The circular law

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Abstract. The circular law theorem states that the empirical spectral distribution of a $n \times n$ random matrix with i.i.d. entries of variance $1/n$ tends to the uniform law on the unit disc of the complex plane as the dimension n tends to infinity. This phenomenon is the non-Hermitian counterpart of the semi circular limit for Wigner random Hermitian matrices, and the quarter circular limit for Marchenko-Pastur random covariance matrices. In these expository notes, we present a proof in a Gaussian case, due to Silverstein, based on a formula by Ginibre, and a proof of the universal case by revisiting the approach of Tao and Vu, based on the Hermitization of Girko, the logarithmic potential, and the control of the small singular values. We also discuss some related models.

These notes constitute an abridged and updated version of the probability survey [**BC12**], prepared at the occasion of the American Mathematical Society short course on Random Matrices, organized by Van H. Vu for the 2013 AMS-MAA Joint Mathematics Meeting held in January 9–13 in San Diego, USA.

Section 1 introduces the notion of eigenvalues and singular values and discusses their relationships. Section 2 states the circular law theorem. Section 3 is devoted to the Gaussian model known as the Complex Ginibre Ensemble, for which the law of the spectrum is known and leads to the circular law. Section 4 provides the proof of the circular law theorem in the universal case, using the approach of Tao and Vu based on the Hermitization of Girko and the logarithmic potential. Section 5 gathers finally some few comments on related problems and models.

All random variables are defined on a unique common probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A typical element of Ω is denoted ω. We write *a.s.*, *a.a.*, and *a.e.* for *almost surely*, *Lebesgue almost all*, and *Lebesgue almost everywhere* respectively.

1. Two kinds of spectra

The *eigenvalues* of $A \in \mathcal{M}_n(\mathbb{C})$ are the roots in $\mathbb{C}$ of its characteristic polynomial $P_A(z) := \det(A - zI)$. We label them $\lambda_1(A), \ldots, \lambda_n(A)$ so that $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$ with growing phases. The *spectral radius* is $|\lambda_1(A)|$. The eigenvalues form the algebraic spectrum of A. The *singular values of A* are defined by

$$s_k(A) := \lambda_k(\sqrt{AA^*})$$

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for all $1 \leq k \leq n$, where $A^* = \bar{A}^\top$ is the conjugate-transpose. We have

$$s_1(A) \geq \cdots \geq s_n(A) \geq 0.$$

The matrices $A, A^\top, A^*$ have the same singular values. The Hermitian matrix

$$H_A := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

is $2n \times 2n$ with eigenvalues $s_1(A), -s_1(A), \ldots, s_n(A), -s_n(A)$. This turns out to be useful because the mapping $A \mapsto H_A$ is linear in A, in contrast with the mapping $A \mapsto \sqrt{AA^*}$. Geometrically, the matrix A maps the unit sphere to an ellipsoid, the half-lengths of its principal axes being exactly the singular values of A. The *operator norm* or *spectral norm* of A is

$$\|A\|_{2\to 2} := \max_{\|x\|_2=1} \|Ax\|_2 = s_1(A) \quad \text{while} \quad s_n(A) = \min_{\|x\|_2=1} \|Ax\|_2.$$

The rank of A is equal to the number of non-zero singular values. If A is non-singular then $s_i(A^{-1}) = s_{n-i}(A)^{-1}$ for all $1 \leq i \leq n$ and $s_n(A) = s_1(A^{-1})^{-1} = \|A^{-1}\|_{2\to 2}^{-1}$.

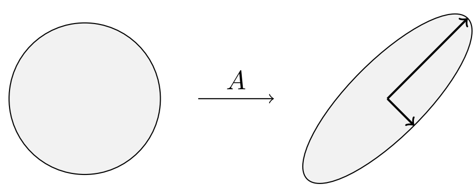

FIGURE 1. Largest and smallest singular values of $A \in \mathcal{M}_2(\mathbb{R})$. Taken from [**CGLP12**].

Since the singular values are the eigenvalues of a Hermitian matrix, we have variational formulas for all of them, often called the Courant-Fischer variational formulas [**HJ94**, Theorem 3.1.2]. Namely, denoting $\mathcal{G}_{n,i}$ the Grassmannian of all i-dimensional subspaces, we have

$$s_i(A) = \max_{E \in \mathcal{G}_{n,i}} \min_{\substack{x \in E \\ \|x\|_2=1}} \|Ax\|_2 = \max_{E,F \in \mathcal{G}_{n,i}} \min_{\substack{(x,y) \in E \times F \\ \|x\|_2=\|y\|_2=1}} \langle Ax, y \rangle.$$

Most useful properties of the singular values are consequences of their Hermitian nature via these variational formulas, which are valid on $\mathbb{R}^n$ and on $\mathbb{C}^n$. In contrast, there are no such variational formulas for the eigenvalues in great generality, beyond the case of normal matrices. If the matrix A is normal[1] (i.e. $A^*A = A^*A$) then $s_i(A) = |\lambda_i(A)|$ for every $1 \leq i \leq n$. Beyond normal matrices, the relationships between the eigenvalues and the singular values are captured by a set of inequalities due to Weyl, which can be obtained by using the Schur unitary triangularization[2], see for instance [**HJ94**, Theorem 3.3.2 page 171].

THEOREM 1.1 (Weyl inequalities). *For every $A \in \mathcal{M}_n(\mathbb{C})$ and $1 \leq k \leq n$,*

$$\prod_{i=1}^k |\lambda_i(A)| \leq \prod_{i=1}^k s_i(A). \tag{1.1}$$

[1]We always use the word *normal* in this way, and never as a synonym for *Gaussian*.

[2]If $A \in \mathcal{M}_n(\mathbb{C})$ then there exists a unitary matrix U such that UAU^* is upper triangular.

The reversed form $\prod_{i=n-k+1}^n s_i(A) \le \prod_{i=n-k+1}^n |\lambda_i(A)|$ for every $1 \le k \le n$ can be deduced easily (exercise!). Equality is achieved for $k = n$ and we have

$$\prod_{k=1}^n |\lambda_k(A)| = |\det(A)| = \sqrt{|\det(A)||\det(A^*)|} = |\det(\sqrt{AA^*})| = \prod_{k=1}^n s_k(A). \quad (1.2)$$

One may deduce from Weyl's inequalities that (see [**HJ94**, Theorem 3.3.13])

$$\sum_{i=1}^n |\lambda_i(A)|^2 \le \sum_{i=1}^n s_i(A)^2 = \mathrm{Tr}(AA^*) = \sum_{i,j=1}^n |A_{i,j}|^2. \quad (1.3)$$

Since $s_1(\cdot) = \|\cdot\|_{2\to 2}$ we have for any $A, B \in \mathcal{M}_n(\mathbb{C})$ that

$$s_1(AB) \le s_1(A)s_1(B) \quad \text{and} \quad s_1(A+B) \le s_1(A) + s_1(B). \quad (1.4)$$

We define the empirical eigenvalues and singular values measures by

$$\mu_A := \frac{1}{n}\sum_{k=1}^n \delta_{\lambda_k(A)} \quad \text{and} \quad \nu_A := \frac{1}{n}\sum_{k=1}^n \delta_{s_k(A)}.$$

Note that μ_A and ν_A are supported respectively in $\mathbb{C}$ and $\mathbb{R}_+$. There is a rigid determinantal relationship between μ_A and ν_A, namely from (1.2) we get

$$\begin{aligned}\int \log|\lambda|\, d\mu_A(\lambda) &= \frac{1}{n}\sum_{i=1}^n \log|\lambda_i(A)| \\ &= \frac{1}{n}\log|\det(A)| \\ &= \frac{1}{n}\sum_{i=1}^n \log(s_i(A)) \\ &= \int \log(s)\, d\nu_A(s).\end{aligned}$$

This identity is at the heart of the Hermitization technique used in sections 4.

The singular values are quite regular functions of the matrix entries. For instance, the Courant-Fischer formulas imply that the map $A \mapsto (s_1(A), \ldots, s_n(A))$ is 1-Lipschitz for the operator norm and the ℓ^∞ norm: for any $A, B \in \mathcal{M}_n(\mathbb{C})$,

$$\max_{1\le i\le n} |s_i(A) - s_i(B)| \le s_1(A - B). \quad (1.5)$$

Recall that $\mathcal{M}_n$ is Hilbert spaces for the scalar product $A \cdot B = \mathrm{Tr}(AB^*)$. The norm $\|\cdot\|_2$ associated to this scalar product, called the trace norm[3], satisfies to

$$\|A\|_2^2 = \mathrm{Tr}(AA^*) = \sum_{i=1}^n s_i(A)^2 = n\int s^2\, d\nu_A(s). \quad (1.6)$$

In the sequel, we say that a sequence of (possibly signed) measures $(\eta_n)_{n\ge 1}$ on $\mathbb{C}$ (respectively on $\mathbb{R}$) tends weakly to a (possibly signed) measure η, and we denote

$$\eta_n \rightsquigarrow \eta,$$

when for all continuous and bounded function $f : \mathbb{C} \to \mathbb{R}$ (respectively $f : \mathbb{R} \to \mathbb{R}$),

$$\lim_{n\to\infty} \int f\, d\eta_n = \int f\, d\eta.$$

[3]Also known as the Hilbert-Schmidt norm, the Schur norm, or the Frobenius norm.

EXAMPLE 1.2 (Spectra of non-normal matrices). *The eigenvalues depend continuously on the entries of the matrix. It turns out that for non-normal matrices, the eigenvalues are more sensitive to perturbations than the singular values. Among non-normal matrices, we find non-diagonalizable matrices, including nilpotent matrices. Let us recall a striking example taken from* [**Śni02**] *and* [**BS06**, Chapter 10]. *Let us consider* $A, B \in \mathcal{M}_n(\mathbb{R})$ *given by*

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \kappa_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where (κ_n) *is a sequence of positive real numbers. The matrix* A *is nilpotent, and* B *is a perturbation with small norm (and rank one!):*

$$\mathrm{rank}(A - B) = 1 \quad \text{and} \quad \|A - B\|_{2\to 2} = \kappa_n.$$

We have $\lambda_1(A) = \cdots = \lambda_{\kappa_n}(A) = 0$ *and thus*

$$\mu_A = \delta_0.$$

In contrast, $B^n = \kappa_n I$ *and thus* $\lambda_k(B) = \kappa_n^{1/n} e^{2k\pi i/n}$ *for all* $1 \leq k \leq n$ *which gives*

$$\mu_B \rightsquigarrow \mathrm{Uniform}\{z \in \mathbb{C} : |z| = 1\}$$

as soon as $\kappa_n^{1/n} \to 1$ *(this allows* $\kappa_n \to 0$*). On the other hand, the identities*

$$AA^* = \mathrm{diag}(1, \ldots, 1, 0) \quad \text{and} \quad BB^* = \mathrm{diag}(1, \ldots, 1, \kappa_n^2)$$

give $s_1(A) = \cdots = s_{n-1}(A) = 1$ *and* $s_1(B) = \cdots = s_{n-1}(B) = 1$ *and thus*

$$\nu_A \rightsquigarrow \delta_1 \quad \text{and} \quad \nu_B \rightsquigarrow \delta_1.$$

This shows the stability of the limiting singular values distribution under additive perturbation of rank 1 *of arbitrary large norm, and the instability of the limiting eigenvalues distribution under an additive perturbation of rank* 1 *and small norm.*

We must keep in mind the fact that the singular values are related to the geometry of the matrix rows. We end up this section with a couple of lemmas relating rows distances and norms of the inverse, which are used in the sequel.

