Complex nodes in Gaussian random waves: quantum waves, cosmology and optics

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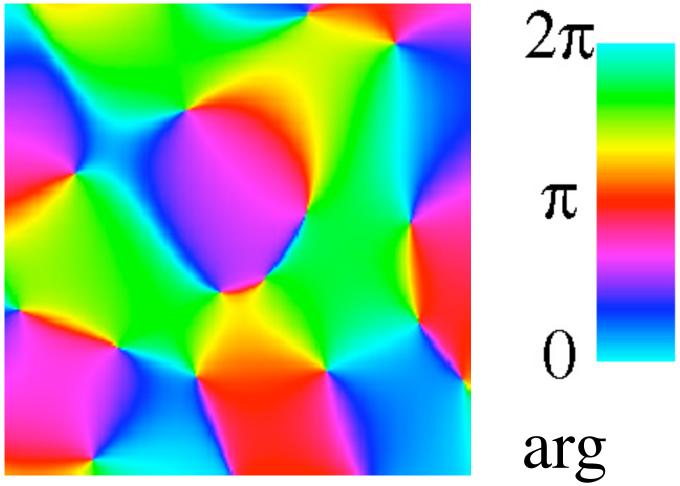
THE ROYAL SOCIETY

Main idea

Many physical wave fields can be modelled using gaussian random functions



It is natural to try to characterise these fields in terms of nodal sets (zero level sets)

For complex scalar fields in 2 dimensions, this defines a point process, and a line process in 3D

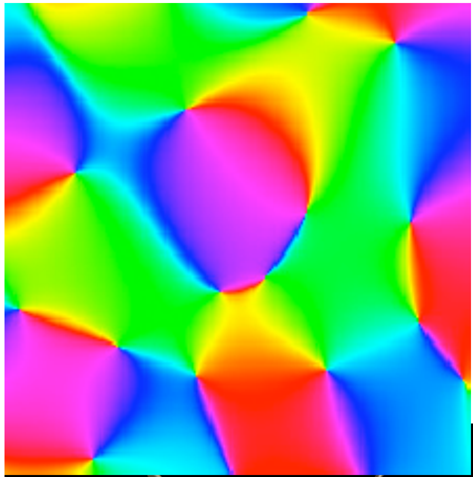
What can this tell us about the physics?

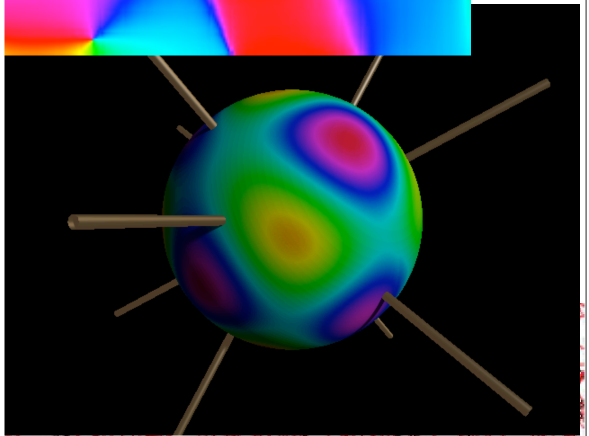
Outline

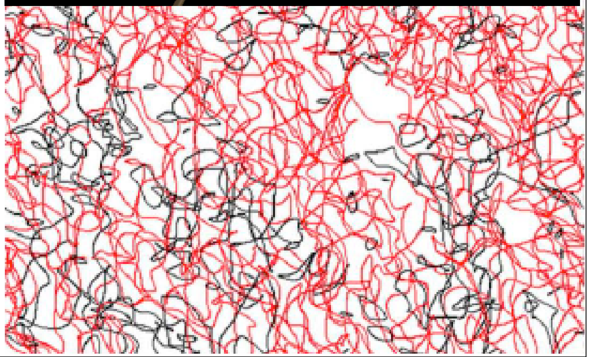
- Nodal points in quantum chaotic wavefunctions & random vector fields

- Cosmic Microwave Background & random complex polynomials

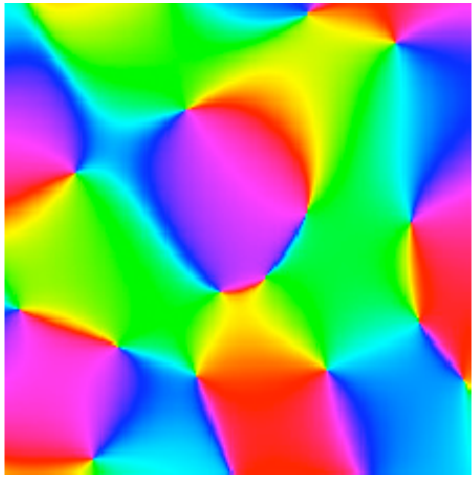
- Tangled nodal lines in 3D random optical waves







Outline



- Nodal points in quantum chaotic wavefunctions & random vector fields

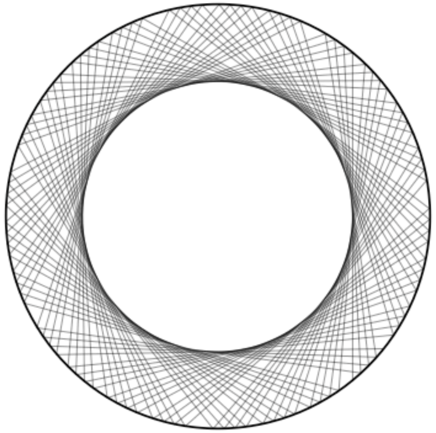
(Berry & MRD, 2000; MRD 2003)
(Hohmann, Kuhl, Stockmann, Urbina, MRD 2009)

- Cosmic Microwave Background & random complex polynomials

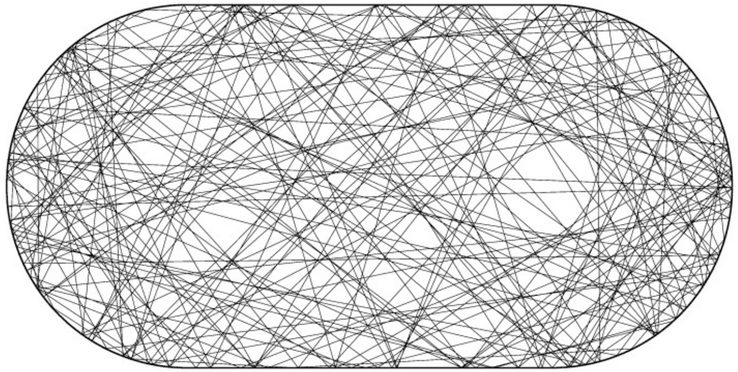
- Tangled nodal lines in 3D random optical waves

Quantum chaotic billiards

Motion of particle in 2D domain, specular reflection: 'billiard'

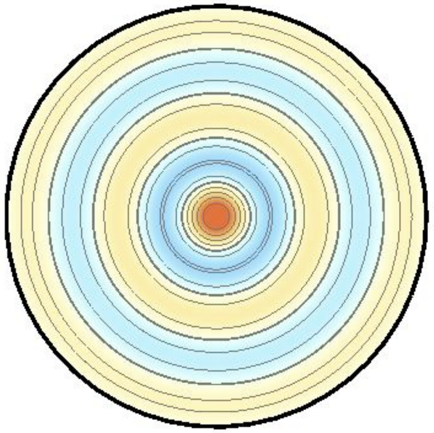


circle: integrable



stadium: ergodic

what are eigenfunctions of the laplacian on these domains? (say Dirichlet bcs)



Bessel function eigenfunctions

$J_n(k_{nm}r)\cos(n\phi)$

eigenvalues from Bessel zeros

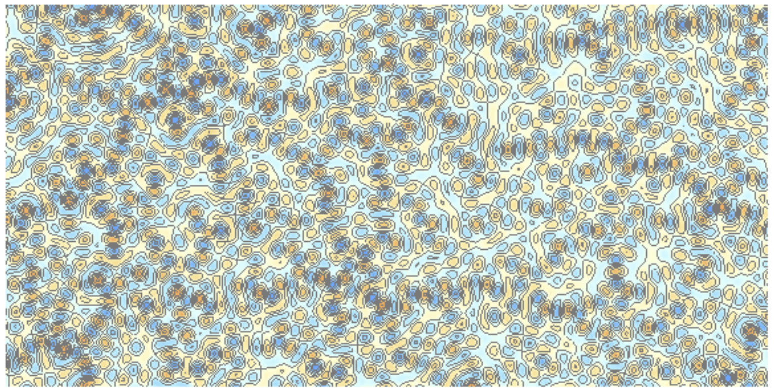
'quantum chaos' - also optics, acoustics, ...

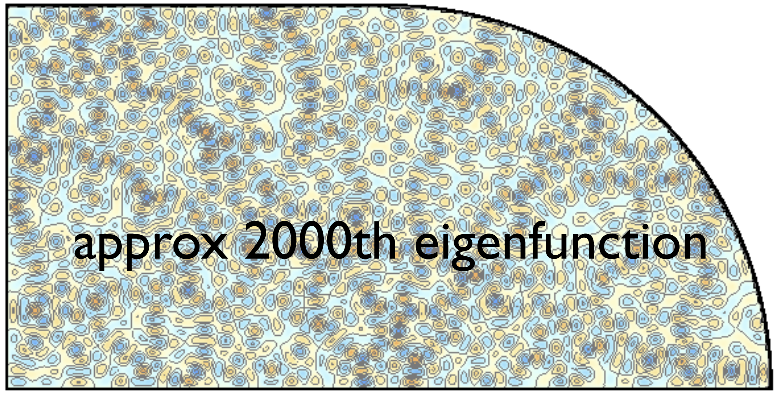
Random wave model

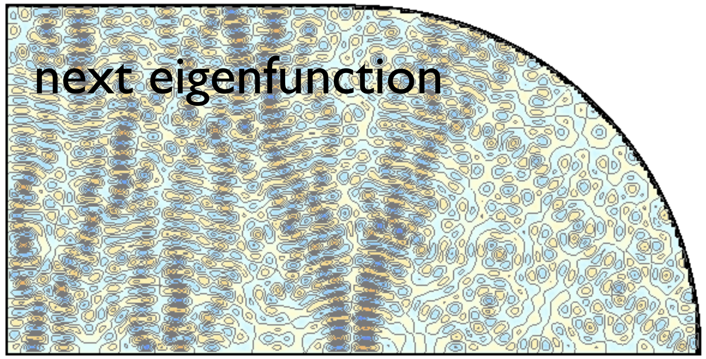
Hypothesis (Berry 1977): a typical ergodic eigenfunction looks like a sample gaussian random function with $\langle f(0) f(r) \rangle = C(r) = J_0(r)$

original conjecture was more general

Which is the stadium eigenfunction?







