our choice of the c_j's implies that $c_j = O(\lambda_j^s)$ for every $s > 0$ allowing us to apply the results of Section 4.3.

4.4.1 Comparison Theorem: $T \to 0^+$

By Theorem 1.0.6

$$p(x,x,T) \underset{T \to 0^+}{\sim} \frac{1}{(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x) T^{j-n/2};$$

here $a_j(x)$ is the j-th *heat invariant*. In particular,

$$a_0(x) = 1,\ a_1(x) = R(x)/6.$$

Therefore,

$$\lim_{T\to 0^+} p(x,x,T)T^{n/2} = \frac{1}{(4\pi)^{n/2}}.$$

In particular,

$$\lim_{T\to 0^+} r_h(x,x)T^{n/2} = \lim_{T\to 0^+} p^*(x,x,T)T^{n/2} = \frac{1}{(4\pi)^{n/2}}$$

Assuming that R_0 *has constant sign*, in terms of σ_v^2 the asymptotics translate to

$$\sigma_v^2 = \sup_{x\in M} \frac{r_h(x,x)}{R_0(x)} \sim \sup_{x\in M} \frac{1}{(4\pi T)^{n/2} R_0(x)} \qquad T \to 0^+.$$

All in all, we obtain the following

Proposition 4.4.1
Assume that the coefficients c_j are chosen as in (4.14). *Then as $T \to 0^+$,*

$$\sigma_v^2 \sim \frac{1}{(4\pi T)^{n/2} \inf_{x\in M}(R_0(x))^2}.$$

That is, as $T \to 0^+$, the probability $\mathbf{P}(a)$ is determined by the value of

$$\inf_{x\in M}(R_0(x))^2.$$

Proposition 4.4.1 is next applied to prove a comparison theorem. Let g_0 and g_1 be two distinct reference metrics on M, normalized to have equal volume, and such that $R_0 \equiv const$ and $R_1 \not\equiv const$.

Theorem 4.4.2
Let g_0 and g_1 be two distinct reference metrics on M, normalized to have equal volume, such that R_0 and R_1 have constant sign, $R_0 \equiv const$ and $R_1 \not\equiv const$. Then there exists $a_0, T_0 > 0$ (that depend on g_0, g_1) such that for any $0 < a < a_0$ and for any $0 < t < T_0$, we have $\mathbf{P}(a,T,g_1) > \mathbf{P}(a,T,g_0)$.

Proof. Set $b_1 := \inf_{x\in M}(R_1(x))$. It follows from Gauss-Bonnet's theorem that

$$R_0 V_0(M) = \int_M R_0\, dV_0 = \int_M R_1\, dV_1 > b_1 V_1(M).$$

Since $V_1(M) = V_0(M)$ and $R_1 \not\equiv const$, it follows that $R_0 > b_1$.
Accordingly, as $T \to 0^+$, we have

$$\frac{\sigma_v^2(g_1,T)}{\sigma_v^2(g_0,T)} \asymp \frac{R_0^2}{b_1^2} > 1.$$

The result now follows from Theorem 4.3.1.

□

It follows that in every conformal class, $\mathbf{P}(a, T, g_0)$ is minimized in the limit $a \to 0, T \to 0^+$ for the metric g_0 of constant curvature.

4.4.2 Comparison Theorem: $T \to \infty$

Let M be a compact surface, where the scalar curvature R_0 *has constant sign*. Let $\lambda_1 = \lambda_1(g_0)$ denote the smallest nonzero eigenvalue of Δ_0. Denote by $m = m(\lambda_1)$ the multiplicity of λ_1, and let

$$F := \sup_{x \in M} \frac{\sum_{j=1}^m \phi_j(x)^2}{R_0(x)^2}. \tag{4.15}$$

The number F is finite by compactness and the assumption that R_0 has constant sign on M.