LEMMA 1.3 (Rows and operator norm of the inverse). *Let* $A \in \mathcal{M}_n(\mathbb{C})$ *with rows* $R_1, \ldots, R_n$. *Define the vector space* $R_{-i} := \mathrm{span}\{R_j : j \neq i\}$. *We have then*

$$n^{-1/2} \min_{1\leq i\leq n} \mathrm{dist}(R_i, R_{-i}) \leq s_n(A) \leq \min_{1\leq i\leq n} \mathrm{dist}(R_i, R_{-i}).$$

PROOF OF LEMMA 1.3. We follow an argument buried in [**RV08**]. Since A and $A^\top$ have same singular values, one can consider the columns $C_1, \ldots, C_n$ of A instead of the rows. For every column vector $x \in \mathbb{C}^n$ and $1 \leq i \leq n$, the triangle inequality and the identity $Ax = x_1 C_1 + \cdots + x_n C_n$ give

$$\|Ax\|_2 \geq \mathrm{dist}(Ax, C_{-i}) = \min_{y \in C_{-i}} \|Ax - y\|_2 = \min_{y \in C_{-i}} \|x_i C_i - y\|_2 = |x_i| \mathrm{dist}(C_i, C_{-i}).$$

If $\|x\|_2 = 1$ then necessarily $|x_i| \geq n^{-1/2}$ for some $1 \leq i \leq n$ and therefore

$$s_n(A) = \min_{\|x\|_2 = 1} \|Ax\|_2 \geq n^{-1/2} \min_{1\leq i\leq n} \mathrm{dist}(C_i, C_{-i}).$$

Conversely, for every $1 \leq i \leq n$, there exists a vector y with $y_i = 1$ such that

$$\operatorname{dist}(C_i, C_{-i}) = \|y_1 C_1 + \cdots + y_n C_n\|_2 = \|Ay\|_2 \geq \|y\|_2 \min_{\|x\|_2=1} \|Ax\|_2 \geq s_n(A)$$

where we used the fact that $\|y\|_2^2 = |y_1|^2 + \cdots + |y_n|^2 \geq |y_i|^2 = 1$. □

LEMMA 1.4 (Rows and trace norm of the inverse). *Let $1 \leq m \leq n$. If $A \in \mathcal{M}_{m,n}(\mathbb{C})$ has full rank, with rows $R_1, \ldots, R_m$ and $R_{-i} := \operatorname{span}\{R_j : j \neq i\}$, then*

$$\sum_{i=1}^{m} s_i(A)^{-2} = \sum_{i=1}^{m} \operatorname{dist}(R_i, R_{-i})^{-2}.$$

PROOF. We follow the proof given in [**TV10b**, Lemma A4]. The orthogonal projection of R_i^* on the subspace R_{-i} is $B^*(BB^*)^{-1}BR_i^*$ where B is the $(m-1)\times n$ matrix obtained from A by removing the row R_i. In particular, we have

$$|R_i|_2^2 - \operatorname{dist}_2(R_i, R_{-i})^2 = \left|B^*(BB^*)^{-1}BR_i^*\right|_2^2 = (BR_i^*)^*(BB^*)^{-1}BR_i^*$$

by the Pythagoras theorem. On the other hand, the Schur block inversion formula states that if M is a $m \times m$ matrix then for every partition $\{1, \ldots, m\} = I \cup I^c$,

$$(M^{-1})_{I,I} = (M_{I,I} - M_{I,I^c}(M_{I^c,I^c})^{-1}M_{I^c,I})^{-1}.$$

We take $M = AA^*$ and $I = \{i\}$, and we note that $(AA^*)_{i,j} = R_i R_j^*$, which gives

$$((AA^*)^{-1})_{i,i} = (R_i R_i^* - (BR_i^*)^*(BB^*)^{-1}BR_i^*)^{-1} = \operatorname{dist}_2(R_i, R_{-i})^{-2}.$$

The desired formula follows by taking the sum over $i \in \{1, \ldots, m\}$. □

2. Circular law

The variance of a random variable Z on $\mathbb{C}$ is $\operatorname{Var}(Z) = \mathbb{E}(|Z|^2) - |\mathbb{E}(Z)|^2$. Let $(X_{ij})_{i,j\geq 1}$ be an infinite table of i.i.d. random variables on $\mathbb{C}$ with variance 1. We consider the square random matrix $X := (X_{ij})_{1\leq i,j\leq n}$ as a random variable in $\mathcal{M}_n(\mathbb{C})$. We start with the classical Marchenko-Pastur theorem for the "empirical covariance matrix" $\frac{1}{n}XX^*$. This theorem is universal in the sense that the limiting distribution does not depend on the law of X_{11}.

THEOREM 2.1 (Marchenko-Pastur quarter circular law). *a.s. $\nu_{n^{-1/2}X} \rightsquigarrow \mathcal{Q}_2$ as $n \to \infty$, where $\mathcal{Q}_2$ is the quarter circular law*[4] *on $[0, 2] \subset \mathbb{R}_+$ with density*

$$x \mapsto \frac{\sqrt{4 - x^2}}{\pi}\mathbf{1}_{[0,2]}(x).$$

The $n^{-1/2}$ normalization is easily understood from the law of large numbers:

$$\begin{aligned}\int s^2 \, d\nu_{n^{-1/2}X}(s) &= \frac{1}{n^2}\sum_{i=1}^{n} s_i(X)^2 \\ &= \frac{1}{n^2}\operatorname{Tr}(XX^*) = \frac{1}{n^2}\sum_{i,j=1}^{n} |X_{i,j}|^2 \to \mathbb{E}(|X_{1,1}|^2). \quad (2.1)\end{aligned}$$

The main subject of these notes is the following counterpart for the eigenvalues.

[4] Actually, it is a quarter ellipse rather than a quarter circle, due to the normalizing factor $1/\pi$. However, one may use different scales to see a true quarter circle, as in figure 2.

THEOREM 2.2 (Girko circular law). *a.s.* $\mu_{n^{-1/2}X} \rightsquigarrow \mathcal{C}_1$ *as* $n \to \infty$, *where* $\mathcal{C}_1$ *is the circular law*[5] *which is the uniform law on the unit disc of* $\mathbb{C}$ *with density*

$$z \mapsto \frac{1}{\pi}\mathbf{1}_{\{z\in\mathbb{C}:|z|\leq 1\}}.$$

If Z is a complex random variable following the circular law $\mathcal{C}_1$ then the random variables $\mathfrak{Re}(Z)$ and $\mathfrak{Im}(Z)$ follow the semi circular law on $[-1,1]$, and are not independent. Additionally, the random variables $|\mathfrak{Re}(Z)|$ and $|\mathfrak{Im}(Z)|$ follow the quarter circular law on $[0,1]$, and $|Z|$ follows the law with density $\rho \mapsto \frac{1}{2}\rho\mathbf{1}_{[0,1]}(\rho)$. As we will see in section 4, we will deduce theorem 2.2 from an extension of theorem 2.1 by using a Hermitization technique.

The circular law theorem 2.2 has a long history. It was established through a sequence of partial results during the period 1965–2009, the general case being finally obtained by Tao and Vu [**TV10b**]. Indeed Mehta [**Meh67**] was the first to obtain a circular law theorem for the expected empirical spectral distribution in the complex Gaussian case, by using the explicit formula for the spectrum due to Ginibre [**Gin65**]. Edelman was able to prove the same kind of result for the far more delicate real Gaussian case [**Ede97**]. Silverstein provided an argument to pass from the expected to the almost sure convergence in the complex Gaussian case [**Hwa86**]. Girko worked on the universal version and came with very good ideas such as the Hermitization technique [**Gir84, Gir94, Gir04a, Gir04b, Gir05**]. Unfortunately, his work was controversial due to a lack of clarity and rigor[6]. In particular, his approach relies implicitly on an unproved uniform integrability related to the behavior of the smallest singular values of random matrices. Let us mention that the Hermitization technique is also present in the work of Widom [**Wid94**] on Toeplitz matrices and in the work of Goldsheid and Khoruzhenko [**GK00**]. Bai [**Bai97**] was the first to circumvent the problem in the approach of Girko, at the price of bounded density assumptions and moments assumptions[7]. Bai improved his approach in his book written with Silverstein [**BS06**]. His approach involves the control of the speed of convergence of the singular values distribution. Śniady considered a universal version beyond random matrices and the circular law, using the notion of $*$-moments and Brown measure of operators in free probability, and a regularization by adding an independent Gaussian Ginibre noise [**Śni02**]. Goldsheid and Khoruzhenko [**GK05**] used successfully the logarithmic potential to derive the analogue of the circular law theorem for random non-Hermitian tridiagonal matrices. The smallest singular value of random matrices was the subject of an impressive activity culminating with the works of Tao and Vu [**TV09b, TV10c, TV10a**] and of Rudelson and Vershynin [**RV08**], using tools from asymptotic geometric analysis and additive combinatorics (Littlewood-Offord problems). These achievements allowed Götze and Tikhomirov [**GT10a**] to obtain the expected circular law theorem up to a small loss in the moment assumption, by using the logarithmic potential. Similar ingredients are present in the work of Pan and Zhou [**PZ10**]. At the same

[5] It is not customary to call it the "disc law". The terminology corresponds to what we draw: a circle for the circular law, a quarter circle (ellipse) for the quarter circular law, even if it is the boundary of the support in the first case, and the density in the second case. See figure 2.

[6] Girko's writing style is also quite original, see for instance the recent paper [**GV10**].

[7] *...I worked for 13 years from 1984 to 1997, which was eventually published in Annals of Probability. It was the hardest problem I have ever worked on.* Zhidong Bai, interview with Atanu Biswas in 2006 [**CZH08**].

time, Tao and Vu, using a refined bound on the smallest singular value and the approach of Bai, deduced the circular law theorem up to a small loss in the moment assumption [**TV08**]. As in the works of Girko, Bai and their followers, the loss was due to a sub-optimal usage of the Hermitization approach. In [**TV10b**], Tao and Vu finally obtained the full circular law theorem 2.2 by using the full strength of the logarithmic potential, and a new control of the count of the small singular values which replaces the speed of convergence estimates of Bai. See also their synthetic paper [**TV09a**]. We will follow essentially their approach in section 4 to prove theorem 2.2.

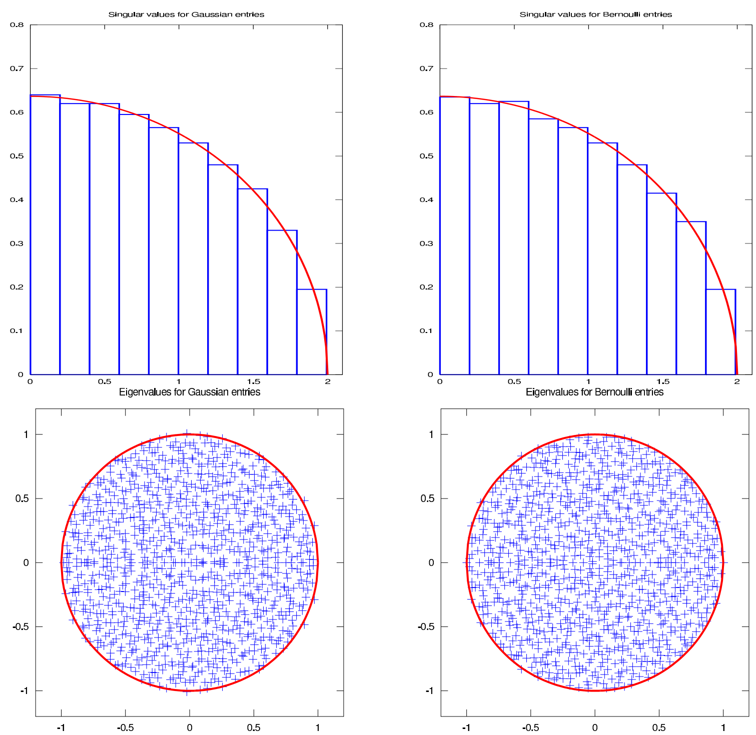

FIGURE 2. Illustration of universality in the quarter circular law and the circular law theorems 2.1 and 2.2. The plots are made with the singular values (upper plots) and eigenvalues (lower plot) for a single random matrix X of dimension $n = 1000$. On the left hand side, X_{11} follows a standard Gaussian law on $\mathbb{R}$, while on the right hand side X_{11} follows a symmetric Bernoulli law on $\{-1, 1\}$.