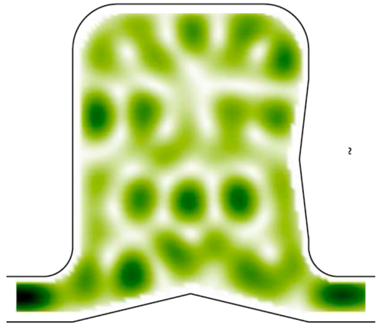
Test this hypothesis by comparing spatial averages of quantities in eigenfunctions with ensemble averages of gaussian random waves

Complex random waves

Ergodic systems without time reversal invariance have complex wavefunctions

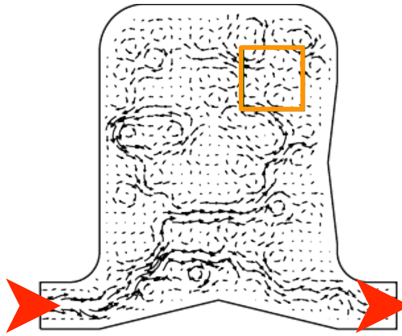
The random wave model in this case is $\psi = f_1 + \mathrm{i} f_2$, with f_1,f_2 iid gaussian random functions

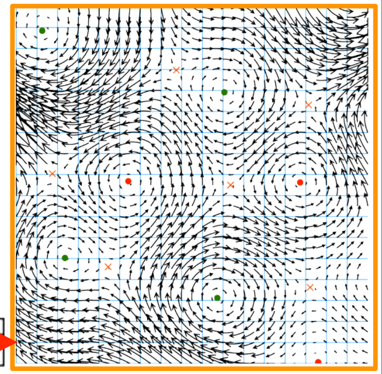
The complex nodes are vortices of probability current flow

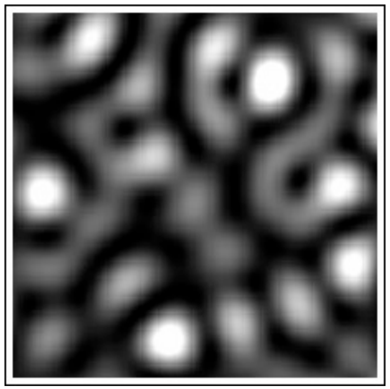


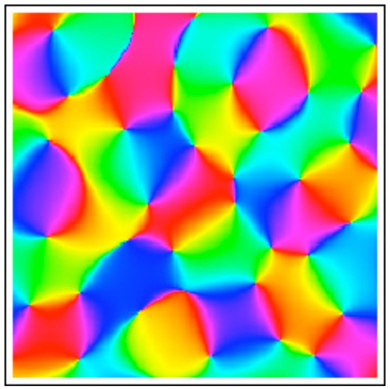
experiment

(Hohmann *et al* 2009)





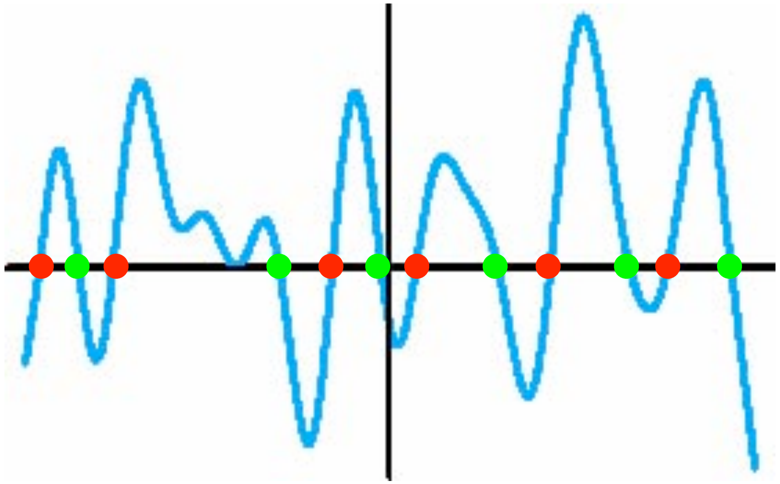




Compare vortex density in measurements vs gaussian random model

Rice: Zeros of 1D real gaussian random function (Rice 1944,1945)

$C(r) = J_0(r)$



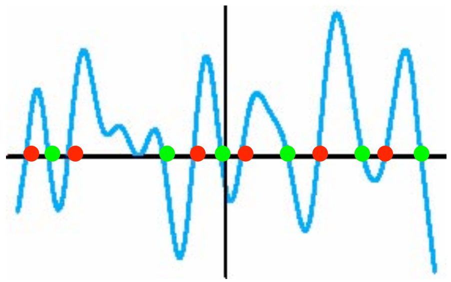
Gaussian random function f is stationary, zero mean, unit variance, with 2-point correlation function

$$\langle f(0) f(r) \rangle = C(r)$$

density of point zeros $\quad d_1 = \langle \delta(f) |f'| \rangle \quad = \dfrac{\sqrt{|C''_0|}}{\pi}$

density of index (sign of gradient) $\quad \langle \delta(f) f' \rangle = 0$

Gaussian random function zero correlation functions



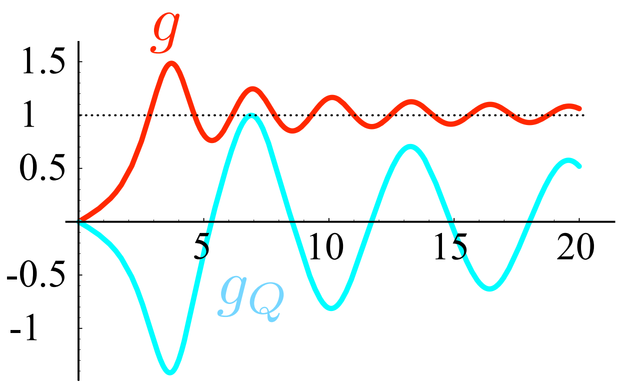
$$g(r) = \frac{1}{d_1^2}\langle \delta(f(0))|f'(0)|\delta(f(r))|f'(r)|\rangle$$

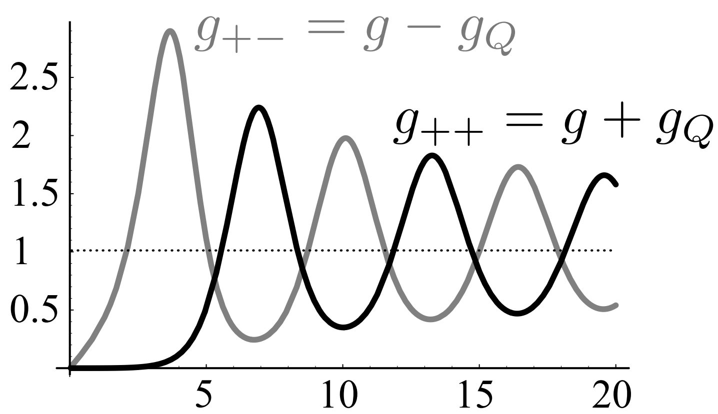
zero-zero correlation function

$$= \frac{B(C)}{d_1^2}(1 + A(C)\arctan A(C))$$

zero index correlation function

$$g_Q(r) = \frac{1}{d_1^2}\langle \delta(f(0))f'(0)\delta(f(r))f'(r)\rangle \quad = \frac{1}{2\pi d_1^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\arccos C$$





n-dimensional vector nodal density and index correlation

Consider nodal points of gaussian random vector fields

$$\boldsymbol{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

Assume $\boldsymbol{f} = (f_1 . f_2, \ldots, f_n)$ is iid, isotropic, stationary, zero mean, unit variance, ...

Generalized Rice formula gives zero density

$$d_n = \langle \delta^n(\boldsymbol{f}) | \det \nabla \boldsymbol{f} | \rangle$$
$$= |C_0''|^{n/2} \frac{n! \, \mathrm{vol} B_n}{(2\pi)^n}$$

Index correlation function

mod signs removed

$$g_Q(r_{AB}) = \tfrac{1}{d_n^2} \langle \delta^n(\boldsymbol{f}_A) \det \nabla \boldsymbol{f}_A \delta^n(\boldsymbol{f}_B) \det \nabla \boldsymbol{f}_B \rangle$$
$$= \frac{(n-1)!}{(2\pi)^n d_n^2 r^{n-1}} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\mathrm{d}}{\mathrm{d}r} \arccos C \right)^n$$

2D vortex-vortex correlation function

(Berry, MRD 2000)

$$g(r_{AB}) = \frac{1}{d_2^2}\langle \delta^2(\psi_A)|\det\nabla\psi_A|\delta^2(\psi_B)|\det\nabla\psi_B|\rangle$$

complicated calculation (due to |•| signs) involves correlation function $C(r)$ and derivatives

$$g(r) \underset{r\to\infty}{\longrightarrow} 1$$

$$\begin{aligned} g(R) &= \frac{1}{d_2^2}\left\langle \delta(\xi_A)\delta(\eta_A)\left|\omega_{z,A}\right|\delta(\xi_B)\delta(\eta_B)\left|\omega_{z,B}\right|\right\rangle \\ &= \frac{2\left(C'^2 + C_0''(1-C^2)\right)}{\pi C_0''(1-C^2)^2}\Bigg(2\sqrt{2-Y+2Z} - \frac{\mathrm{i}}{\sqrt{2ZU}}\Big[(4-U)ZF_p - 4ZE_p \\ &\qquad + 2YU\Pi_p + 2\sqrt{Z}\left(-(1+X+Y)F_m + UE_m + 2Y\Pi_m\right)\Big]\Bigg), \end{aligned} \tag{32}$$

where $C_0'' \equiv C''(0) = d_2/2\pi$, and

$$\begin{aligned} F_p &= F\left(\mathrm{i}\ \mathrm{arcsinh}\left[\sqrt{V/2}\right] \,|\, U/V\right), \\ F_m &= F\left(-\mathrm{i}\ \mathrm{arcsinh}\left[\sqrt{2/V}\right] \,|\, V/U\right), \\ E_p &= E\left(\mathrm{i}\ \mathrm{arcsinh}\left[\sqrt{V/2}\right] \,|\, U/V\right), \\ E_m &= E\left(-\mathrm{i}\ \mathrm{arcsinh}\left[\sqrt{2/V}\right] \,|\, V/U\right), \\ \Pi_p &= \Pi\left(2/V;\mathrm{i}\ \mathrm{arcsinh}\left[\sqrt{V/2}\right] \,|\, U/V\right), \\ \Pi_m &= \Pi\left(V/2;-\mathrm{i}\ \mathrm{arcsinh}\left[\sqrt{2/V}\right] \,|\, V/U\right), \end{aligned} \tag{33}$$

where F,E,Π in (33) are the (incomplete) elliptic functions of the first, second and third kinds respectively (with the conventions for elliptic functions being those used by *Mathematica* [27]). Also,