Proposition 4.4.3
Let the coefficients c_j be as in (4.14). Denote by $\sigma_v^2(T)$ the corresponding supremum of the variance of v. Then

$$\lim_{T \to \infty} \frac{\sigma_v^2(T)}{F e^{-\lambda_1 T}} = 1. \tag{4.16}$$

Proof. Recall that

$$r_v(x, x) = \frac{e^*(x, x, T)}{R_0(x)^2}.$$

We write $e^*(x, x, T) = e_1(x, T) + e_2(x, T)$, where

$$e_1(x, T) = e^{-\lambda_1 T} \sum_{j=1}^{m} \phi_j(x)^2,$$

and

$$e_2(x, T) = \sum_{j=m+1}^{\infty} e^{-\lambda_j T} \phi_j(x)^2.$$

Clearly, as $T \to \infty$, we have

$$\lim_{T \to \infty} e^{\lambda_1 T} \sup_{x \in M} \frac{e_1(x, T)}{R_0(x)^2} = F,$$

where F was defined in (4.15). It suffices to show that as $T \to \infty$,

$$\frac{e_2(x, T)}{R_0(x)^2} = o\left(e^{-\lambda_1 T}\right) \tag{4.17}$$

Note that by compactness, there exists $C_1 > 0$ such that $(1/C_1) \le R_0^2(x) \le C_1$ for all $x \in M$. Accordingly, it suffices to establish (4.17) for $\sup_{x \in M} e_2(x, T)$.
We let $\mu := \lambda_{m+1} - \lambda_m$; note that $\lambda_m = \lambda_1$ by the definition of m. We have

$$e_2(x, T) = e^{-\lambda_1 T} \sum_{j=m+1}^{\infty} e^{-(\lambda_j - \lambda_1) T} \phi_j(x)^2. \tag{4.18}$$

Let k be the smallest number such that $\lambda_k > 2\lambda_1$.
We rewrite the sum in (4.18) as $e_2 = e_3 + e_4$, where the first term e_3 is given by

$$e_3(x,T) := \sum_{j=m+1}^{k-1} e^{-(\lambda_j-\lambda_1)T}\phi_j(x)^2 \le e^{-\mu T} \sup_{x\in M} \sum_{j=m+1}^{k-1} \phi_j(x)^2, \tag{4.19}$$

where the last supremum (which we denote by H) is finite by compactness.
The second term e_4 is given by

$$e_4(x,T) := \sum_{j=k}^{\infty} e^{-(\lambda_j-\lambda_1)T}\phi_j(x)^2 \le \sum_{j=k}^{\infty} e^{-\lambda_j T/2}\phi_j(x)^2 \le \sup_{x\in M} e^*(x,x,T/2). \tag{4.20}$$

We remark that as $T\to\infty$, $e^*(x,x,T/2)\to 0$ exponentially fast, uniformly in x.
Combining (4.19) and (4.20), we find that

$$e_2(x,T) = O\left(\left[H\cdot e^{-\mu T} + e^*(x,x,T/2)\right] e^{-\lambda_1 T}\right),$$

establishing (4.17) for $\sup_{x\in M} e_2(x,T)$ and finishing the proof of Proposition 4.4.3. □

Theorem 4.4.4
Let g_0 and g_1 be two reference metrics (of equal area) on a compact surface M, such that R_0 and R_1 have constant sign, and such that $\lambda_1(g_0) > \lambda_1(g_1)$. Then there exist $a_0 > 0$ and $0 < T_0 < \infty$ (that depend on g_0, g_1), such that for all $a < a_0$ and $T > T_0$ we have $\mathbf{P}(a,T;g_0) < \mathbf{P}(a,T;g_1)$.

Proof. By Proposition 4.4.3, we find that for $T > T_1 = T_1(g_0,g_1)$ there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{\sigma_v^2(T,g_1)e^{\lambda_1(g_1)T}}{\sigma_v^2(T,g_2)e^{\lambda_1(g_2)T}} \le C$$

Accordingly, if we choose T_2 so that $e^{(\lambda_1(g_1)-\lambda_1(g_2))T_2} > C$, and take $T > \max\{T_1,T_2\}$, we find that Theorem 4.4.4 follows from the formula above and Theorem 4.3.1. □

It was proved by Hersch in [**?**] that for $M = S^2$, if we denote by g_0 the round metric on S^2, then $\lambda_1(g_0) > \lambda_1(g_1)$ for any other metric g_1 on S^2 of equal area. This immediately implies the following

Corollary 4.4.5
Let g_0 be the round metric on S^2, and let g_1 be any other metric of equal area. Then, there exist $a_0 > 0$ and $T_0 > 0$ (depending on g_1) such that for all $a < a_0$ and $T > T_0$ we have $\mathbf{P}(a,T;g_0) < \mathbf{P}(a,T;g_1)$.