The a.s. tightness of $\mu_{n^{-1/2}X}$ is easily understood since Weyl's inequality give

$$\int |\lambda|^2 \, d\mu_{n^{-1/2}X}(\lambda) = \frac{1}{n^2} \sum_{i=1}^{n} |\lambda_i(X)|^2 \leq \frac{1}{n^2} \sum_{i=1}^{n} s_i(X)^2 = \int s^2 \, d\nu_{n^{-1/2}X}(s).$$

The convergence in the couple of theorems above is the weak convergence of probability measures with respect to continuous bounded functions. We recall that this mode of convergence does not capture the convergence of the support. It implies only that a.s.

$$\underline{\lim}_{n\to\infty} s_1(n^{-1/2}X) \geq 2 \quad \text{and} \quad \underline{\lim}_{n\to\infty} |\lambda_1(n^{-1/2}X)| \geq 1.$$

However, it can be shown that if $\mathbb{E}(X_{1,1}) = 0$ and $\mathbb{E}(|X_{1,1}|^4) < \infty$ then a.s.

$$\lim_{n\to\infty} s_1(n^{-1/2}X) = 2 \quad \text{and} \quad \lim_{n\to\infty} |\lambda_1(n^{-1/2}X)| = 1,$$

see [**BS10**]. The asymptotic factor 2 between the operator norm and the spectral radius indicates in a sense that X is a non-normal matrix asymptotically as $n \to \infty$ (note that if X_{11} is absolutely continuous then X is absolutely continuous and thus $XX^* \neq X^*X$ a.s. which means that X is non-normal a.s.). The law of the modulus under the circular law has density $\rho \mapsto 2\rho\mathbf{1}_{[0,1]}(\rho)$ which differs completely from the shape of the quarter circular law $s \mapsto \pi^{-1}\sqrt{4-s^2}\,\mathbf{1}_{[0,2]}(s)$, see figure 3. The integral of "log" for both laws is the same.

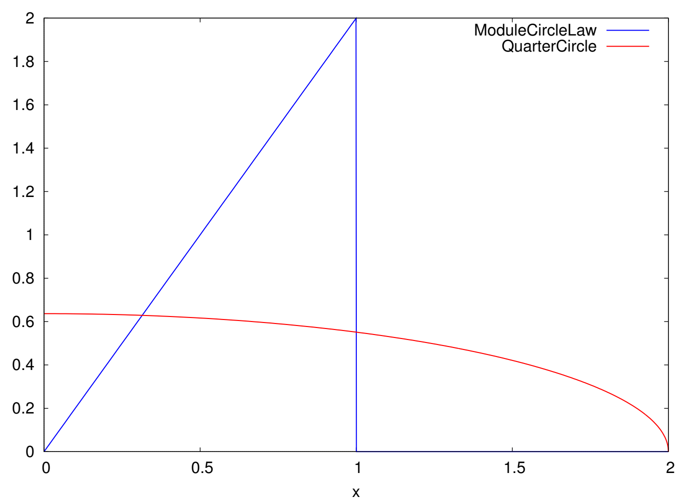

FIGURE 3. Comparison between the quarter circular distribution of theorem 2.1 for the singular values, and the modulus under the circular law of theorem 2.2 for the eigenvalues. The supports and the shapes are different. This difference indicates the asymptotic non-normality of these matrices. The integral of the function $t \mapsto \log(t)$ is the same for both distributions.

3. Gaussian case

This section is devoted to the case where $X_{11} \sim \mathcal{N}(0, \frac{1}{2}I_2)$. From now on, we denote G instead of X in order to distinguish the Gaussian case from the general case. We say that G belongs to the *Complex Ginibre Ensemble*. The Lebesgue density of the $n \times n$ random matrix $G = (G_{i,j})_{1\leq i,j\leq n}$ in $\mathcal{M}_n(\mathbb{C}) \equiv \mathbb{C}^{n\times n}$ is

$$A \in \mathcal{M}_n(\mathbb{C}) \mapsto \pi^{-n^2} e^{-\sum_{i,j=1}^n |A_{ij}|^2}. \tag{3.1}$$

This law is a Boltzmann distribution with energy

$$A \mapsto \sum_{i,j=1}^n |A_{ij}|^2 = \mathrm{Tr}(AA^*) = \|A\|_2^2 = \sum_{i=1}^n s_i^2(A).$$

This law is unitary invariant, in the sense that if U and V are $n\times n$ unitary matrices then UGV and G are equally distributed. If H_1 and H_2 are two independent copies of GUE[8] then $(H_1 + iH_2)/\sqrt{2}$ has the law of G. Conversely, the matrices $(G+G^*)/\sqrt{2}$ and $(G-G^*)/\sqrt{2}i$ are independent and belong to the GUE.

The singular values of G are the square root of the eigenvalues of the positive semidefinite Hermitian matrix GG^*. The matrix GG^* is a complex Wishart matrix, and belongs to the complex Laguerre Ensemble ($\beta = 2$). The empirical distribution of the singular values of $n^{-1/2}G$ tends to the Marchenko-Pastur quarter circular distribution (Gaussian case in theorem 2.1). This section is rather devoted to the study of the eigenvalues of G, and in particular to the proof of the circular law theorem 2.2 in this Gaussian settings.

The set of elements of $\mathcal{M}_n(\mathbb{C})$ with multiple eigenvalues has zero Lebesgue measure in $\mathbb{C}^{n\times n}$. In particular, the set of non-diagonalizable elements of $\mathcal{M}_n(\mathbb{C})$ has zero Lebesgue measure in $\mathbb{C}^{n\times n}$. Since G is absolutely continuous, we have a.s. $GG^* \neq G^*G$ (non-normality) and G is diagonalizable with distinct eigenvalues. Following Ginibre [**Gin65**] – see also [**Meh04, For10**, Chapter 15] and [**KS11**] – one may then compute the joint density of the eigenvalues $\lambda_1(G),\ldots,\lambda_n(G)$ of G by integrating (3.1) over the non-eigenvalues variables. The result is stated in theorem 3.1 below. It is worthwhile to mention that in contrast with Hermitian unitary invariant ensembles, the computation of the spectrum law is problematic if one replaces the square potential by a more general potential, see [**KS11**]. The law of G is invariant by the multiplication of the entries with a common phase, and thus the law of the spectrum of G has also the same property. In the sequel we set

$$\Delta_n := \{(z_1,\ldots,z_n) \in \mathbb{C}^n : |z_1| \geq \cdots \geq |z_n|\}.$$

THEOREM 3.1 (Spectrum law). *$(\lambda_1(G),\ldots,\lambda_n(G))$ has density $n!\varphi_n \mathbf{1}_{\Delta_n}$ where*

$$\varphi_n(z_1,\ldots,z_n) = \frac{\pi^{-n^2}}{1!2!\cdots n!} \exp\left(-\sum_{k=1}^n |z_k|^2\right) \prod_{1\leq i<j\leq n} |z_i - z_j|^2.$$

In particular, for every symmetric Borel function $F : \mathbb{C}^n \to \mathbb{R}$,

$$\mathbb{E}[F(\lambda_1(G),\ldots,\lambda_n(G))] = \int_{\mathbb{C}^n} F(z_1,\ldots,z_n)\varphi_n(z_1,\ldots,z_n)\,dz_1\cdots dz_n.$$

We will use theorem 3.1 with symmetric functions of the form

$$F(z_1,\ldots,z_n) = \sum_{i_1,\ldots,i_k \text{ distinct}} f(z_{i_1})\cdots f(z_{i_k}).$$

The Vandermonde determinant comes from the Jacobian of the diagonalization. The eigenvalues of G can be seen as a Coulomb gas of particles on $\mathbb{C}$, experiencing logarithmic pair repulsion, and confined by a quadratic potential. The spectrum is also a Gaussian determinantal point process, see [**HKPV09**, Chapter 4].

[8]Up to scaling, a random $n\times n$ Hermitian matrix H belongs to the Gaussian Unitary Ensemble (GUE) when its density is proportional to $\exp(-\frac{1}{2}\mathrm{Tr}(H^2)))$. Equivalently, the random variables $\{H_{ii}, H_{ij} : 1 \leq i \leq n, i < j \leq n\}$ are indep. with $H_{ii} \sim \mathcal{N}(0,1)$ and $H_{ij} \sim \mathcal{N}(0, \frac{1}{2}I_2)$ for $i \neq j$.

Let $z \in \mathbb{C} \mapsto \gamma(z) = \pi^{-1}e^{-|z|^2}$ be the density of the standard Gaussian $\mathcal{N}(0, \frac{1}{2}I_2)$ on $\mathbb{C}$. For every $1 \leq k \leq n$, the "k-point correlation" is

$$\varphi_{n,k}(z_1, \ldots, z_k) := \int_{\mathbb{C}^{n-k}} \varphi_n(z_1, \ldots, z_n)\, dz_{k+1} \cdots dz_n.$$

THEOREM 3.2 (k-points correlations). *For every $1 \leq k \leq n$,*

$$\varphi_{n,k}(z_1, \ldots, z_k) = \frac{(n-k)!}{n!}\pi^{-k^2}\gamma(z_1)\cdots\gamma(z_k)\det\left[K(z_i, z_j)\right]_{1\leq i,j\leq k}$$

where

$$K(z_i, z_j) := \sum_{\ell=0}^{n-1}\frac{(z_i z_j^*)^\ell}{\ell!} = \sum_{\ell=0}^{n-1} H_\ell(z_i)H_\ell(z_j)^* \quad \textit{with} \quad H_\ell(z) := \frac{1}{\sqrt{\ell!}}z^\ell.$$

In particular, by taking $k = n$ we get

$$\varphi_{n,n}(z_1, \ldots, z_n) = \varphi_n(z_1, \ldots, z_n) = \frac{1}{n!}\pi^{-n^2}\gamma(z_1)\cdots\gamma(z_n)\det\left[K(z_i, z_j)\right]_{1\leq i,j\leq n}.$$

PROOF. Following [**Meh04**, Chapter 15 page 271 equation 15.1.29], we use

$$\prod_{1\leq i<j\leq n}|z_i - z_j|^2 = \prod_{1\leq i<j\leq n}(z_i - z_j)\prod_{1\leq i<j\leq n}(z_i - z_j)^*$$

and

$$\det\left[z_j^{i-1}\right]_{1\leq i,j\leq n}\det\left[(z_j^*)^{i-1}\right]_{1\leq i,j\leq n} = \frac{1}{n!}\det\left[K(z_i, z_j)\right]_{1\leq i,j\leq n}.$$

□

The next result relies on the fact that $\mathbb{E}\mu_G$ has density $\varphi_{n,1}$.