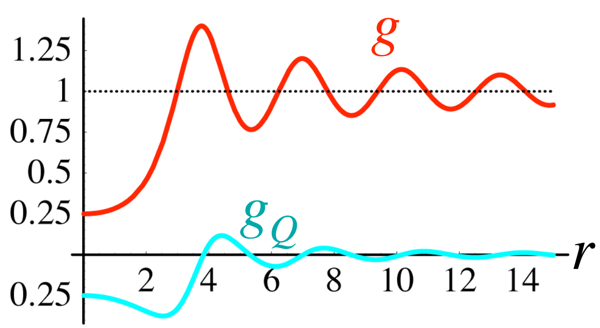
$$\begin{aligned} U &= 1+X-Y+Z, \\ V &= 1-X-Y+Z, \end{aligned} \tag{34}$$

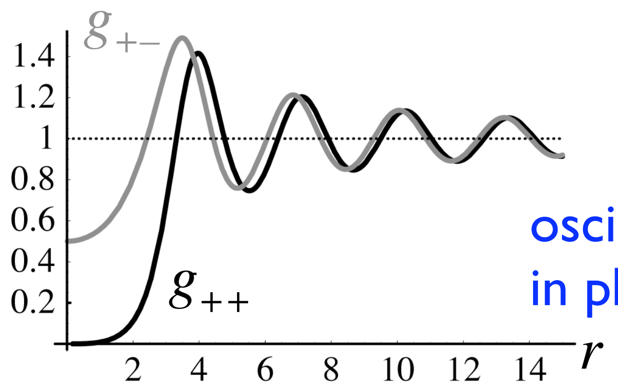
and finally,

$$\begin{aligned} X &= \frac{\left(C'^3 + C_0''(1-C^2)(C'+RC'') + RCC'^2C_0''\right)\left(C'^3 + C_0''(1-C^2)(C'-RC'') - RCC'^2C_0''\right)}{R^2C_0''^2\left(C_0''(1-C^2)+C'^2\right)^2}, \\ Y &= \frac{C'^2\left(CC'^2 + C''(1-C^2)\right)^2}{R^2C_0''^2\left(C_0''(1-C^2)+C'^2\right)^2}, \\ Z &= \frac{(1-C^2)(R^2C_0''^2 - C'^2)\left(C'^2+(1-C)(C_0''+C'')\right)\left(C'^2+(1+C)(C_0''-C'')\right)}{R^2C_0''^2\left(C_0''(1-C^2)+C'^2\right)^2}. \end{aligned} \tag{35}$$

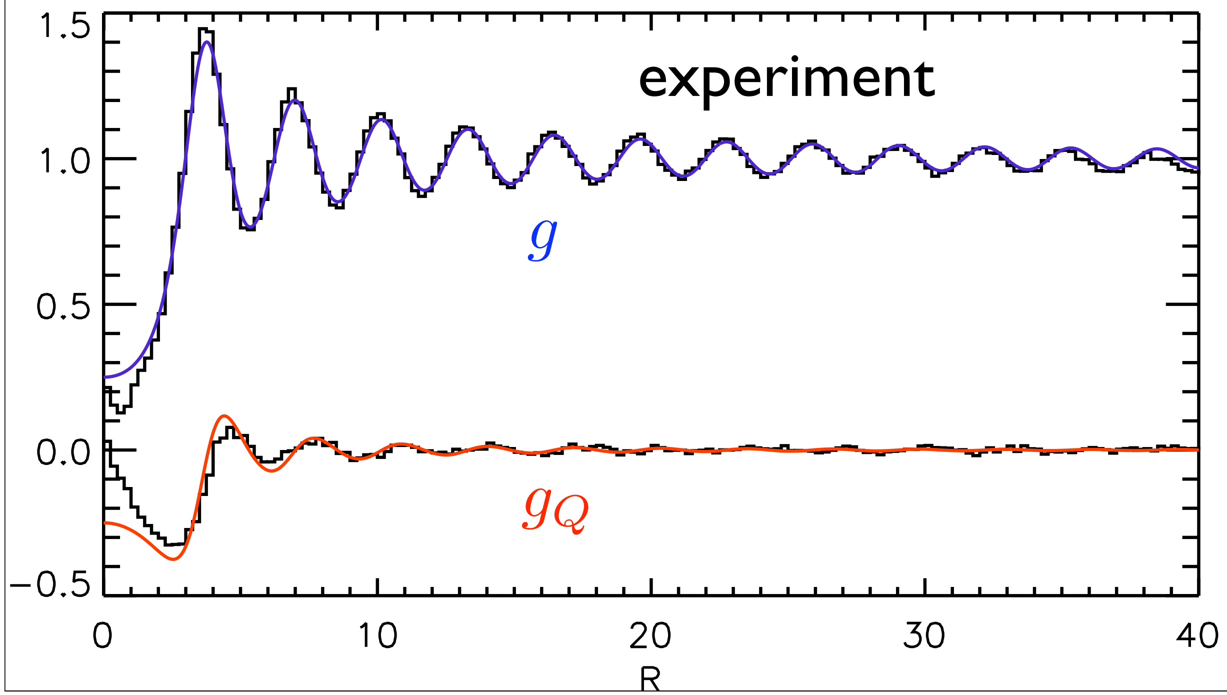
Nodal correlation functions

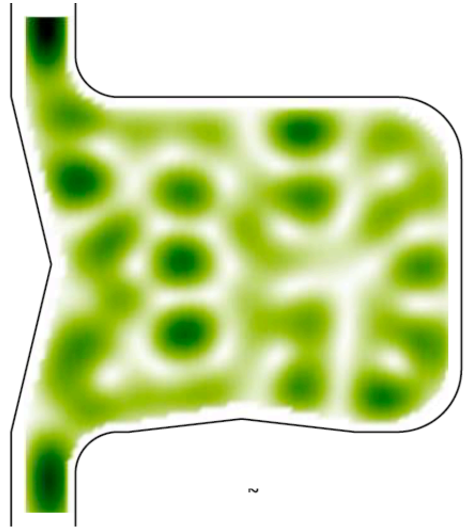
Bessel correlation





oscillations in phase





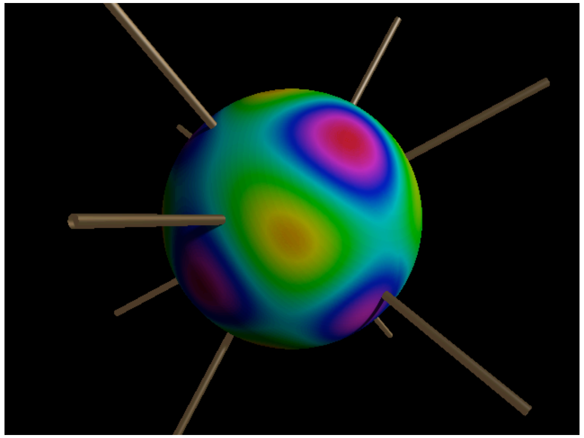
Hohmann *et al* 2009

Outline

- Nodal points in quantum chaotic wavefunctions & random vector fields

- Cosmic Microwave Background & random complex polynomials

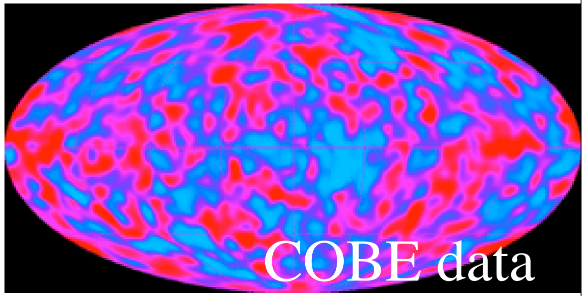
  (MRD 2005, MRD & Land 2008)



- Tangled nodal lines in 3D random optical waves

Cosmic Microwave Background (CMB)

Observational cosmology: Physics Nobel Prize 2006 (Smoot & Mather) - all physics is in the spherical map of temperature fluctuations



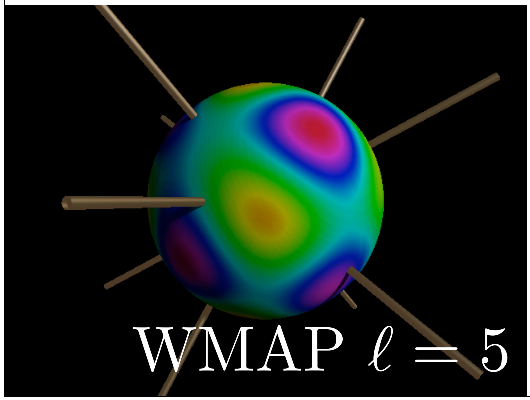
$$f(\theta,\phi) = \sum_{\ell=2}^{\infty} C_\ell \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(\theta,\phi)$$

THE BIG QUESTION

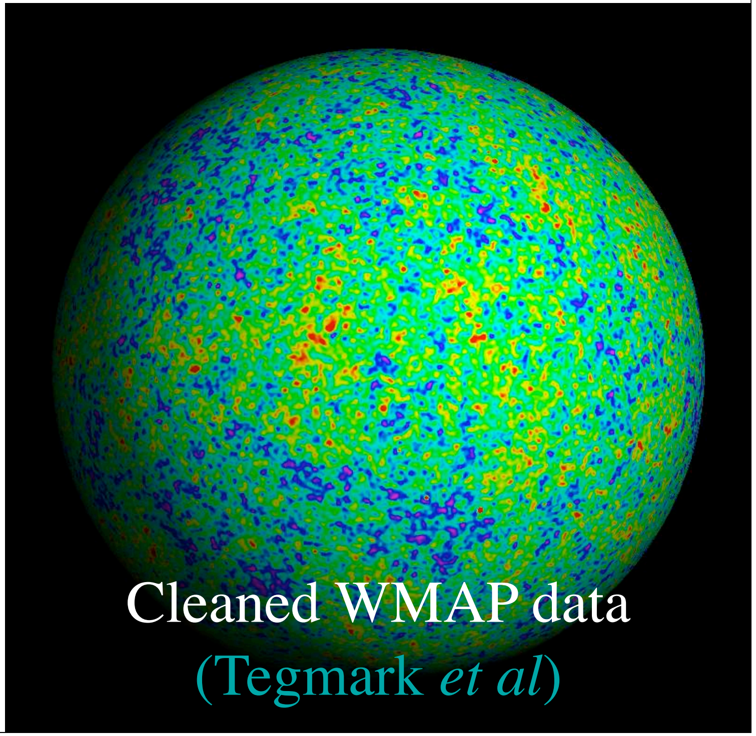
- what is the power spectrum C_ℓ?