4.5 L^∞ curvature bounds on Surfaces

In section 4.3 we studied the probability of the curvature *changing sign* after a small conformal perturbation, on S^2 and on surfaces of genus greater than one. On the torus

$\mathbf{T}^2$, however, Gauss-Bonnet theorem implies that the curvature has to change sign for every metric, so that question is meaningless.
Accordingly, on $\mathbf{T}^2$ we investigate the probability of another event that is considered very frequently in comparison geometry: the probability that scalar curvature satisfies the L^∞ curvature bounds $||R_1||_\infty < u$, where $u > 0$ is a parameter. Metrics satisfying such bounds for fixed u are called *metrics of bounded geometry.* The argument on $\mathbf{T}^2$ is then modified to study the following natural analogue of the problem on $\mathbf{T}^2$: estimating the probability that $||R_1 - R_0||_\infty < u$. That question is considered on S^2, and on surfaces of genus greater than one.
In this section we *do not* assume that R_0 is constant; *nor* do we assume that R_0 has constant sign.

Definition 4.5.1
We shall consider the following three centered random fields on the surface S:

i.- The random conformal multiple $f(x)$ given by (4.2). We denote its covariance function by $r_f(x,y)$, and we define $\sigma_f^2 = \sup_{x\in S} r_f(x,x)$.

ii.- The random field $h = \Delta_0 f$ defined in (4.7). We denote its covariance function by $r_h(x,y)$, and we define $\sigma_h^2 = \sup_{x\in S} r_h(x,x)$.

iii.- The random field $w = \Delta_0 f + R_0 f = h + R_0 f$. We denote its covariance function by $r_w(x,y)$, and we define $\sigma_w^2 = \sup_{x\in S} r_w(x,x)$. Note that on flat $\mathbf{T}^2$, $R_0 \equiv 0$ and therefore $h \equiv w$.

The random fields f, h and w have constant variance on round S^2; also f and $h = w$ have constant variance on flat $\mathbf{T}^2$.

We shall prove the following theorem:

Theorem 4.5.2
Assume that the random metric is chosen so that the random fields f, h, w are a.s. C^0. Let $a \to 0$ and $u \to 0$ so that

$$\frac{u}{a} \to \infty. \tag{4.21}$$

Then

$$\log \mathbb{P}\Big\{\|R_1 - R_0\|_\infty > u\Big\} \sim -\frac{u^2}{2a^2\sigma_w^2}. \tag{4.22}$$

Remark 4.5.3
On the flat 2-torus, $\sigma_h = \sigma_w$.

Proof.
In the proof, we shall use (Borell-TIS) Theorem 3.1.2; we require that the random fields f, h, w are a.s. C^0. Assuming that $R_0 \in C^0(S)$, sufficient conditions for that are formulated in Corollary 3.2.2; we remark that if h is a.s. C^0, then so is w.

Let S will denote a compact orientable surface ($\mathcal{S}^2, \mathbf{T}^2$ or of genus $\gamma \geq 2$) where the random fields are defined.

Step 1.
Case $S = T^2$.
We introduce a (large) parameter S that will be chosen later. On $\mathbf{T}^2$, we let B_S denote the "bad" event where f is *large*

$$B_S = \{||f||_\infty > S\}. \tag{4.23}$$

Applying Theorem 3.1.2, we find that there exists a constant α_f such that the following estimate holds:

$$\mathbb{P}\{B_S\} = O\left(e^{\alpha_f S - \frac{S^2}{2\sigma_f^2}}\right). \tag{4.24}$$