THEOREM 3.3 (Mean circular Law). *$\mathbb{E}\mu_{n^{-1/2}G} \rightsquigarrow \mathcal{C}_1$ as $n \to \infty$, where $\mathcal{C}_1$ is the uniform law on the unit disc of $\mathbb{C}$ with density $z \mapsto \pi^{-1}\mathbf{1}_{\{z\in\mathbb{C}:|z|\leq 1\}}$.*

PROOF. From theorem 3.2, with $k = 1$, we get that the density of $\mathbb{E}\mu_G$ is

$$\varphi_{n,1} : z \mapsto \gamma(z)\left(\frac{1}{n}\sum_{\ell=0}^{n-1}|H_\ell|^2(z)\right) = \frac{1}{n\pi}e^{-|z|^2}\sum_{\ell=0}^{n-1}\frac{|z|^{2\ell}}{\ell!}.$$

Now, following Mehta [**Meh04**, Chapter 15 page 272], for $r^2 < n$,

$$e^{r^2} - \sum_{\ell=0}^{n-1}\frac{r^{2\ell}}{\ell!} = \sum_{\ell=n}^{\infty}\frac{r^{2\ell}}{\ell!} \leq \frac{r^{2n}}{n!}\sum_{\ell=0}^{\infty}\frac{r^{2\ell}}{(n+1)^\ell} = \frac{r^{2n}}{n!}\frac{n+1}{n+1-r^2}$$

while for $r^2 > n$,

$$\sum_{\ell=0}^{n-1}\frac{r^{2\ell}}{\ell!} \leq \frac{r^{2(n-1)}}{(n-1)!}\sum_{\ell=0}^{n-1}\left(\frac{n-1}{r^2}\right)^\ell \leq \frac{r^{2(n-1)}}{(n-1)!}\frac{r^2}{r^2-n+1}.$$

By taking $r^2 = |\sqrt{n}z|^2$ we obtain that for every compact $C \subset \mathbb{C}$

$$\lim_{n\to\infty}\sup_{z\in C}\left|n\varphi_{n,1}(\sqrt{n}z) - \pi^{-1}\mathbf{1}_{[0,1]}(|z|)\right| = 0.$$

The n in front of $\varphi_{n,1}$ is due to the fact that we are on $\mathbb{C}$: $d\sqrt{n}xd\sqrt{n}y = ndxdy$. □

The sequence $(H_k)_{k\in\mathbb{N}}$ forms an orthonormal basis (orthogonal polynomials) of square integrable analytic functions on $\mathbb{C}$ for the standard Gaussian on $\mathbb{C}$. The uniform law on the unit disc (known as the circular law) is the law of $\sqrt{V}e^{2i\pi W}$ where V and W are i.i.d. uniform random variables on the interval $[0,1]$.

THEOREM 3.4 (Circular law). *a.s.* $\mu_{n^{-1/2}G} \rightsquigarrow \mathcal{C}_1$ *as* $n\to\infty$, *where* $\mathcal{C}_1$ *is the uniform law on the unit disc of* $\mathbb{C}$ *with density* $z\mapsto \pi^{-1}\mathbf{1}_{\{z\in\mathbb{C}:|z|\leq 1\}}$.

PROOF. We reproduce Silverstein's argument [**Hwa86**], which is similar to the quick proof of the strong law of large numbers for independent random variables with bounded fourth moment. It suffices to establish the result for compactly supported continuous bounded functions. Let us pick such a function f and set

$$S_n := \int_{\mathbb{C}} f\,d\mu_{n^{-1/2}G} \quad \text{and} \quad S_\infty := \pi^{-1}\int_{|z|\leq 1} f(z)\,dxdy.$$

Suppose for now that we have

$$\mathbb{E}[(S_n-\mathbb{E}S_n)^4] = O(n^{-2}). \tag{3.2}$$

By monotone convergence (or by the Fubini-Tonelli theorem),

$$\mathbb{E}\sum_{n=1}^\infty (S_n-\mathbb{E}S_n)^4 = \sum_{n=1}^\infty \mathbb{E}[(S_n-\mathbb{E}S_n)^4] < \infty$$

and consequently $\sum_{n=1}^\infty (S_n-\mathbb{E}S_n)^4<\infty$ a.s. which implies $\lim_{n\to\infty} S_n-\mathbb{E}S_n=0$ a.s. Since $\lim_{n\to\infty}\mathbb{E}S_n = S_\infty$ by theorem 3.3, we get that a.s.

$$\lim_{n\to\infty} S_n = S_\infty.$$

Finally, one can swap the universal quantifiers on ω and f thanks to the separability of the set of compactly supported continuous bounded functions $\mathbb{C}\to\mathbb{R}$ equipped with the supremum norm. To establish (3.2), we set

$$S_n-\mathbb{E}S_n = \frac{1}{n}\sum_{i=1}^n Z_i \quad \text{with} \quad Z_i := f\Big(\lambda_i\Big(n^{-1/2}G\Big)\Big).$$

Next, we obtain, with $\sum_{i_1,\ldots}$ running over distinct indices in $1,\ldots,n$,

$$\begin{aligned}\mathbb{E}\Big[(S_n-\mathbb{E}S_n)^4\Big] &= \frac{1}{n^4}\sum_{i_1}\mathbb{E}[Z_{i_1}^4]\\ &+\frac{4}{n^4}\sum_{i_1,i_2}\mathbb{E}[Z_{i_1}Z_{i_2}^3]\\ &+\frac{3}{n^4}\sum_{i_1,i_2}\mathbb{E}[Z_{i_1}^2Z_{i_2}^2]\\ &+\frac{6}{n^4}\sum_{i_1,i_2,i_3}\mathbb{E}[Z_{i_1}Z_{i_2}Z_{i_3}^2]\\ &+\frac{1}{n^4}\sum_{i_1,i_2,i_3,i_3,i_4}\mathbb{E}[Z_{i_1}Z_{i_3}Z_{i_3}Z_{i_4}].\end{aligned}$$

The first three terms of the right hand side are $\mathcal{O}(n^{-2})$ since $\max_{1\leq i\leq n}|Z_i|\leq \|f\|_\infty$. On the other hand, the expressions of $\varphi_{n,3}$ and $\varphi_{n,4}$ provided by theorem 3.2 allow to show that the remaining two terms are also $\mathcal{O}(n^{-2})$, see [**Hwa86**, page 151]. □

THEOREM 3.5 (Layers). *We have the following equality in distribution:*

$$(|\lambda_1(G)|,\dots,|\lambda_n(G)|) \overset{d}{=} (Z_{(1)},\dots,Z_{(n)})$$

where $Z_{(1)},\dots,Z_{(n)}$ *is the non-increasing reordering of a sequence* $Z_1,\dots,Z_n$ *of independent random variables with*[9] $Z_k^2 \sim \Gamma(k,1)$ *for every* $1 \le k \le n$.

SKETCH OF PROOF. Set $z_i = r_i e^{i\theta_i}$ (the i in front of θ_i means $\sqrt{-1}$), and write

$$|z_i - z_j|^2 = (z_i - z_j)\overline{(z_i - z_j)} = (r_i e^{i\theta_i} - r_j e^{i\theta_j})(r_i e^{-i\theta_i} - r_j e^{-i\theta_j}).$$

Following Kostlan [**Kos92**], this expansion allows to integrate the phases in the joint density of the spectrum given by theorem 3.1, providing the desired result. □

Note that[10] $(\sqrt{2}Z_k)^2 \sim \chi^2(2k)$. Since $Z_k^2 \overset{d}{=} E_1 + \cdots + E_k$ where $E_1,\dots,E_k$ are i.i.d. exponential random variables of unit mean, we get, for every $r > 0$,

$$\mathbb{P}\big(|\lambda_1(G)| \le \sqrt{n}r\big) = \prod_{1\le k\le n} \mathbb{P}\left(\frac{E_1 + \cdots + E_k}{n} \le r^2\right)$$

The law of large numbers suggests that $r = 1$ is a critical value. The central limit theorem suggests that $|\lambda_1(G)|$ behaves when $n \gg 1$ as the maximum of i.i.d. Gaussians, for which the fluctuations follow the Gumbel law. Indeed, a quantitative central limit theorem and the Borel-Cantelli lemma provides the follow result. The full proof is in Rider [**Rid03**]. This Gumbel fluctuations of the spectral radius $|\lambda_1(G)|$ differs from the Tracy-Widom fluctuations of the operator norm $s_1(G)$.

THEOREM 3.6 (Convergence and fluctuation of the spectral radius).

$$\mathbb{P}\Big(\lim_{n\to\infty} |\lambda_1(n^{-1/2}G)| = 1\Big) = 1.$$

Moreover, if $\gamma_n := \log(n/2\pi) - 2\log(\log(n))$ *then*

$$\sqrt{4n\gamma_n}\left(|\lambda_1(n^{-1/2}G)| - 1 - \sqrt{\frac{\gamma_n}{4n}}\right) \xrightarrow[n\to\infty]{d} \mathcal{G}$$

where $\mathcal{G}$ *is the Gumbel law with cumulative distribution function* $x \mapsto e^{-e^{-x}}$ *on* $\mathbb{R}$.

The circular law theorem for the Complex Ginibre Ensemble can be obtained from a large deviation principle. Indeed, setting $V(z) = |z|^2$, we have

$$e^{-n\sum_{i=1}^n V(z_i)} \prod_{i<j} |z_i - z_j|^2 = e^{-n^2\left(\frac{1}{n}\sum_{i=1}^n V(z_i) + \frac{2}{n^2}\sum_{i<j}\log|z_i - z_j|\right)}$$

we discover an empirical version of the logarithmic energy functional penalized by the "external" potential V. Indeed, it has been shown by Hiai and Petz [**PH98**] and by Ben Arous and Zeitouni [**BAZ98**] (see also [**Har12**]) that $\frac{1}{n}\sum_{i=1}^n \delta_{z_i}$ satisfies a large deviations principle at speed n^2 for the weak topology with rate function $\mu \mapsto \mathcal{I}(\mu) - \mathcal{I}(\mathcal{C}_1)$ where $\mathcal{I}$ is the logarithmic energy with external field V:

$$\mathcal{I}(\mu) = \iint \left(V(z) + V(\lambda) + \log\frac{1}{|\lambda - z|}\right) d\mu(z)d\mu(\lambda). \qquad (3.3)$$

This is coherent with the fact that the circular law $\mathcal{C}_1$ is the minimum of the logarithmic energy among the probability measures on $\mathbb{C}$ with fixed variance [**ST97**].

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[9] Here $\Gamma(a,\lambda)$ stands for the law on $\mathbb{R}_+$ with Lebesgue density $x \mapsto \lambda^a\Gamma(a)^{-1}x^{a-1}e^{-\lambda x}$.

[10] Here $\chi^2(n)$ stands for the law of $\|V\|_2^2$ where $V \sim \mathcal{N}(0, I_n)$.

The fluctuations around the circular law can be described by in various ways, see for instance [**RV07, RS06, Rid04**] and [**SS12**] and references therein. We end up this section by mentioning that Ginibre has considered also in his original paper [**Gin65**] the case where X_{11} is Gaussian on $\mathbb{R}$ or on the quaternions, and these two cases are more delicate, see [**BC12**] for additional comments and references.

4. Universal case

This section is devoted to the proof of the circular law theorem 2.2 following [**TV10b**]. The universal Marchenko-Pastur theorem 2.1 can be proved by using powerful Hermitian techniques such as truncation, centralization, the method of moments, or the Cauchy-Stieltjes trace-resolvent transform. It turns out that all these techniques fail for the eigenvalues of non-normal random matrices. Indeed, the key to prove the circular law theorem 2.2 is to use a bridge pulling back the problem to the Hermitian world. This is called *Hermitization*.