ANOTHER QUESTION

- does $f(\theta,\phi)$ have any other structure?



approach using *Maxwell multipole vectors*

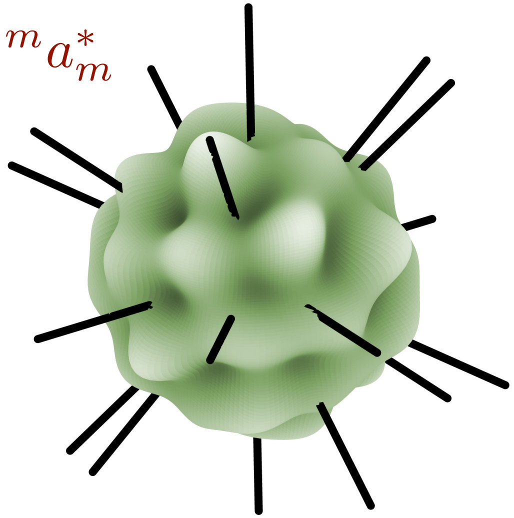


Maxwell multipole vectors

Real eigenfunction of laplacian on sphere

$$f(\theta,\phi) = \sum_{m=-\ell}^{\ell} a_m Y_\ell^m(\theta,\phi) \qquad a_{-m} = (-1)^m a_m^*$$

$$= \text{const} \times r^{2\ell+1} D_{\boldsymbol{u}_1} \cdots D_{\boldsymbol{u}_\ell} \frac{1}{r}$$



Maxwell multipole representation

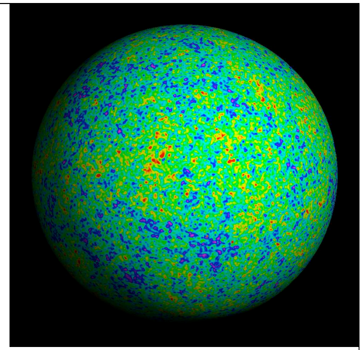
$D_{\boldsymbol{u}_j}$ directional derivative, direction $\boldsymbol{u}_j$

r radial coord

$2j$ directions $\pm\boldsymbol{u}_j$ correspond to complex roots, on Riemann sphere, of **SU(2) polynomial**:

$$p_f = p(\zeta) = \sum_{m=-\ell}^{\ell} a_m (-1)^{\ell+m} \begin{pmatrix} 2\ell \\ \ell+m \end{pmatrix}^{1/2} \zeta^{\ell+m}$$

$$\zeta = \mathrm{e}^{\mathrm{i}\phi} \tan\theta/2$$



The CMB - a random spherical function?

Pick a particular mode labelled by ℓ,

$$f_\ell = f(\theta,\phi) = \sum_{m=-\ell}^{\ell} a_m Y_\ell^m(\theta,\phi)$$

Simplest cosmological theory suggests that coefficients a_m are independent, identically gaussian distributed (variance ℓ-dep?)

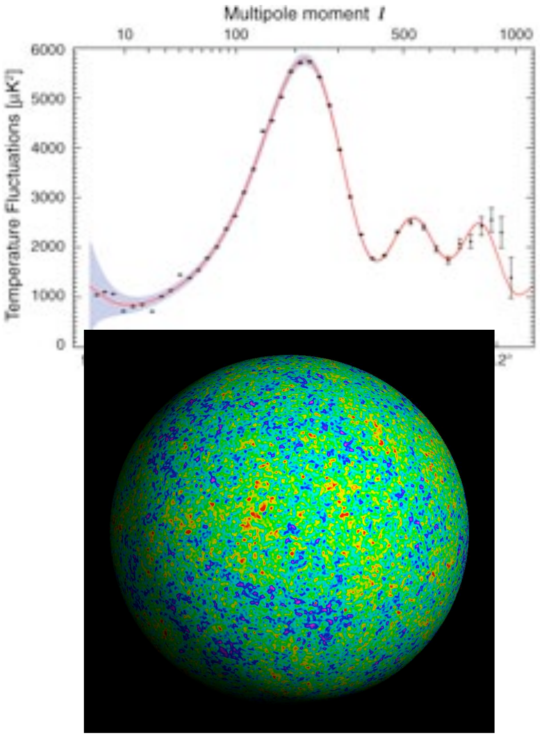
- only the norm $C_\ell^2 = \sum_m |a_m|^2$ is determined

not the direction in $2\ell+1$-D

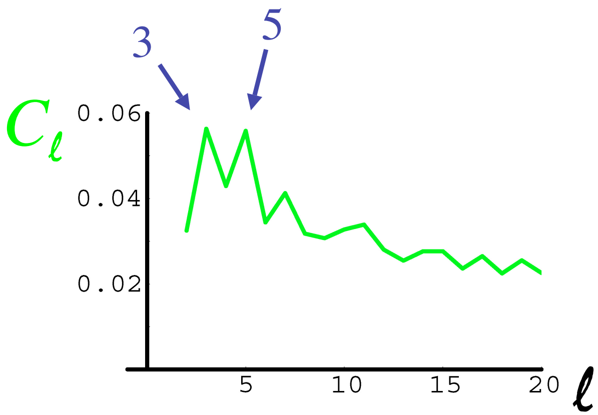
Multipole vectors provide a basis-independent means of testing the data against this hypothesis

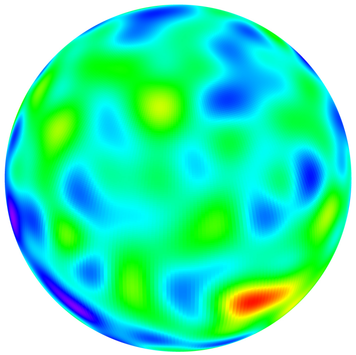
Spherical modes of the CMB

Concentrate attention on Maxwell’s multipole vectors for modes with small ℓ (potential numerical problems for high ℓ)



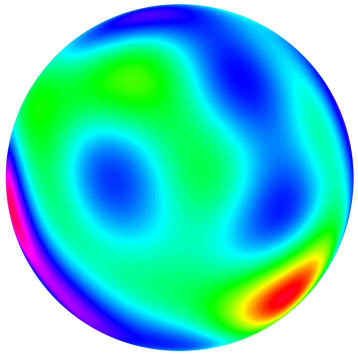
total cleaned data





$2 \leq \ell \leq 20$

(Copi et al 2004, Land & Magueijo 2005)



$2 \leq \ell \leq 8$

Statistically isotropic spherical functions

Any ensemble of spherical functions, of fixed ℓ, whose statistics depend only on the length $C_\ell^2 = \sum_m |a_m|^2$, have equivalent multipole vector statistics

unitary invariant (not only rotation)

We can use any such distribution to calculate the statistics; it is convenient to choose the a_m independent

=> identically distributed gaussian variables
(*cf* derivation of Maxwell distribution)

ensemble averaging

$$\langle a_m^* a_n \rangle = \delta_{m,n} \qquad \Rightarrow \qquad \langle a_m a_n \rangle = (-1)^m \delta_{m,-n}$$

since $a_{-m} = (-1)^m a_m^*$

Correlations between Maxwell's multipoles

Therefore want to find the statistics of the zeros of the random SU(2) polynomial

(related rand polys: Bogomolny *et al*, Hannay, Prosen 1996,...)

$$p_f = p(\zeta) = \sum_{m=-\ell}^{\ell} a_m (-1)^{\ell+m} \begin{pmatrix} 2\ell \\ \ell+m \end{pmatrix}^{1/2} \zeta^{\ell+m}$$

$$a_{-m} = (-1)^m a_m^*$$

roots $\zeta_i, \zeta_{i+\ell}$ antipodal

with the a_m coefficients iid gaussians

=> (with $p_i \equiv p(\zeta_i)$, ...)

$$\langle p_i^* p_j \rangle = (1 + \zeta_i^* \zeta_j)^{2\ell} \qquad \langle p_i p_j \rangle = (\zeta_i - \zeta_j)^{2\ell}$$

... other correlations (involving $p_i' \equiv \mathrm{d}p/\mathrm{d}\zeta|_{\zeta_i}$, etc)