Case $S = S^2$ or S_γ with $\gamma \geq 2$.
On S^2 and on surfaces of genus ≥ 2 we modify the definition slightly, and let B_S denote the "bad" event that either f or h is large

$$B_S = \{||f||_\infty > S\} \cup \{||h||_\infty > S\}. \tag{4.25}$$

By Theorem 3.1.2 we find that there exist two constants α_f and α_h such that

$$\mathbb{P}\{B_S\} = O\left(e^{\alpha_f S - \frac{S^2}{2\sigma_f^2}} + e^{\alpha_h S - \frac{S^2}{2\sigma_h^2}}\right). \tag{4.26}$$

We have shown that

$$\mathbb{P}\{B_S\} = \begin{cases} O\left(e^{\alpha_f S - \frac{S^2}{2\sigma_f^2}}\right), & S = \mathbf{T}^2; \\ O\left(e^{\alpha_f S - \frac{S^2}{2\sigma_f^2}} + e^{\alpha_h S - \frac{S^2}{2\sigma_h^2}}\right), & \text{otherwise}. \end{cases} \tag{4.27}$$

Step 2.
We denote $A_{u,a}$ the event $\{||R_1 - R_0||_\infty > u\}$; clearly,

$$\mathbb{P}\{A_{u,a}\} = \mathbb{P}\{A_{u,a} \cap B_S\} + \mathbb{P}\{A_{u,a} \cap B_S^c\}. \tag{4.28}$$

We estimate $\mathbb{P}\{A_{u,a} \cap B_S\}$ trivially, $\mathbb{P}\{A_{u,a} \cap B_S\} \leq \mathbb{P}\{B_S\}$. Therefore,

$$\mathbb{P}\{A_{u,a} \cap B_S\} = O(\mathbb{P}\{B_S\}).$$

We next estimate $\text{Prob}(A_{u,a} \cap B_S^c)$. Recall that in dimension two, it follows from (4.5) that

$$R_1 - R_0 = R_0(e^{-af} - 1) - ae^{-af}\Delta_0 f = R_0(e^{-af} - 1) - ae^{-af}h. \tag{4.29}$$

Note that on $\mathbf{T}^2$ we have $R_0 = 0$, and the first term on the right vanishes, hence we get

$$R_1 = -ae^{-af}h$$

in that case.

Case $S = T^2$.
Let S satisfy

$$S = o\left(\frac{1}{a}\right) \qquad \text{and} \qquad \frac{u}{a} = o(S). \tag{4.30}$$

We remark that such S exists because $u \to 0$. Indeed, since $u \to 0$ there exists $b \to 0$ with $u < b < 1$. Set $S = b/a$. Then $u/a < S < 1/a$. Therefore, $S = o\left(\frac{1}{a}\right)$ and $\frac{u}{a} = o(S)$.

On B_S^c, we have $|f(x)| = O(S)$, hence $e^{-af(x)} = 1 + O(aS)$. Let us set from now on

$$\kappa := \frac{u}{a(1 + O(aS))}. \tag{4.31}$$

With this notation

$$\begin{aligned} \mathbb{P}\left\{A_{u,a} \cap B_S^c\right\} &= \mathbb{P}\left\{\left[\|ae^{-af}h\|_\infty > u\right] \cap B_S^c\right\} \\ &= \mathbb{P}\left\{[\|h\|_\infty > \kappa] \cap B_S^c\right\} \\ &= \mathbb{P}\left\{\|h\|_\infty > \kappa\right\} - \mathbb{P}\left\{[\|h\|_\infty > \kappa] \cap B_S\right\} \\ &= \mathbb{P}\left\{\|h\|_\infty > \kappa\right\} + O(\mathbb{P}\left\{B_S\right\}), \end{aligned} \tag{4.32}$$

the last summand being already estimated in (4.27). Plugging (4.27) and (4.32) into (4.28) we obtain

$$\mathbb{P}\left\{A_{u,a}\right\} = \mathbb{P}\left\{\|h\|_\infty > \kappa\right\} + O\left(\mathbb{P}\left\{B_S\right\}\right). \tag{4.33}$$