4.1. Logarithmic potential and Hermitization. Let $\mathcal{P}(\mathbb{C})$ be the set of probability measures on $\mathbb{C}$ which integrate $\log|\cdot|$ in a neighborhood of infinity. The *logarithmic potential* U_μ of $\mu \in \mathcal{P}(\mathbb{C})$ is the function $U_\mu : \mathbb{C} \to (-\infty, +\infty]$ defined for all $z \in \mathbb{C}$ by

$$U_\mu(z) = \int_{\mathbb{C}} \log \frac{1}{|z-\lambda|}\, d\mu(\lambda) = \left(\log \frac{1}{|\cdot|} * \mu\right)(z). \tag{4.1}$$

For the circular law $\mathcal{C}_1$ of density $\pi^{-1}\mathbf{1}_{\{z\in\mathbb{C}:|z|\leq 1\}}$, we have, for every $z \in \mathbb{C}$,

$$U_{\mathcal{C}_1}(z) = \begin{cases} -\log|z| & \text{if } |z| > 1, \\ \frac{1}{2}(1-|z|^2) & \text{if } |z| \leq 1, \end{cases} \tag{4.2}$$

see e.g. [**ST97**]. Let $\mathcal{D}'(\mathbb{C})$ be the set of Schwartz-Sobolev distributions. We have $\mathcal{P}(\mathbb{C}) \subset \mathcal{D}'(\mathbb{C})$. Since $\log|\cdot|$ is Lebesgue locally integrable on $\mathbb{C}$, the Fubini-Tonelli theorem implies that U_μ is Lebesgue locally integrable on $\mathbb{C}$. In particular, $U_\mu < \infty$ a.e. and $U_\mu \in \mathcal{D}'(\mathbb{C})$.

Let us define the first order linear differential operators in $\mathcal{D}'(\mathbb{C})$

$$\partial := \frac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \overline{\partial} := \frac{1}{2}(\partial_x + i\partial_y) \tag{4.3}$$

and the Laplace operator

$$\Delta = 4\partial\overline{\partial} = 4\overline{\partial}\partial = \partial_x^2 + \partial_y^2.$$

Each of these operators coincide on smooth functions with the usual differential operator acting on smooth functions. By using Green's or Stockes' theorems, one may show, for instance via the Cauchy-Pompeiu formula, that for any smooth and compactly supported function $\varphi : \mathbb{C} \to \mathbb{R}$,

$$-\int_{\mathbb{C}} \Delta\varphi(z) \log|z|\, dxdy = 2\pi\varphi(0) \tag{4.4}$$

where $z = x + iy$. Now (4.4) can be written, in $\mathcal{D}'(\mathbb{C})$,

$$\Delta \log|\cdot| = 2\pi\delta_0$$

In other words, $\frac{1}{2\pi}\log|\cdot|$ is the fundamental solution of the Laplace equation on $\mathbb{R}^2$. Note that $\log|\cdot|$ is harmonic on $\mathbb{C} \setminus \{0\}$. It follows that in $\mathcal{D}'(\mathbb{C})$,

$$\Delta U_\mu = -2\pi\mu, \tag{4.5}$$

i.e. for every smooth and compactly supported "test function" $\varphi:\mathbb{C}\to\mathbb{R}$,

$$-\langle \Delta U_\mu,\varphi\rangle_{\mathcal{D}'(\mathbb{C})}=\int_{\mathbb{C}}\Delta\varphi(z)U_\mu(z)\,dxdy=2\pi\int_{\mathbb{C}}\varphi(z)\,d\mu(z)=\langle 2\pi\mu,\varphi\rangle_{\mathcal{D}'(\mathbb{C})}\quad(4.6)$$

where $z=x+iy$, i.e. $-\frac{1}{2\pi}U_\cdot$ is the Green operator on $\mathbb{R}^2$ (Laplacian inverse).

Lemma 4.1 (Unicity). *For every $\mu,\nu\in\mathcal{P}(\mathbb{C})$, if $U_\mu=U_\nu$ a.e. then $\mu=\nu$.*

Proof. Since $U_\mu=U_\nu$ in $\mathcal{D}'(\mathbb{C})$, we get $\Delta U_\mu=\Delta U_\nu$ in $\mathcal{D}'(\mathbb{C})$. Now (4.5) gives $\mu=\nu$ in $\mathcal{D}'(\mathbb{C})$, and thus $\mu=\nu$ as measures since μ and ν are Radon measures. Note: the lemma remains valid if $U_\mu=U_\nu+h$ for some harmonic $h\in\mathcal{D}'(\mathbb{C})$. ☐

If $A\in\mathcal{M}_n(\mathbb{C})$ with characteristic polynomial $P_A(z):=\det(A-zI)$, then

$$U_{\mu_A}(z)=\int_{\mathbb{C}}\log\frac{1}{|\lambda-z|}\,d\mu_A(\lambda)=-\frac{1}{n}\log|\det(A-zI)|=-\frac{1}{n}\log|P_A(z)|$$

for every $z\in\mathbb{C}\setminus\{\lambda_1(A),\ldots,\lambda_n(A)\}$. We have also the alternative expression[11]

$$U_{\mu_A}(z)=-\frac{1}{n}\log\det(\sqrt{(A-zI)(A-zI)^*})=-\int_0^\infty\log(t)\,d\nu_{A-zI}(t).\quad(4.7)$$

One may retain from this determinantal Hermitization that for any $A\in\mathcal{M}_n(\mathbb{C})$,

knowledge of ν_{A-zI} for a.a. $z\in\mathbb{C}$ $\Rightarrow$ knowledge of μ_A

Note that from (4.5), for every smooth compactly supported function $\varphi:\mathbb{C}\to\mathbb{R}$,

$$2\pi\int\varphi\,d\mu_A=\int_{\mathbb{C}}(\Delta\varphi)\log|P_A|\,dxdy.$$

The identity (4.7) bridges the eigenvalues with the singular values, and is at the heart of the next lemma, which allows to deduce the convergence of μ_A from the one of ν_{A-zI}. The strength of this Hermitization lies in the fact that contrary to the eigenvalues, one can control the singular values with the entries of the matrix using powerful methods such as the method of moments or the trace-resolvent Cauchy-Stieltjes transform. The price paid here is the introduction of the auxiliary variable z. Moreover, we cannot simply deduce the convergence of the integral from the weak convergence of ν_{A-zI} since the logarithm is unbounded on $\mathbb{R}_+$. We circumvent this problem by requiring uniform integrability. We recall that on a Borel measurable space $(E,\mathcal{E})$, a Borel function $f:E\to\mathbb{R}$ is *uniformly integrable* for a sequence of probability measures $(\eta_n)_{n\geq1}$ on E when

$$\lim_{t\to\infty}\sup_{n\geq1}\int_{\{|f|>t\}}|f|\,d\eta_n=0.$$

We will use this property as follows: if $\eta_n\rightsquigarrow\eta$ as $n\to\infty$ for some probability measure η and if f is continuous and uniformly integrable for $(\eta_n)_{n\geq1}$ then f is η-integrable and

$$\lim_{n\to\infty}\int f\,d\eta_n=\int f\,d\eta.$$

The idea of using Hermitization goes back at least to Girko [**Gir84**]. However, the proofs of lemmas 4.2-4.3 are inspired from the work of Tao and Vu [**TV10b**].

[11] Girko uses the name "V-transform of μ_A", where V stands for "Victory".

LEMMA 4.2 (Hermitization). *Let $(A_n)_{n\geq 1}$ be a sequence of complex random matrices where A_n is $n\times n$ for every $n\geq 1$. Suppose that there exists a family of (non-random) probability measures $(\nu_z)_{z\in\mathbb{C}}$ on $\mathbb{R}_+$ such that, for a.a. $z\in\mathbb{C}$, a.s.*

(i) $\nu_{A_n-zI}\rightsquigarrow\nu_z$ *as* $n\to\infty$

(ii) $\log$ *is uniformly integrable for* $(\nu_{A_n-zI})_{n\geq 1}$.

Then there exists a probability measure $\mu\in\mathcal{P}(\mathbb{C})$ such that

(j) *a.s.* $\mu_{A_n}\rightsquigarrow\mu$ *as* $n\to\infty$

(jj) *for a.a.* $z\in\mathbb{C}$,

$$U_\mu(z)=-\int_0^\infty\log(s)\,d\nu_z(s).$$

PROOF OF LEMMA 4.2. We first observe that one can swap the quantifiers "a.a." on z and "a.s." on ω in front of (i-ii). Namely, let us call $P(z,\omega)$ the property "the function log is uniformly integrable for $\left(\nu_{A_n(\omega)-zI}\right)_{n\geq 1}$ and $\nu_{A_n(\omega)-zI}\rightsquigarrow\nu_z$". The assumptions of the lemma provide a measurable Lebesgue negligible set C in $\mathbb{C}$ such that for all $z\notin C$ there exists a probability one event E_z such that for all $\omega\in E_z$, the property $P(z,\omega)$ is true. From the Fubini-Tonelli theorem, this is *equivalent* to the existence of a probability one event E such that for all $\omega\in E$, there exists a Lebesgue negligible measurable set C_ω in $\mathbb{C}$ such that for all $z\notin C_\omega$, the property $P(z,\omega)$ is true.

From now on, we fix an arbitrary $\omega\in E$. For every $z\notin C_\omega$, we define the probability measure $\nu:=\nu_z$ and the triangular arrays $(a_{n,k})_{1\leq k\leq n}$, $(b_{n,k})_{1\leq k\leq n}$ by

$$a_{n,k}:=|\lambda_k(A_n(\omega)-zI)|\quad\text{and}\quad b_{n,k}:=s_k(A_n(\omega)-zI).$$

Note that $\mu_{A_n(\omega)-zI}=\mu_{A_n(\omega)}*\delta_{-z}$. Thanks to Weyl's inequalities (1.1) and to the assumptions (i-ii), one can use lemma 4.3 below, which gives that $(\mu_{A_n(\omega)})_{n\geq 1}$ is tight, that for a.a. $z\in\mathbb{C}$, $\log|z-\cdot|$ is uniformly integrable for $(\mu_{A_n(\omega)})_{n\geq 1}$, and

$$\lim_{n\to\infty}U_{\mu_{A_n(\omega)}}(z)=-\int_0^\infty\log(s)\,d\nu_z(s)=U(z).$$

Consequently, if the sequence $\left(\mu_{A_n(\omega)}\right)_{n\geq 1}$ admits two probability measures μ_ω and μ'_ω as accumulation points for the weak convergence, then both μ_ω and μ'_ω belong to $\mathcal{P}(\mathbb{C})$ and $U_{\mu_\omega}=U=U_{\mu'_\omega}$ a.e., which gives $\mu_\omega=\mu'_\omega$ thanks to lemma 4.1. Therefore, the sequence $\left(\mu_{A_n(\omega)}\right)_{n\geq 1}$ admits at most one accumulation point for the weak convergence. Since the sequence $\left(\mu_{A_n(\omega)}\right)_{n\geq 1}$ is tight, the Prohorov theorem implies that $\left(\mu_{A_n(\omega)}\right)_{n\geq 1}$ converges weakly to some probability measure $\mu_\omega\in\mathcal{P}(\mathbb{C})$ such that $U_{\mu_\omega}=U$ a.e. Since U is deterministic, it follows that $\omega\mapsto\mu_\omega$ is deterministic by lemma 4.1 again. □

The following lemma is the skeleton of lemma 4.2 (no matrices). It states essentially a propagation of a uniform logarithmic integrability for a couple of triangular arrays, provided that a logarithmic majorization holds between the arrays.