2-point multipole vector correlation function

set the 2 points to be $\zeta_1 = 0, \zeta_2 = r$ (real); then

$$\rho_2(0,r) = (\pi^2 D^{5/2})^{-1} \Big((2\ell D - 4buv - (b^2+v^2)(a-1-u^2)) \times (dD - 2cuv(a+1-u^2) - (c^2+av^2)(a-1-u^2)) + (2\ell D - 2cuv - buv(a+1-u^2) - v^2(a-1+u^2) - bc(a-1-u^2))^2 + (wD - 2bcu - uv^2(a+1-u^2) - bv(a-1+u^2) - cv(a-1-u^2))^2 \Big)$$

with $D = \det \mathbf{A} = (a-1-u^2-2u)(a-1-u^2+2u)$ and

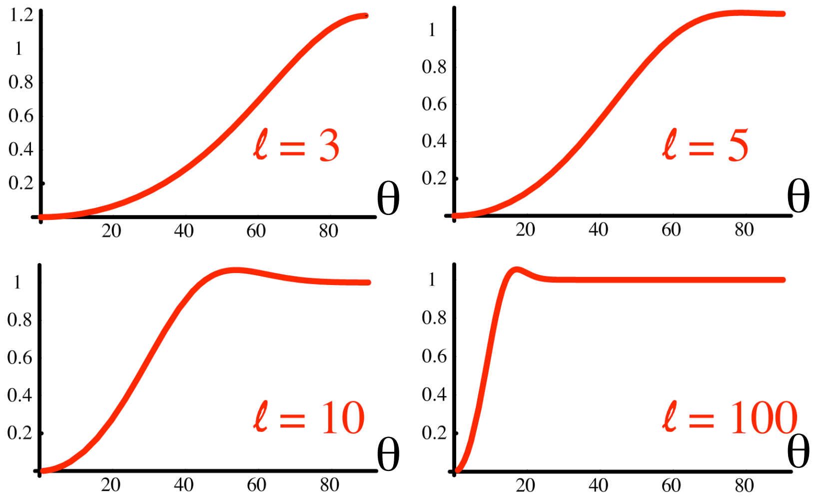
$$a = (1+r^2)^{2\ell},\ b = 2\ell r,\ c = 2\ell r(1+r^2)^{2\ell-1},\ d = 2\ell(1+2\ell r^2)(1+r^2)^{2\ell-2},$$
$$u = r^{2\ell},\ v = -2\ell r^{2\ell-1},\ w = -2\ell(2\ell-1)r^{2\ell-2}.$$

on Riemann/direction sphere (angular separation θ),

$$\rho_2(\theta) = \frac{27(1-\cos^2\theta)}{2(3+\cos^2\theta)^{5/2}} \text{ for } \ell = 2$$

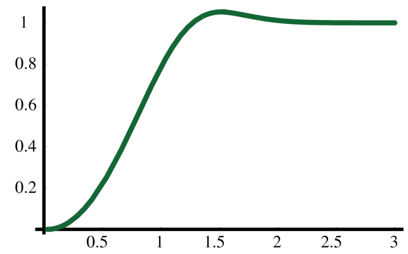
2-pole correlation function for higher ℓ

Other ℓ...
(always symmetric about θ =90°)



In high-ℓ limit, $\rho_2(0, r)$ approaches $g(\ell^{1/2} r)$, where

$$g(R) = \frac{(\sinh^2 R^2 + R^4)\cosh R^2 - 2R^2 \sinh R^2}{\sinh^3 R^2}$$



(found originally as limit for general random SU(2) polynomials with similar method - Hannay 1996)

Full ℓ-pole joint probability distribution function

In terms of roots ζ_i on Riemann sphere/complex plane

modulus of polynomial discriminant (accounts for repulsion)

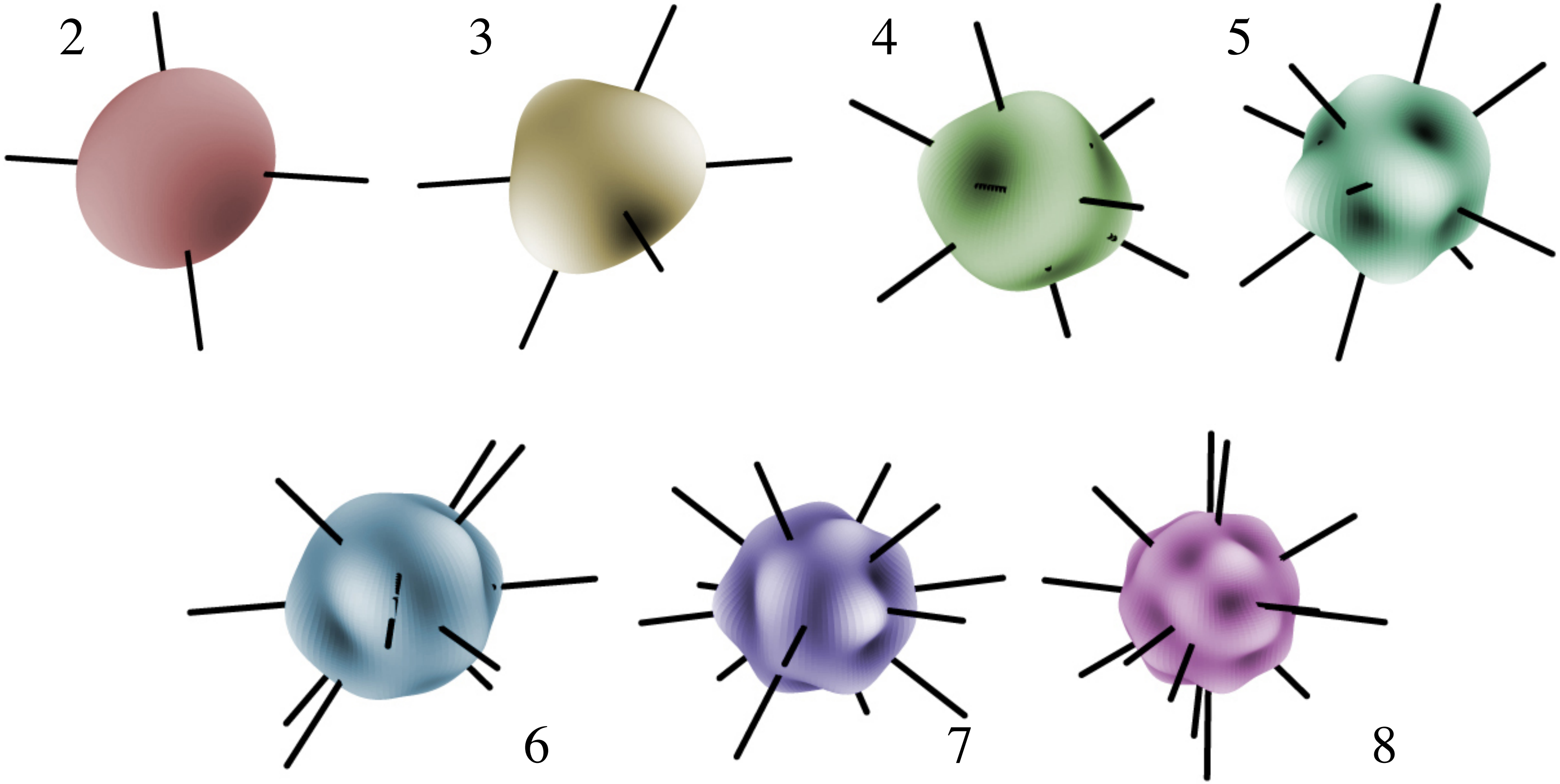
$$P_\ell(\{\zeta_i\}) = \text{const} \times \frac{\prod_{i=1}^{\ell} |\zeta_i|^{-2} \prod_{1=i<k}^{2\ell} |\zeta_i - \zeta_k|}{\left(\sum_{\sigma\in S_{2\ell}} \prod_{i=1}^{2\ell} (1+\zeta_i \zeta^*_{\sigma(i)})\right)^{(2\ell+1)/2}}$$

sum over permutations of roots

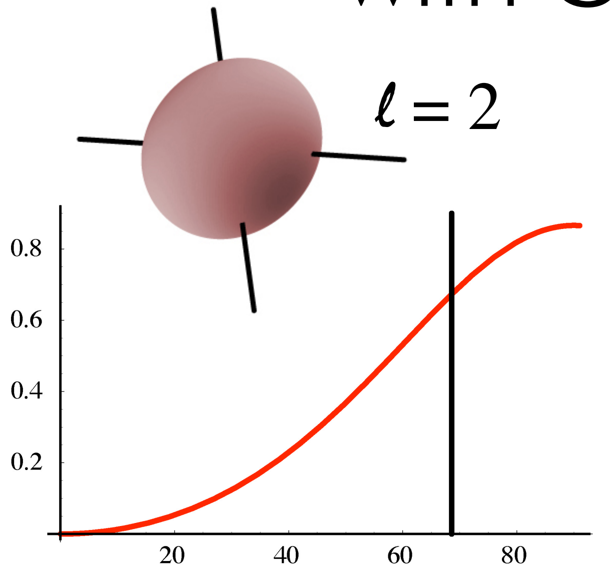
Similar in form to general SU(2) polynomial (Hannay 1996) and more general random polynomials (Bogomolny, Bohigas, & Leboeuf 1996)