We will choose S so that $O\left(\mathbb{P}\left\{B_S\right\}\right)$ is negligible. If we do this, it would only remain to estimate $\mathbb{P}\left\{\|h\|_\infty > \kappa\right\}$ since we will have $\mathbb{P}\left\{A_{u,a}\right\} \sim \mathbb{P}\left\{\|h\|_\infty > \kappa\right\}$. To this end we note that by symmetry,

$$\mathbb{P}\left\{\|h\|_{\mathbf{T}^2} > \kappa\right\} \le \mathbb{P}\left\{\|h\|_\infty > \kappa\right\} \le 2\mathbb{P}\left\{\|h\|_{\mathbf{T}^2} > \kappa\right\}, \tag{4.34}$$

and the factor 2 is negligible on the *logarithmic* scale.
To evaluate

$$\mathbb{P}\left\{\|h\|_{\mathbf{T}^2} > \kappa\right\} \tag{4.35}$$

we note that (4.21) together with (4.30) imply that

$$\kappa \to \infty. \tag{4.36}$$

Indeed,

$$\frac{1}{\kappa} = \frac{a}{u} + \frac{a\,O(aS)}{u} = \frac{a}{u} + aS\,O\left(\frac{a}{u}\right)$$

and both aS and a/u go to 0 as $a \to 0$.
Thus, we may apply Theorem 3.1.2 to obtain

$$\mathbb{P}\left\{\|h\|_{\mathbf{T}^2} > \kappa\right\} \le e^{\alpha_h \kappa - \frac{\kappa^2}{2\sigma_h^2}}$$

To get a lower bound for (4.35), we proceed as in section 4.3 and choose $x_0 \in S$ where $\sigma_h^2 = \sup_x r_h(x,x)$ is attained. Then

$$\mathbb{P}\left\{\|h\|_{\mathbf{T}^2} > \kappa\right\} > \mathbb{P}\left\{h(x_0) > \kappa\right\} = \Psi\left(\kappa\right) > \left(\frac{1}{\kappa} - \frac{1}{\kappa^3}\right) e^{-\frac{\kappa^2}{2\sigma_h^2}}.$$

So far we have shown that

$$\log \mathbb{P}\left\{\|h\|_\infty > \kappa\right\} \sim \log \mathbb{P}\left\{\|h\|_{\mathbf{T}^2} > \kappa\right\} \sim -\frac{\kappa^2}{2\sigma_h^2}.$$

Moreover, since $\kappa \sim u/a$ for small a, we obtain the desired result

$$\log \mathbb{P}\left\{\|h\|_\infty > \kappa\right\} \sim -\frac{u^2}{2a\sigma_h^2}.$$

To finish the argument it only remains to show that $\mathbb{P}\left\{A_{u,a}\right\} \sim \mathbb{P}\left\{\|h\|_\infty > \kappa\right\}$. In order to do this, it suffices to show that $\mathbb{P}\left\{\|h\|_\infty > \kappa\right\} \to 0$ faster than $\mathbb{P}\left\{B_S\right\} \to 0$. To show this, notice that

$$\mathbb{P}\left\{\|h\|_\infty > \kappa\right\} = O\left(e^{\alpha_h \kappa - \frac{\kappa^2}{2\sigma_h^2}}\right) \quad \text{and} \quad \mathbb{P}\left\{B_S\right\} = O\left(e^{\alpha_h S - \frac{S^2}{2\sigma_h^2}}\right). \tag{4.37}$$

Observe also that

$$\frac{\kappa/S}{u/(aS)} = \frac{\kappa}{u/a} = \frac{1}{1+O(aS)} \to 1$$

shows that $\kappa/S \to 0$ because $u/a = o(S)$. Since $\kappa \to 0$ faster that $S \to 0$, (4.37) give us the result we were looking for.