LEMMA 4.3 (Logarithmic majorization and uniform integrability). *Let $(a_{n,k})_{1\leq k\leq n}$, $(b_{n,k})_{1\leq k\leq n}$ be two triangular arrays in $\mathbb{R}_+$, and the discrete probability measures*

$$\mu_n:=\frac{1}{n}\sum_{k=1}^n\delta_{a_{n,k}}\quad\text{and}\quad\nu_n:=\frac{1}{n}\sum_{k=1}^n\delta_{b_{n,k}}.$$

If the following properties hold

(i) $a_{n,1} \geq \cdots \geq a_{n,n}$ *and* $b_{n,1} \geq \cdots \geq b_{n,n}$ *for* $n \gg 1$,
(ii) $\prod_{i=1}^{k} a_{n,i} \leq \prod_{i=1}^{k} b_{n,i}$ *for every* $1 \leq k \leq n$ *for* $n \gg 1$,
(iii) $\prod_{i=k}^{n} b_{n,i} \leq \prod_{i=k}^{n} a_{n,i}$ *for every* $1 \leq k \leq n$ *for* $n \gg 1$,
(iv) $\nu_n \rightsquigarrow \nu$ *as* $n \to \infty$ *for some probability measure* ν,
(v) $\log$ *is uniformly integrable for* $(\nu_n)_{n\geq 1}$,

then

(j) $\log$ *is uniformly integrable for* $(\mu_n)_{n\geq 1}$ *(in particular,* $(\mu_n)_{n\geq 1}$ *is tight),*
(jj) *we have, as* $n \to \infty$,

$$\int_0^\infty \log(t)\, d\mu_n(t) = \int_0^\infty \log(t)\, d\nu_n(t) \to \int_0^\infty \log(t)\, d\nu(t),$$

and in particular, for every accumulation point μ *of* $(\mu_n)_{n\geq 1}$,

$$\int_0^\infty \log(t)\, d\mu(t) = \int_0^\infty \log(t)\, d\nu(t).$$

PROOF. An elementary proof can be found in [**BCC11a**, Lemma C2]. Let us give an alternative argument. From the de la Vallée Poussin criterion (see e.g. [**DM78**, Theorem 22]), assumption (v) is equivalent to the existence of a non-decreasing convex function $J : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} J(t)/t = \infty$, and

$$\sup_n \int J(|\log(t)|)\, d\nu_n(t) < \infty.$$

On the other hand, assumption (i-ii-iii) implies that for every real valued function φ such that $t \mapsto \varphi(e^t)$ is non-decreasing and convex, we have, for every $1 \leq k \leq n$,

$$\sum_{i=1}^{k} \varphi(a_{n,k}) \leq \sum_{i=1}^{k} \varphi(b_{n,k}),$$

see [**HJ94**, Theorem 3.3.13]. Hence, applying this for $k = n$ and $\varphi = J$,

$$\sup_n \int J(|\log(t)|)\, d\mu_n(t) < \infty.$$

We obtain by this way (j). Statement (jj) follows trivially. □

REMARK 4.4 (Logarithmic potential and logarithmic energy). *The term "logarithmic potential" comes from the fact that* U_μ *is the electrostatic potential of* μ *viewed as a distribution of charged particles in the plane* $\mathbb{C} = \mathbb{R}^2$ [**ST97**]. *The so called logarithmic energy of this distribution of charged particles is*

$$\mathcal{E}(\mu) := \int_{\mathbb{C}} U_\mu(z)\, d\mu(z) = \int_{\mathbb{C}} \int_{\mathbb{C}} \log \frac{1}{|z-\lambda|}\, d\mu(z) d\mu(\lambda). \tag{4.8}$$

The circular law minimizes $\mathcal{E}(\cdot)$ *under a second moment constraint* [**ST97**]. *If* $\mathrm{supp}(\mu) \subset \mathbb{R}$ *then* $\mathcal{E}(\mu)$ *matches up to a sign and an additive constant the Voiculescu free entropy for one variable in free probability theory* [**Voi94**, Proposition 4.5]. *It appears also in the rate function of a natural large deviation principle* (3.3).

REMARK 4.5 (From converging potentials to weak convergence). *As for the Fourier transform, the pointwise convergence of logarithmic potentials along a sequence of probability measures implies the weak convergence of the sequence to a probability measure. We need however some strong tightness. More precisely, if* $(\mu_n)_{n\geq 1}$ *is a sequence in* $\mathcal{P}(\mathbb{C})$ *and if* $U : \mathbb{C} \to (-\infty, +\infty]$ *is such that*

(i) *for a.a.* $z \in \mathbb{C}$, $\lim_{n\to\infty} U_{\mu_n}(z) = U(z)$,

(ii) $\log(1+|\cdot|)$ *is uniformly integrable for* $(\mu_n)_{n\geq 1}$,

then there exists $\mu \in \mathcal{P}(\mathbb{C})$ *such that* $U_\mu = U$ *a.e. and* $\mu = -\frac{1}{2\pi}\Delta U$ *in* $\mathcal{D}'(\mathbb{C})$ *and*

$$\mu_n \rightsquigarrow \mu.$$

The proof is very similar to the proof of lemmas 4.2-4.3 and can be found in [**BC12**].

4.2. Proof of the circular law. The proof of theorem 2.2 is based on the Hermitization lemma 4.2. Assumption (i) of lemma 4.2 states that for all $z \in \mathbb{C}$, there exists a probability measure ν_z depending only on z such that a.s. $\nu_{n^{-1/2}X-zI} \rightsquigarrow \nu_z$ as $n \to \infty$. For $z = 0$ this is exactly the quarter circular Marchenko-Pastur theorem 2.1. Indeed, this assumption (i) can be proved using the Hermitian techniques used for theorem 2.1, namely truncation, centralization, the method of moments or the Cauchy-Stieltjes trace-resolvent transform, see [**DS07**] and [**BC12**]. It remains to check the uniform integrability assumption (ii) of lemma 4.2. From Markov's inequality, it suffices to show that for all $z \in \mathbb{C}$, there exists $p > 0$ such that a.s.

$$\varlimsup_{n\to\infty} \int s^{-p}\, d\nu_{n^{-1/2}X-zI}(s) < \infty \quad \text{and} \quad \varlimsup_{n\to\infty} \int s^{p}\, d\nu_{n^{-1/2}X-zI}(s) < \infty. \tag{4.9}$$

The second statement in (4.9) with $p \leq 2$ follows from the strong law of large numbers (2.1) together with (1.5), which gives $s_i(n^{-1/2}X - zI) \leq s_i(n^{-1/2}X) + |z|$ for all $1 \leq i \leq n$. The first statement in (4.9) concentrates most of the difficulty behind theorem 2.2. In the next two sub-sections, we will prove and comment the following couple of key lemmas taken from [**TV10b**] and [**TV08**] respectively.

Lemma 4.6 (Count of small singular values). *There exist* $c_0 > 0$ *and* $0 < \gamma < 1$ *such that a.s. for* $n \gg 1$ *and* $n^{1-\gamma} \leq i \leq n-1$ *and all* $M \in \mathcal{M}_n(\mathbb{C})$*, we have*

$$s_{n-i}(n^{-1/2}X + M) \geq c_0 \frac{i}{n}.$$

Lemma 4.6 is more meaningful when i is close to $n^{1-\gamma}$. For $i = n-1$, it gives only a lower bound on s_1. The linearity in i is what we may expect on spacing.

Lemma 4.7 (Lower bound on s_n). *For every* $d > 0$ *there exists* $b > 0$ *such that if* $M \in \mathcal{M}_n(\mathbb{C})$ *is deterministic with* $s_1(M) \leq n^d$ *then a.s. for* $n \gg 1$,

$$s_n(X + M) \geq n^{-b}.$$

Let us abridge $s_i(n^{-1/2}X - zI)$ into s_i. Applying lemmas 4.6-4.7 with $M = -zI$ and $M = -z\sqrt{n}I$ respectively, we get, for any $c > 0$, $z \in \mathbb{C}$, a.s. for $n \gg 1$,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} s_i^{-p} &\leq \frac{1}{n}\sum_{i=1}^{n-\lfloor n^{1-\gamma}\rfloor} s_i^{-p} + \frac{1}{n}\sum_{i=n-\lfloor n^{1-\gamma}\rfloor+1}^{n} s_i^{-p} \\ &\leq c_0^{-p}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n}{i}\right)^p + 2n^{-\gamma}n^{bp}.\end{aligned}$$

The first term of the right hand side is a Riemann sum for $\int_0^1 s^{-p}\,ds$ which converges as soon as $0 < p < 1$. We finally obtain the first statement in (4.9) as soon as $0 < p < \min(\gamma/b, 1)$. Now the Hermitization lemma 4.2 ensures that there exists a probability measure $\mu \in \mathcal{P}(\mathbb{C})$ such that a.s. $\mu_Y \rightsquigarrow \mu$ as $n \to \infty$ and for all $z \in \mathbb{C}$,

$$U_\mu(z) = -\int_0^\infty \log(s)\, d\nu_z(s).$$

Since ν_z does not depend on the law of X_{11} (we say that it is then universal), it follows that μ also does not depend on the law of X_{11}. It remains to show that $\int_0^\infty \log(s)\,d\nu_z(s)$ is equal to the logarithmic potential of $\mathcal{C}_1$ given by (4.2). This can be done either indirectly by using the circular law for the complex Gaussian case (theorem 3.4), or by a direct computation as in [**PZ10**, Lemma 3].

4.3. Count of small singular values. This sub-section is devoted to lemma 4.6 used in the proof of theorem 2.2 to check the uniform integrability assumption in lemma 4.2. Lemma 4.6 says that $\nu_{n^{-1/2}X-zI}([0,\eta]) \leq \eta/C$ for every $\eta \geq 2Cn^{-\gamma}$. In this form, we see that lemma 4.6 is an upper bound on the counting measure $n\nu_{n^{-1/2}X-zI}$ on a small interval $[0,\eta]$. This type of estimate (local Wegner estimates) has already been studied. Notably, an alternative proof of lemma 4.6 can be obtained following the work of [**ESY10**] on the resolvent of Wigner matrices.