Behaviour of cosmic multipoles

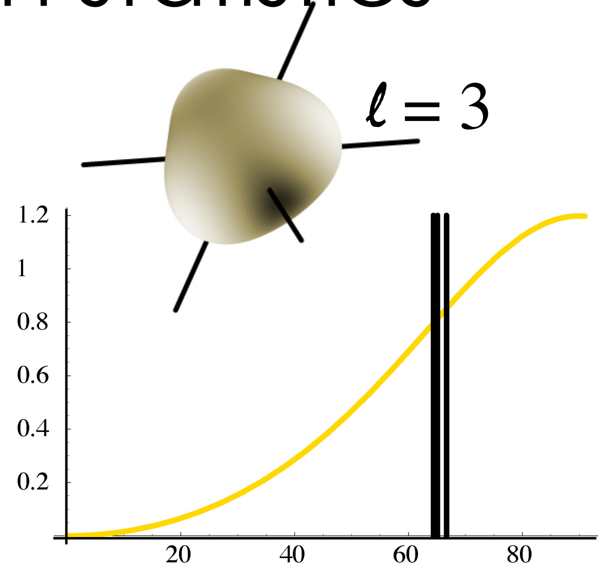
$2 \leq \ell \leq 8$



CMB multipoles data comparison with Gaussian statistics



$\theta_{CMB} = 68.6°$

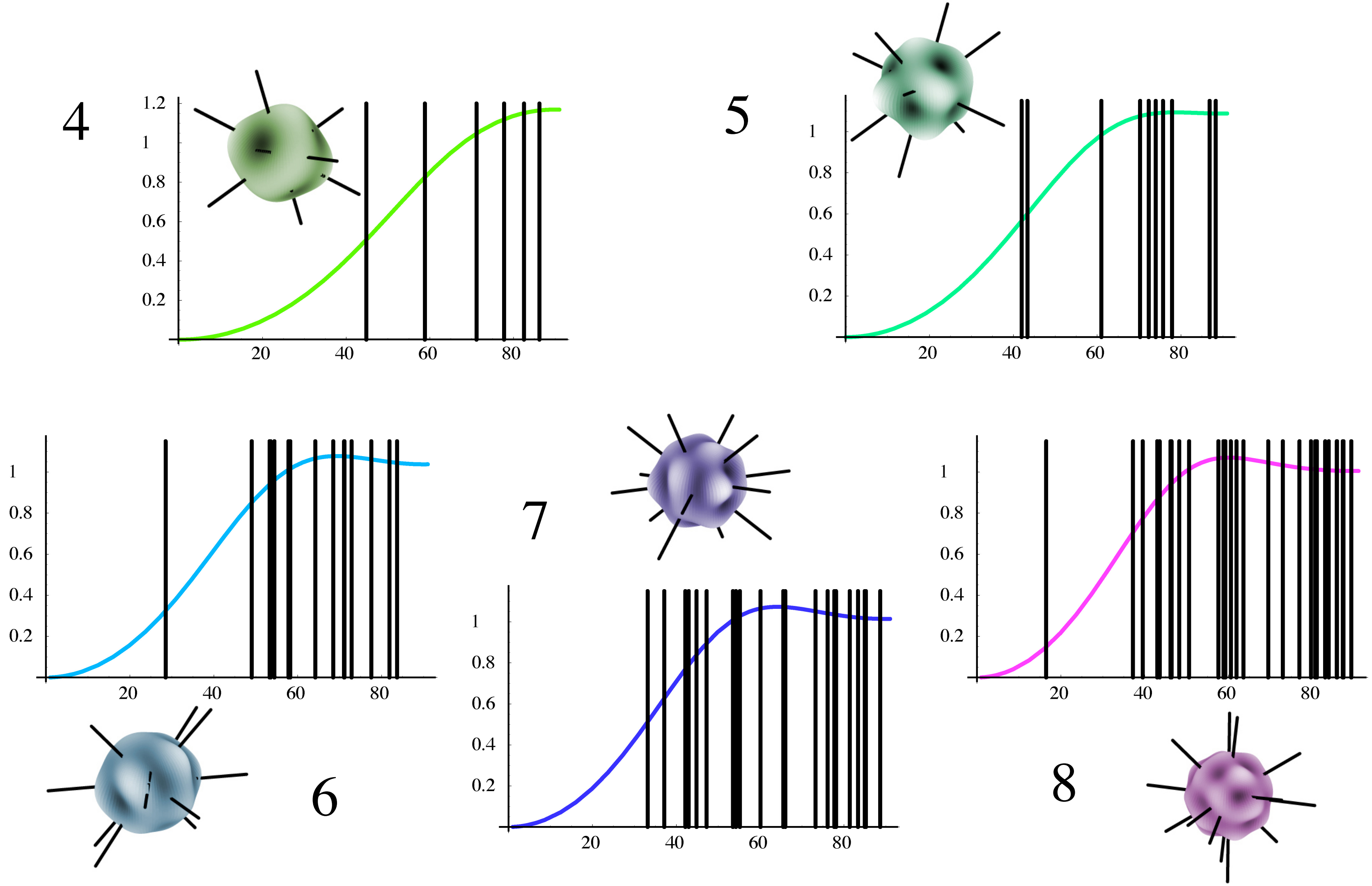


$\langle\theta_{CMB}\rangle = 65.4°$

Preferred orientation for 2 or 3 multipole axes is mutually orthogonal since they repel.

Observed multipoles apparently prefer ~65° orientation.

2-point function comparison for higher ℓ



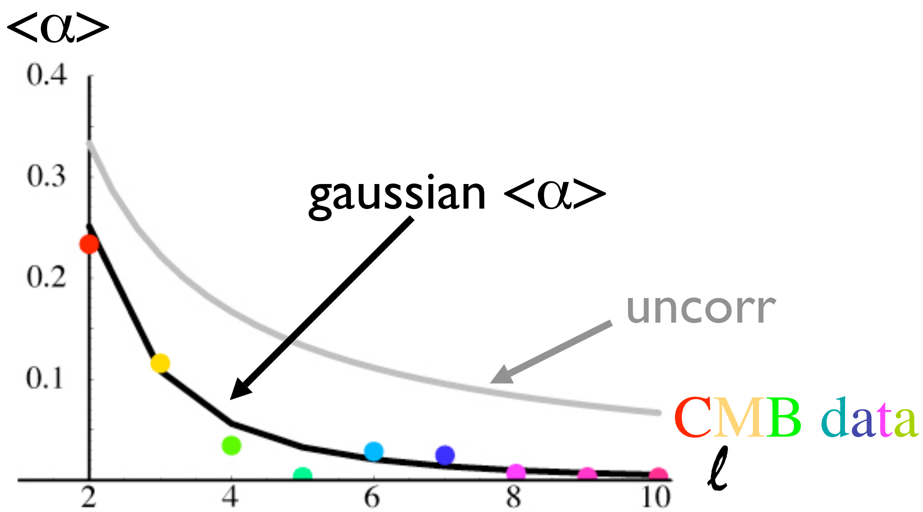
Quadrupole anisotropy of multipole vectors

Compare anisotropy of multipole vector distributions for different ℓ on an equal footing, using traceless part of moment of inertia tensor:

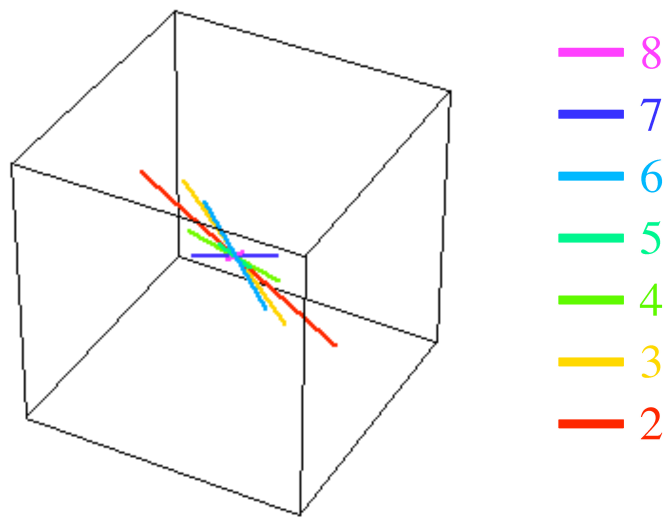
$$\mathbf{Q}_\ell = \frac{1}{\ell}\sum_{i=1}^{\ell} \boldsymbol{u}_i \otimes \boldsymbol{u}_i - \tfrac{1}{3}\mathbf{1}_3$$

anisotropy

$\alpha = \mathrm{Tr}\,\mathbf{Q}_\ell^2$



max eigenvalue/vector orientation and length



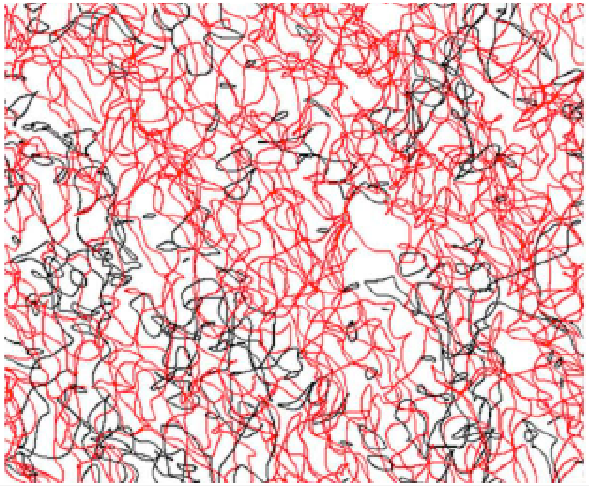
Outline

- Nodal points in quantum chaotic wavefunctions & random vector fields

- Cosmic Microwave Background & random complex polynomials

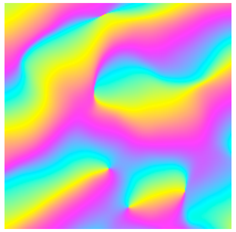
- Tangled nodal lines in 3D random optical waves

(O'Holleran, MRD & Padgett 2008; *submitted*)



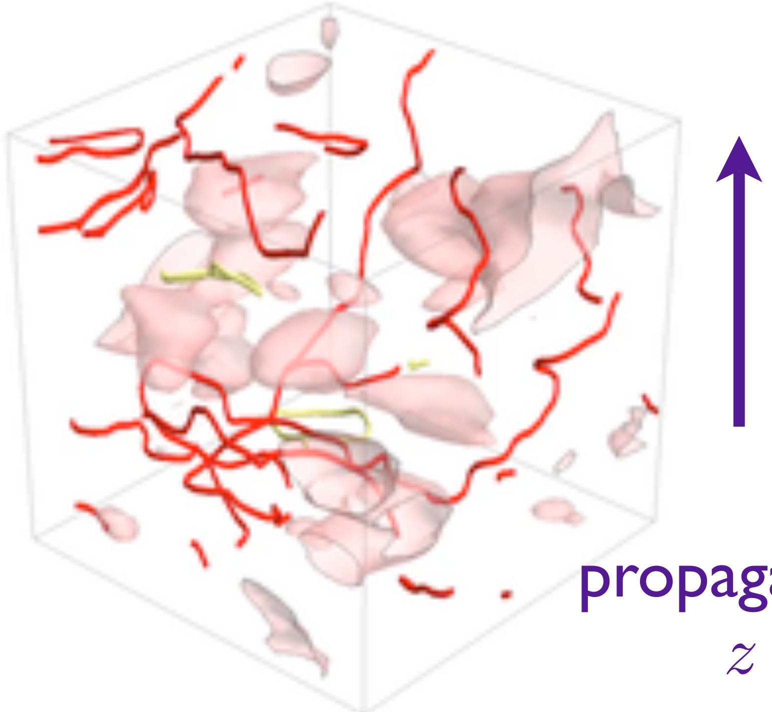
3D singularity topology in experimental speckle fields