<u>*Case $S = S^2$ or S_γ with $\gamma \geq 2$.*</u>

We next consider the case $S = \mathcal{S}^2$ or $S = S_\gamma, \gamma \geq 2$. We want to estimate the probability of the event $\{||R_1 - R_0||_\infty > u\} \cap B_S^c$. Recall from (4.29) that

$$R_1 - R_0 = R_0(e^{-af} - 1) - ae^{-af}h.$$

By the definition of B_S, on B_S^c, we have for $x \in S$, $|f(x)| = O(S)$ and

$$|h(x)| = |\Delta_0 f(x)| = O(S).$$

Again, we choose S so that $aS = o(1)$, and it follows easily from the Taylor expansion of e^{-af} and the definition of w that

$$R_1 - R_0 = -aw - O(aS)(af + ah) = -aw + O(a^2S^2). \tag{4.38}$$

As we did when dealing with $\mathbf{T}^2$, we will choose S later so that

$$\mathbb{P}\left\{A_{u,a} \cap B_S\right\} = o\left(\mathbb{P}\left\{A_{u,a} \cap B_S^c\right\}\right); \tag{4.39}$$

this is only possible under the assumption (4.21) of the present theorem. The equality (4.39) implies that it will be sufficient to evaluate $\mathbb{P}\left\{A_{u,a} \cap B_S^c\right\}$.

On S^2, the isotropic random field w has constant variance σ_w^2 that will be computed later; on $S_\gamma, \gamma \geq 2$ the variance $r_w(x,x)$ is no longer constant, and we denote by σ_w^2 its supremum $\sup_{x\in S_\gamma} r_w(x,x)$.
Therefore (cf. (4.32))

$$\begin{aligned}\mathbb{P}\left\{A_{u,a} \cap B_S^c\right\} &= \mathbb{P}\left\{\left[\|w + O(aS^2)\|_\infty > \frac{u}{a}\right] \cap B_S^c\right\} \\ &= \mathbb{P}\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\} + O(\mathbb{P}\left\{B_S\right\}).\end{aligned}$$

Assuming that (4.39) holds and taking (4.28) into account, we obtain

$$\begin{aligned}\mathbb{P}\left\{A_{u,a}\right\} &= \mathbb{P}\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\} \\ &\quad + O\left(e^{\alpha_f S - \frac{S^2}{2\sigma_f^2}} + e^{\alpha_h S - \frac{S^2}{2\sigma_h^2}}\right).\end{aligned} \tag{4.40}$$

We choose S so that $\frac{u}{a} = o(S)$ but $S = o\left(\frac{\sqrt{u}}{a}\right)$, so that this choice is possible since $\sqrt{u}$ is much larger than u, as u is small. We then have

$$aS^2 = o\left(\frac{u}{a}\right),$$

so that

$$\mathbb{P}\left\{\|w\|_\infty > \frac{u}{a} + O(aS^2)\right\} = \mathbb{P}\left\{\|w\|_\infty > \frac{u}{a}(1 + o(1))\right\}.$$

As in section 4.3, we shall estimate the quantity $\mathbb{P}\left\{\left\{\|w\|_\infty > \frac{u}{a}(1 + o(1))\right\}\right\}$ from above and below by separate arguments. We let

$$\tau = \tau(u, a, S) := u/a + O(aS^2) = (u/a)(1 + o(1)).$$

By Borel-TIS Theorem 3.1.2, there exists α_w such that

$$\mathbb{P}\left\{||w||_\infty > \tau\right\} \leq e^{\alpha_w \tau - \frac{\tau^2}{2\sigma_w^2}}. \tag{4.41}$$

This concludes the proof of the upper bound in (4.22) in this case.
To get a lower bound in (4.22), consider the point $x_0 \in S$ where $r_w(x,x)$ attains its maximum, $r_w(x_0, x_0) = \sigma_w^2$. Consider the event $\{|w(x_0)| > \tau\}$.
We find that trivially

$$\mathbb{P}\left\{||w||_\infty > \tau\right\} \geq \mathbb{P}\left\{|w(x_0)| > \tau\right\} \geq \left(\frac{C_1}{\tau} - \frac{C_2}{\tau^3}\right) e^{-\frac{\tau^2}{2\sigma_w^2}}. \tag{4.42}$$

We next pass to the limit $u \to 0, u/a \to \infty$; then $\tau \cdot a/u \to 1$. Taking logarithm in (4.41) and (4.42) and comparing the upper and lower bound, we establish (4.22) for surfaces of genus ≥ 2. This concludes the proof of Theorem 4.5.2.

□