Proof of lemma 4.6. We follow the proof of Tao and Vu [**TV10b**]. Up to increasing γ, it is enough to prove the statement for all $2n^{1-\gamma} \leq i \leq n-1$ for some $\gamma \in (0,1)$ to be chosen later. To lighten the notations, we denote by $s_1 \geq \cdots \geq s_n$ the singular values of $Y := n^{-1/2}X + M$. We fix $2n^{1-\gamma} \leq i \leq n-1$ and we consider the matrix Y' formed by the first $m := n - \lceil i/2 \rceil$ rows of $\sqrt{n}Y$. Let $s'_1 \geq \cdots \geq s'_m$ be the singular values of Y'. By the Cauchy-Poincaré interlacing[12], we get

$$n^{-1/2}s'_{n-i} \leq s_{n-i}$$

Next, by lemma 1.4 we obtain

$$s_1'^{-2} + \cdots + s_{n-\lceil i/2\rceil}'^{-2} = \mathrm{dist}_1^{-2} + \cdots + \mathrm{dist}_{n-\lceil i/2\rceil}^{-2},$$

where $\mathrm{dist}_j := \mathrm{dist}(R_j, H_j)$ is the distance from the j^{th} row R_j of Y' to H_j, the subspace spanned by the other rows of Y'. In particular, we have

$$\frac{i}{2n}s_{n-i}^{-2} \leq i s_{n-i}'^{-2} \leq \sum_{j=n-\lceil i\rceil}^{n-\lceil i/2\rceil} s_j'^{-2} \leq \sum_{j=1}^{n-\lceil i/2\rceil} \mathrm{dist}_j^{-2}. \tag{4.10}$$

Now H_j is independent of R_j and $\dim(H_j) \leq n - \frac{i}{2} \leq n - n^{1-\gamma}$, and thus, for the choice of γ given in the forthcoming lemma 4.8,

$$\sum_{n\gg 1} \mathbb{P}\left(\bigcup_{i=2n^{1-\gamma}}^{n-1} \bigcup_{j=1}^{n-\lceil i/2\rceil} \left\{\mathrm{dist}_j \leq \frac{\sqrt{i}}{2\sqrt{2}}\right\}\right) < \infty$$

(note that the exponential bound in lemma 4.8 kills the polynomial factor due to the union bound over i,j). Consequently, by the first Borel-Cantelli lemma, we obtain that a.s. for $n \gg 1$, all $2n^{1-\gamma} \leq i \leq n-1$, and all $1 \leq j \leq n - \lceil i/2 \rceil$,

$$\mathrm{dist}_j \geq \frac{\sqrt{i}}{2\sqrt{2}} \geq \frac{\sqrt{i}}{4}$$

Finally, (4.10) gives $s_{n-i}^2 \geq (i^2)/(32n^2)$, i.e. the desired result with $c_0 := 1/(4\sqrt{2})$. □

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[12]If $A \in \mathcal{M}_n(\mathbb{C})$ and $1 \leq m \leq n$ and if $B \in \mathcal{M}_{m,n}(\mathbb{C})$ is obtained from A by deleting $r := n-m$ rows, then $s_i(A) \geq s_i(B) \geq s_{i+r}(A)$ for every $1 \leq i \leq m$. In particular, $[s_m(B), s_1(B)] \subset [s_n(A), s_1(A)]$, i.e. the smallest singular value increases while the largest singular value is diminished. See [**HJ94**, Corollary 3.1.3]

Lemma 4.8 (Distance of a random vector to a subspace). *There exist $\gamma > 0$ and $\delta > 0$ such that for all $n \gg 1$, $1 \le i \le n$, any deterministic vector $v \in \mathbb{C}^n$ and any subspace H of $\mathbb{C}^n$ with $1 \le \dim(H) \le n - n^{1-\gamma}$, we have, denoting $R := (X_{i1}, \dots, X_{in}) + v$,*

$$\mathbb{P}\left(\operatorname{dist}(R, H) \le \frac{1}{2}\sqrt{n - \dim(H)}\right) \le \exp(-n^{\delta}).$$

The exponential bound above is not optimal, but is more than enough for our purposes: in the proof of lemma 4.6, a large enough polynomial bound suffices.

Proof. The argument is due to Tao and Vu [**TV10b**, Proposition 5.1]. We first note that if H' is the vector space spanned by H, v and $\mathbb{E}R$, then

$$\dim(H') \le \dim(H) + 2 \quad \text{and} \quad \operatorname{dist}(R, H) \ge \operatorname{dist}(R, H') = \operatorname{dist}(R', H'),$$

where $R' := R - \mathbb{E}(R)$. We may thus directly suppose without loss of generality that $v = 0$ and $\mathbb{E}(X_{ik}) = 0$. Then, it is easy to check that (see computation below)

$$\mathbb{E}(\operatorname{dist}(R, H)^2) = n - \dim(H).$$

The lemma is thus a statement on the deviation probability of $\operatorname{dist}(R, H)$. We first perform a truncation. Let $0 < \varepsilon < 1/3$. Markov's inequality gives

$$\mathbb{P}(|X_{ik}| \ge n^{\varepsilon}) \le n^{-2\varepsilon}.$$

Hence, from Hoeffding's deviation inequality[13], for $n \gg 1$,

$$\mathbb{P}\left(\sum_{k=1}^{n} \mathbf{1}_{\{|X_{ik}| \le n^{\varepsilon}\}} < n - n^{1-\varepsilon}\right) \le \exp(-2n^{1-2\varepsilon}(1 - n^{-\varepsilon})^2) \le \exp(-n^{1-2\varepsilon}).$$

It is thus sufficient to prove that the result holds by conditioning on

$$E_m := \{|X_{i1}| \le n^{\varepsilon}, \dots, |X_{im}| \le n^{\varepsilon}\} \quad \text{with} \quad m := \lceil n - n^{1-\varepsilon} \rceil.$$

Let $\mathbb{E}_m[\,\cdot\,] := \mathbb{E}[\,\cdot\,|E_m; \mathcal{F}_m]$ denote the conditional expectation given E_m and the filtration $\mathcal{F}_m$ generated by $X_{i,m+1}, \dots, X_{i,n}$. Let W be the subspace spanned by

$$H, \quad u = (0, \dots, 0, X_{i,m+1}, \dots, X_{i,n}), \quad w = (\mathbb{E}_m[X_{i1}], \dots, \mathbb{E}_m[X_{im}], 0, \dots, 0).$$

Then, by construction $\dim(W) \le \dim(H) + 2$ and W is $\mathcal{F}_m$-measurable. If we set $Y = (X_{i1} - \lambda, \dots, X_{im} - \lambda, 0, \dots, 0) = R - u - w$ and $\lambda = \mathbb{E}_m[X_{i1}]$ then

$$\operatorname{dist}(R, H) \ge \operatorname{dist}(R, W) = \operatorname{dist}(Y, W).$$

Next we have

$$\sigma^2 := \mathbb{E}_m\big[Y_1^2\big] = \mathbb{E}\Big[\big(X_{i1} - \mathbb{E}\big[X_{i1} \bigm| |X_{i1}| \le n^{\varepsilon}\big]\big)^2 \Bigm| |X_{i1}| \le n^{\varepsilon}\Big] = 1 - o(1).$$

Now, let us consider the disc $D := \{z \in \mathbb{C} : |z| \le n^{\varepsilon}\}$ and the convex function $f : D^m \to \mathbb{R}_+$ defined by $f(x) = \operatorname{dist}((x, 0, \dots, 0), W)$, which is also 1-Lipschitz: $|f(x) - f(x')| \le \operatorname{dist}(x, x')$. Talagrand's concentration inequality[14] gives then

$$\mathbb{P}_m(|\operatorname{dist}(Y, W) - M_m| \ge t) \le 4\exp\left(-\frac{t^2}{16n^{2\varepsilon}}\right), \tag{4.11}$$

[13] If $X_1, \dots, X_n$ are independent and bounded real r.v. and if $S_n := X_1 + \cdots + X_n$ then $\mathbb{P}(S_n - \mathbb{E}S_n \le tn) \le \exp(-2n^2t^2/(d_1^2 + \cdots + d_n^2))$ for any $t \ge 0$, where $d_i := \max(X_i) - \min(X_i)$. See [**McD89**, Theorem 5.7].

[14] If $X_1, \dots, X_n$ are i.i.d. r.v. on $D := \{z \in \mathbb{C} : |z| \le r\}$ and if $f : D^n \to \mathbb{R}$ is convex, 1-Lipschitz, with median M, then $\mathbb{P}(|f(X_1, \dots, X_n) - M| \ge t) \le 4\exp(-\frac{t^2}{16r^2})$ for any $t \ge 0$. See [**Tal95**] and [**Led01**, Corollary 4.9].

where M_m is the median of $\mathrm{dist}(Y,W)$ under $\mathbb{E}_m$. In particular,

$$M_m \geq \sqrt{\mathbb{E}_m \mathrm{dist}^2(Y,W)} - cn^{\varepsilon}.$$

Also, if P denotes the orthogonal projection on the orthogonal of W, we find

$$\begin{aligned}\mathbb{E}_m \mathrm{dist}^2(Y,W) &= \sum_{k=1}^{m} \mathbb{E}_m\left[Y_k^2\right] P_{kk} \\ &= \sigma^2\left(\sum_{k=1}^{n} P_{kk} - \sum_{k=m+1}^{n} P_{kk}\right) \\ &\geq \sigma^2(n - \dim(W) - (n-m)) \\ &\geq \sigma^2\big(n - \dim(H) - n^{1-\varepsilon} - 2\big)\end{aligned}$$

Pick some $0 < \gamma < \varepsilon$. Then, from the above expression for any $1/2 < c < 1$ and $n \gg 1$, $M_m \geq c\sqrt{n - \dim(H)}$. We set $t = (c - 1/2)\sqrt{n - \dim(H)}$ in (4.11). $\square$

4.4. Smallest singular value. This sub-section is devoted to lemma 4.7 which was used in the proof of theorem 2.2 to get the uniform integrability in lemma 4.2. The full proof of lemma 4.7 by Tao and Vu in [**TV08**] is based on Littlewood-Offord type problems. The main difficulty is the possible presence of atoms in the law of the entries (in this case X is non-invertible with positive probability). Regarding the assumptions, the finite second moment hypothesis on X_{11} is not crucial and can be considerably weakened. For the sake of simplicity, we give here a simplified proof when the law of X_{11} has a bounded density on $\mathbb{C}$ or on $\mathbb{R}$ (which implies that $X + M$ is invertible with probability one).

Proof of lemma 4.7. By the first Borel-Cantelli lemma, it suffices to show that for every $a, d > 0$ (actually $a > 1$ is enough), there exists $b > 0$ such that if $M \in \mathcal{M}_n(\mathbb{C})$ is deterministic with $s_1(M) \leq n^d$ then

$$\mathbb{P}(s_n(X+M) \leq n^{-b}) \leq n^{-a}.$$

As already mentioned, we will only prove this when X_{11} has a bounded density. For every $x, y \in \mathbb{C}^n$ and $S \subset \mathbb{C}^n$, we set $x \cdot y := x_1\overline{y_1} + \cdots + x_n\overline{y_n}$ and $\|x\|_2 := \sqrt{x \cdot x}$ and $\mathrm{dist}(x,S) := \min_{y \in S} \|x - y\|_2$. Let $R_1, \ldots, R_n$ be the rows of $X + M$ and set

$$R_{-i} := \mathrm{span}\{R_j; j \neq i\}$$

for every $1 \leq i \leq n$. The lower bound in lemma 1.3 gives

$$\min_{1 \leq i \leq n} \mathrm{dist}(R_i, R_{-i}) \leq \sqrt{n}\, s_n(X+M)$$

and consequently, by the union bound, for any $u \geq 0$,

$$\mathbb{P}(\sqrt{n}\, s_n(X+M) \leq u) \leq n \max_{1 \leq i \leq n} \mathbb{P}(\mathrm{dist}(R_i, R_{-i}) \leq u).$$

Let us fix $1 \leq i \leq n$. Let Y_i be a unit vector orthogonal to R_{-i}. Such a vector is not unique, but we may just pick one which is independent of R_i. This defines a random variable on the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{C}^n : \|x\|_2 = 1\}$. By the Cauchy-Schwarz inequality,

$$|R_i \cdot Y_i| \leq \|\pi_i(R_i)\|_2 \|Y_i\|_2 = \mathrm{dist}(R_i, R_{-i})$$

where π_i is the orthogonal projection on the orthogonal complement of R_{-i}. Let ν_i be the distribution of Y_i on $\mathbb{S}^{n-1}$. Since Y_i and R_i are independent, for any $u \geq 0$,

$$\mathbb{P}(\mathrm{dist}(R_i, R_{-i}) \leq u) \leq \mathbb{P}(|R_i \cdot Y_i| \leq u) = \int_{\mathbb{S}^{n-1}} \mathbb{P}(|R_i \cdot y| \leq u)\, d\nu_i(y).$$

Let us assume that X_{11} has a bounded density φ on $\mathbb{C}$. Since $\|y\|_2 = 1$ there exists an index $j_0 \in \{1, \dots, n\}$ such that $y_{j_0} \neq 0$ with $|y_{j_0}|^{-1} \leq \sqrt{n}$. The complex random variable $R_i \cdot y$ is a sum of independent complex random variables and one of them is $X_{ij_0}\,\overline{y_{j_0}}$, which is absolutely continuous with a density bounded above by $\sqrt{n}\,\|\varphi\|_\infty$. Consequently, by a basic property of convolutions of probability measures, the complex random variable $R_i \cdot y$ is also absolutely continuous with a density φ_i bounded above by $\sqrt{n}\,\|\varphi\|_\infty$, and thus

$$\mathbb{P}(|R_i \cdot y| \leq u) = \int_{\mathbb{C}} \mathbf{1}_{\{|s| \leq u\}} \varphi_i(s)\, ds \leq \pi u^2 \sqrt{n}\,\|\varphi\|_\infty.$$

Therefore, for every $b > 0$, we obtain the desired result

$$\mathbb{P}(s_n(X + M) \leq n^{-b-1/2}) = \mathcal{O}(n^{3/2-2b}).$$

This scheme remains valid in the case where X_{11} has a bounded density on $\mathbb{R}$. □

5. Comments

Weak convergence in probability. For simplicity, the main mode of convergence considered in these notes for the empirical spectral distributions of random matrices is the a.s. weak convergence. It is often useful to consider another mode of convergence, which is the weak convergence in probability. The Hermitization lemma 4.2 is also available for this mode of convergence. The details are in [**BC12**].