Laser light, randomized by propagation through ground glass screen

transverse xy section

in rescaled coordinates, distribution of tangent directions is isotropic

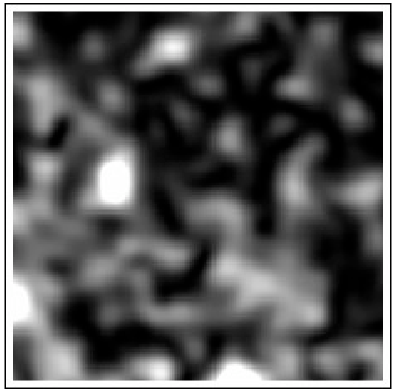


propagation
z

surfaces enclose intensities over 50% maximum

Singularity densities in gaussian random wave superpositions

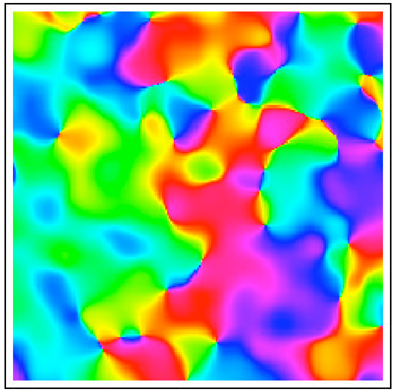
usual model for fully developed speckle:
superposition of plane waves with
independent random directions and phases



central limit theorem → field limits to gaussian random function

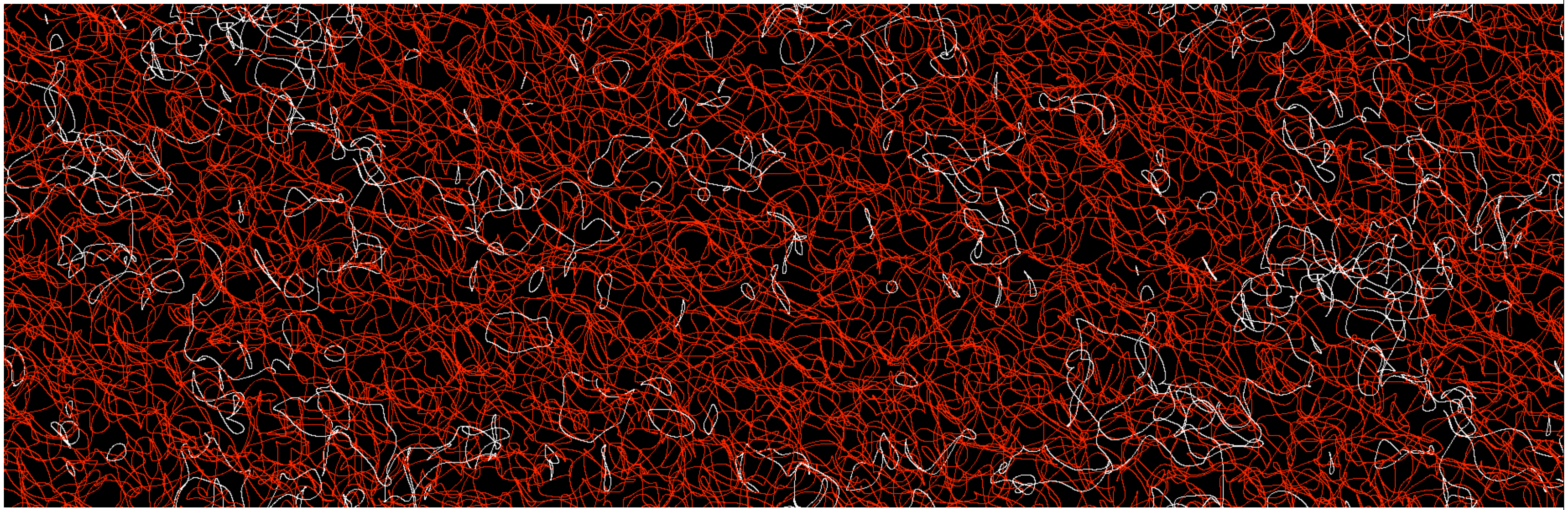
statistics completely determined by power spectrum, chosen here to be gaussian

$$\exp(-K_r^2\Lambda^2/2)$$



(Fourier transform is 2-point field correlation function by Wiener-Khinchin theorem)

Numerical singularity line tangle



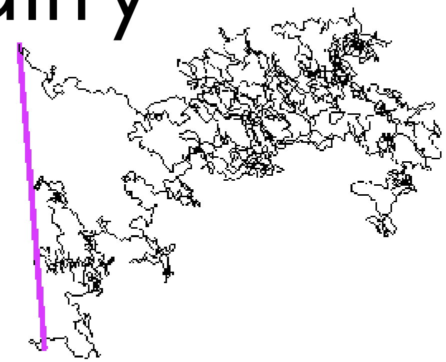
Periodic 3D cell, superposed 27 x 27 Fourier grid

729 wave superposition, Gaussian spectrum

Distinguish closed loops (white)
from periodic lines (red)

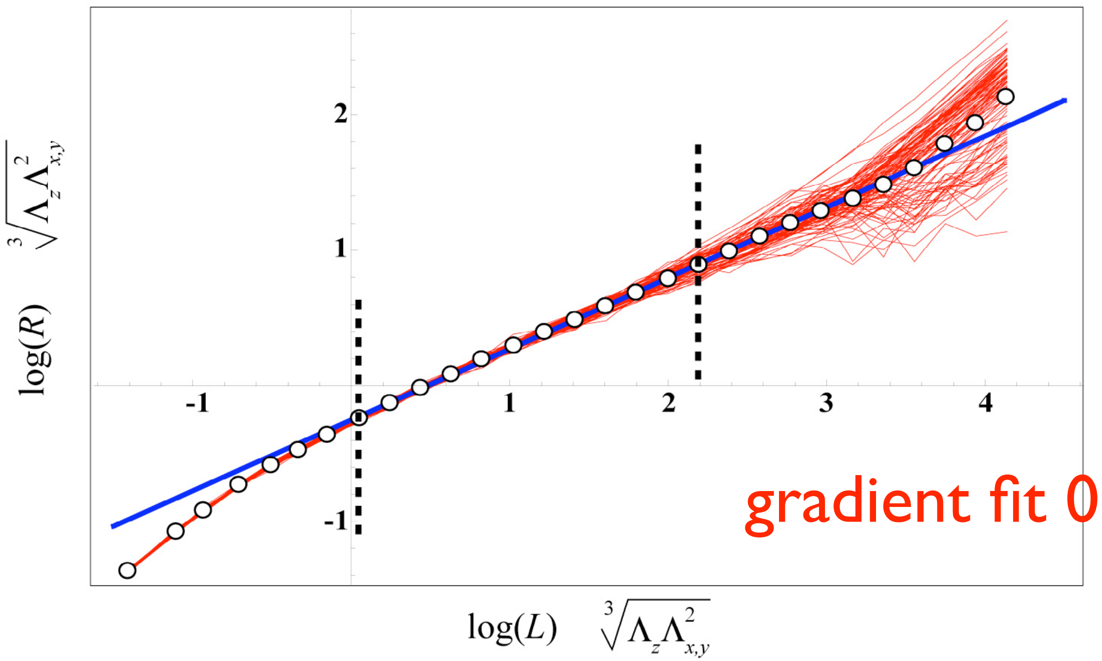
ratio
~ 73 : 27

Singularity line fractality



Scaling of arclength L against pythagorean distance R

100 lines from different simulations

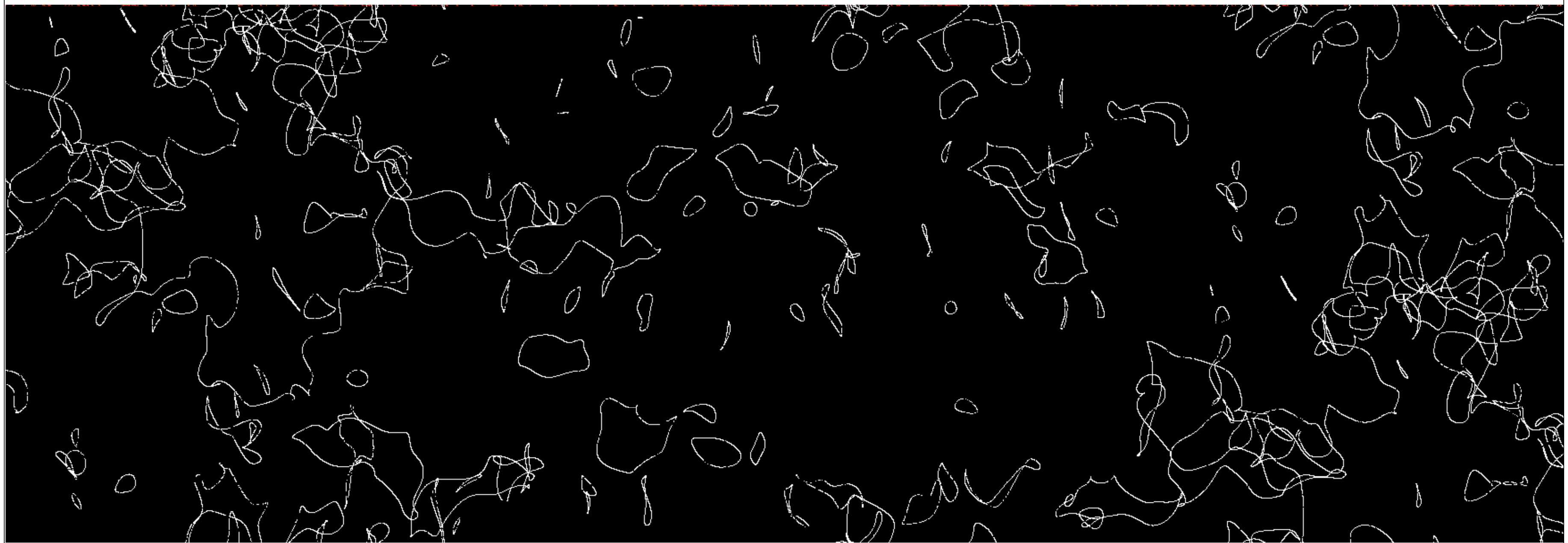


gradient fit 0.52 ± .01

nodal lines in random waves appear to be brownian curves

Loop length distribution

27% of the lines in the tangle are closed loops

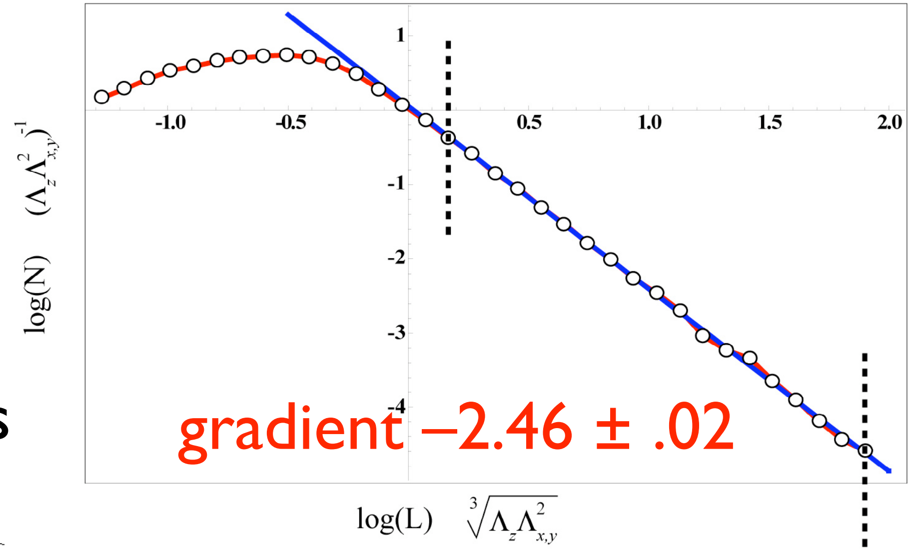


What is the loop length distribution?

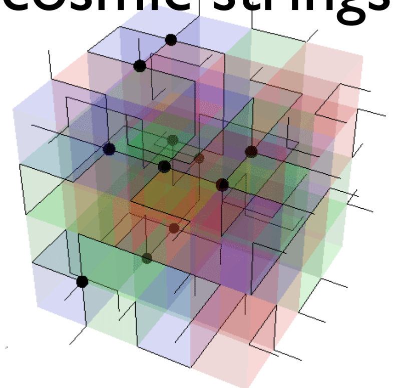
Loop length scaling

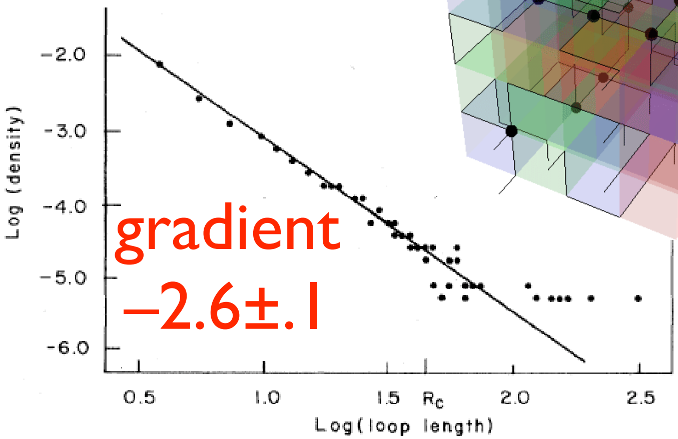
log-log histogram of loop lengths for ~80 000 loops from different runs

Cubic lattice model of $\mathbb{Z}_3$ phases modelling cosmic strings



gradient $-2.46 \pm .02$





gradient $-2.6 \pm .1$

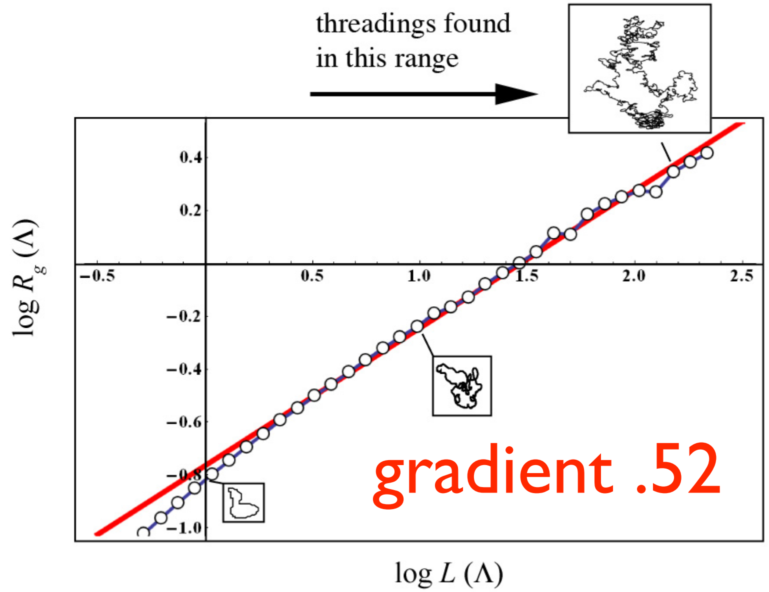
(Vachaspati & Vilenkin 1984)

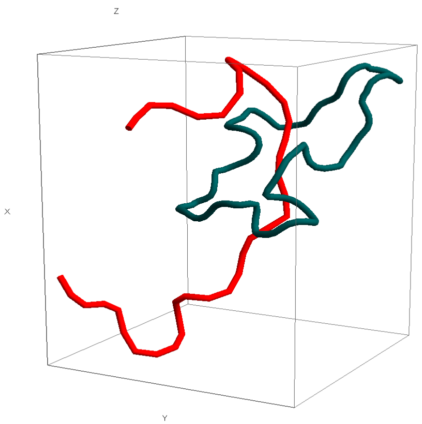
Gradient of –5/2 consistent with brownian fractality and global scale invariance

Random singularity topology

Scaling of closed loop size
(radius of gyration)

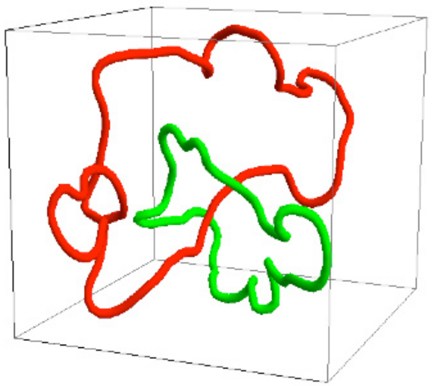
Probability of loop being
threaded by another line
increases with loop size



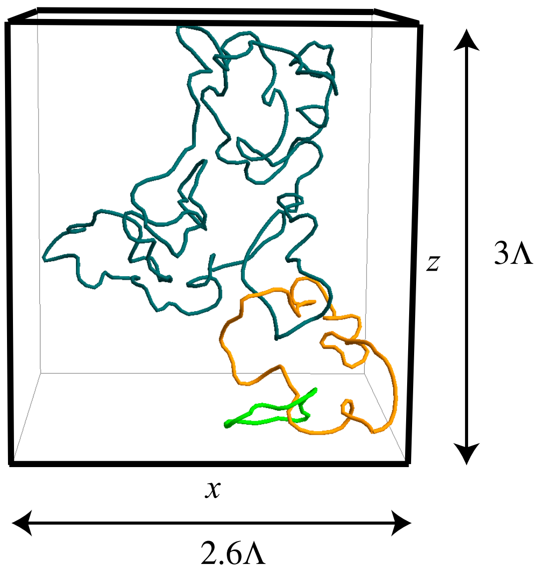


threading by
periodic line

Hopf link



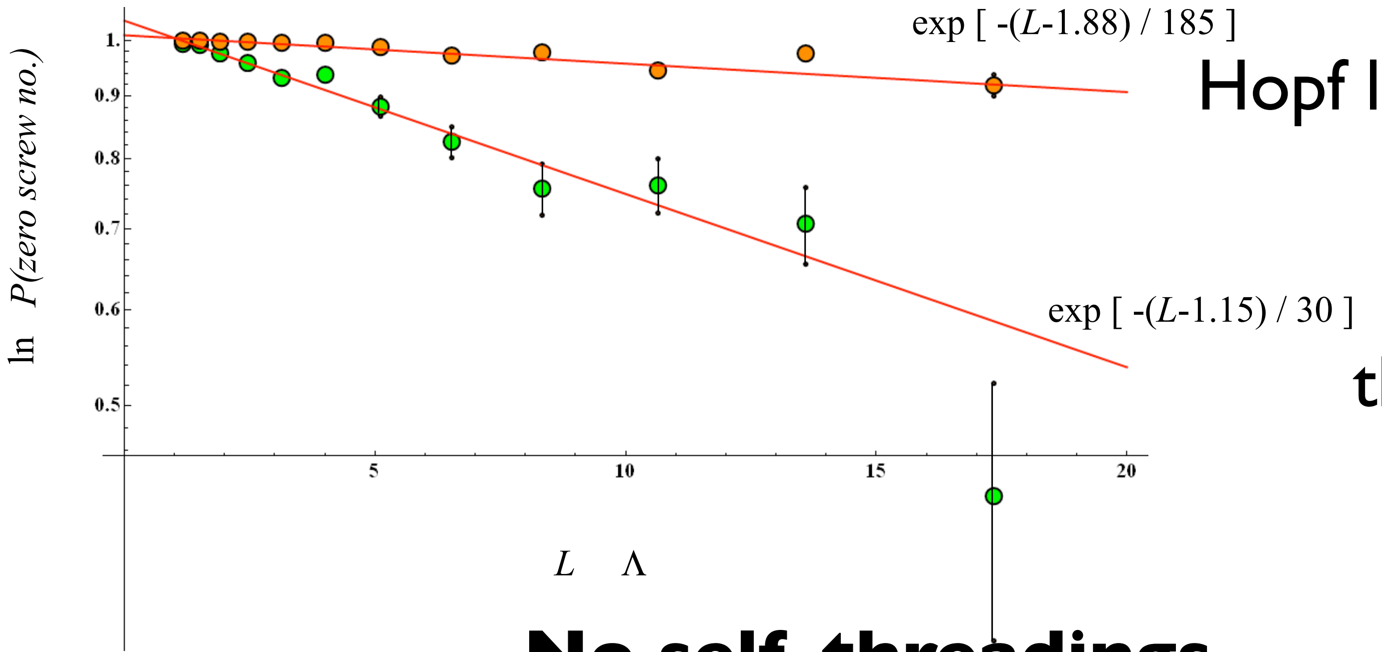
One
3-loop
link
found



Random topology scaling

Probability of being unthreaded ~ $\exp(-L/A\Lambda)$

A depends on type of threading



Hopf link

all threadings

No self-threadings, i.e. knots, found