Replacement principle. A variant of the Hermitization lemma 4.2, known as the "replacement principle" states that if $(A_n)_{n\geq1}$ and $(B_n)_{n\geq1}$ are sequences where A_n and B_n are random variables in $\mathcal{M}_n(\mathbb{C})$ and such that for a.a. $z \in \mathbb{C}$ a.s.

(k) $\lim_{n\to\infty} U_{\mu_{A_n}}(z) - U_{\mu_{B_n}}(z) = 0$

(kk) $\log(1 + \cdot)$ is uniformly integrable for $(\nu_{A_n})_{n\geq1}$ and $(\nu_{B_n})_{n\geq1}$

then a.s. $\mu_{A_n} - \mu_{B_n} \rightsquigarrow 0$ as $n \to \infty$. The details are in [**TV10b**, Theorem 2.1]. This replacement principle is the key to obtain a universality principle going beyond the circular law. Namely, following [**TV10b**] and [**Bor11**], if X and G are the random matrices considered in sections 3-4 obtained from infinite tables with i.i.d. entries, and if $(M_n)_{n\geq1}$ is a deterministic sequence such that $M_n \in \mathcal{M}_n(\mathbb{C})$ and

$$\varlimsup_{n\to\infty} \int s^p\, d\nu_{M_n}(s) < \infty$$

for some $p > 0$, then a.s. $\mu_{n^{-1/2}X+M_n} - \mu_{n^{-1/2}G+M_n} \rightsquigarrow 0$ as $n \to \infty$.

Logarithmic potential and Cauchy-Stieltjes transform. The Cauchy-Stieltjes transform $m_\mu : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ of a probability measure μ on $\mathbb{C}$ is

$$m_\mu(z) := \int_{\mathbb{C}} \frac{1}{\lambda - z}\, d\mu(\lambda).$$

Since $1/|\cdot|$ is Lebesgue locally integrable on $\mathbb{C}$, the Fubini-Tonelli theorem implies that $m_\mu(z)$ is finite for a.a. $z \in \mathbb{C}$, and moreover m_μ is locally Lebesgue integrable on $\mathbb{C}$ and thus belongs to $\mathcal{D}'(\mathbb{C})$. Note that for a matrix $A \in \mathcal{M}_n(\mathbb{C})$, we have $m_{\mu_A}(z) = \frac{1}{n}\mathrm{Tr}((A - zI)^{-1})$, which is the normalized trace of the resolvent of A

at point z, finite outside the spectrum of A. Suppose now that $\mu \in \mathcal{P}(\mathbb{C})$. The logarithmic potential is related to the Cauchy-Stieltjes transform via the identity

$$m_\mu = 2\partial U_\mu$$

in $\mathcal{D}'(\mathbb{C})$. In particular, since $4\partial\overline{\partial} = 4\overline{\partial}\partial = \Delta$ in $\mathcal{D}'(\mathbb{C})$, we obtain, still in $\mathcal{D}'(\mathbb{C})$,

$$2\overline{\partial} m_\mu = -\Delta U_\mu = -2\pi\mu.$$

Thus we can recover μ from m_μ. Note that for any $\varepsilon > 0$, m_μ is bounded on

$$D_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, \operatorname{supp}(\mu)) > \varepsilon\}.$$

If $\operatorname{supp}(\mu)$ is one-dimensional then one may completely recover μ from the knowledge of m_μ on D_ε as $\varepsilon \to 0$. Note also that m_μ is analytic outside $\operatorname{supp}(\mu)$, and is thus characterized by its real part or its imaginary part on arbitrary small balls in the connected components of $\operatorname{supp}(\mu)^c$. If $\operatorname{supp}(\mu)$ is not one-dimensional then one needs the knowledge of m_μ inside the support to recover μ. An elegant solution to this problem is to define a quaternionic Cauchy-Stieltjes transform, replacing the complex variable z by a quaternionic variable h. This idea appears in various works such as [**FZ97a, GNJNP07, RC09, Rog10, BCC11b**]. The quaternionic Cauchy-Stieltjes transform is a powerful tool well suited for non-Hermitian random matrices, and which may replace completely the logarithmic potential, see [**BC12**].

Free probability, Brown spectral measure, and $\star$-moments. The circular law has an interpretation in free probability theory, a sub-domain of operator algebra theory connected to random matrices, see the books by Voiculescu, Dykema and Nica [**VDN92**] and by Anderson, Guionnet, and Zeitouni [**AGZ10**].

Let $\mathcal{M}$ be an algebra of bounded operators on a Hilbert space H, with unit 1, stable by the adjoint operation $*$. Let $\tau : \mathcal{M} \to \mathbb{C}$ be a linear map such that $\tau(1) = 1$ and $\tau(aa^*) = \tau(a^*a) \geq 0$. For $a \in \mathcal{M}$, define $|a| = \sqrt{aa^*}$. If $b \in \mathcal{M}$ is self-adjoint, i.e. $b^* = b$, the spectral measure μ_b of b is the unique probability measure on the real line satisfying, for any integer $k \in \mathbb{N}$,

$$\tau(b^k) = \int t^k d\mu_b(t).$$

Brown spectral measure. For any $a \in \mathcal{M}$ we define $\nu_a = \mu_{|a|}$, which is a probability measure on $\mathbb{R}_+$. In the spirit of (4.7), the Brown spectral measure [**Bro86**] of $a \in \mathcal{M}$ is the unique probability measure μ_a on $\mathbb{C}$, which satisfies for a.a. $z \in \mathbb{C}$,

$$\int \log|z - \lambda|\, d\mu_a(\lambda) = \int \log(s)\, d\nu_{a-z}(s).$$

In distribution, it is given by the formula[15]

$$\mu_a = \frac{1}{2\pi}\Delta \int \log(s)\, d\nu_{a-z}(s). \tag{5.1}$$

The fact that the above definition is indeed a probability measure requires a proof, which can be found in [**HS07**]. Our notation is consistent: first, if a is self-adjoint, then the Brown spectral measure coincides with the spectral measure. Secondly, if $\mathcal{M} = \mathcal{M}_n(\mathbb{C})$ and $\tau := \frac{1}{n}\mathrm{Tr}$ is the normalized trace on $\mathcal{M}_n(\mathbb{C})$, then we retrieve our usual definition for ν_A and μ_A. It is interesting to point out that the identity (5.1) which is a consequence of the definition of the eigenvalues when $\mathcal{M} = \mathcal{M}_n(\mathbb{C})$ serves

[15] The so-called Fuglede-Kadison determinant of a is $\exp \int \log(t)\, d\mu_{|a|}(t)$, see [**FK52**].

as a definition in the general setting of operator algebras. All these definitions can be extended beyond bounded operators, see Haagerup and Schultz [**HS07**].

Failure of the method of moments. For non-Hermitian matrices, the spectrum does not necessarily belong to the real line, and in general, the limiting spectral distribution is not supported in the real line. The problem here is that the moments are not enough to characterize laws on $\mathbb{C}$. For instance, if Z is a complex random variable following the uniform law $\mathcal{C}_\kappa$ on the centered disc $\{z \in \mathbb{C}; |z| \leq \kappa\}$ of radius κ then for every $r \geq 0$, $\mathbb{E}(Z^r) = 0$ and thus $\mathcal{C}_\kappa$ is not characterized by its moments. Any rotational invariant law on $\mathbb{C}$ with light tails shares with $\mathcal{C}_\kappa$ the same sequence of null moments. One can try to circumvent the problem by using "mixed moments" which uniquely determine μ by the Weierstrass theorem. Namely, for every $A \in \mathcal{M}_n(\mathbb{C})$, if $A = UTU^*$ is the Schur unitary triangularization of A then for every integers $r, r' \geq 0$ and with $z = x + iy$ and $\tau = \frac{1}{n}\mathrm{Tr}$,

$$\int_{\mathbb{C}} z^r \overline{z}^{r'}\, d\mu_A(z) = \frac{1}{n}\sum_{i=1}^n \lambda_i^r(A)\overline{\lambda_i(A)}^{r'} = \tau(T^r \overline{T}^{r'}) \neq \tau(T^r T^{*r'}) = \tau(A^r A^{*r'}).$$

Indeed equality holds true when $\overline{T} = T^*$, i.e. when T is diagonal, i.e. when A is normal. This explains why the method of moments looses its strength for non-normal operators. To circumvent the problem, one may think about using the notion of $\star$-moments. Note that if A is normal then for every word $A^{\varepsilon_1}\cdots A^{\varepsilon_k}$ where $\varepsilon_1, \ldots, \varepsilon_n \in \{1, *\}$, we have $\tau(A^{\varepsilon_1}\cdots A^{\varepsilon_k}) = \tau(A^{k_1}A^{*k_2})$ where k_1, k_2 are the number of occurrence of A and A^*.

$\star$-distribution. The $\star$-distribution of $a \in \mathcal{M}$ is the collection of all its $\star$-moments:

$$\tau(a^{\varepsilon_1}a^{\varepsilon_2}\cdots a^{\varepsilon_n}),$$

where $n \geq 1$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{1, *\}$. The element $c \in \mathcal{M}$ is *circular* when it has the $\star$-distribution of $(s_1 + is_2)/\sqrt{2}$ where s_1 and s_2 are free semi circular variables with spectral measure of Lebesgue density $x \mapsto \frac{1}{\pi}\sqrt{4 - x^2}\mathbf{1}_{[-2,2]}(x)$.

The $\star$-distribution of $a \in \mathcal{M}$ allows to recover the moments of $|a - z|^2 = (a - z)(a - z)^*$ for all $z \in \mathbb{C}$, and thus ν_{a-z} for all $z \in \mathbb{C}$, and thus the Brown measure μ_a of a. Actually, for a random matrix, the $\star$-distribution contains, in addition to the spectral measure, an information on the eigenvectors of the matrix.

We say that a sequence of matrices $(A_n)_{n\geq 1}$ where A takes it values in $\mathcal{M}_n(\mathbb{C})$ converges in $\star$-moments to $a \in \mathcal{M}$, if all $\star$-moments converge to the $\star$-moments of $a \in \mathcal{M}$. For example, if $G \in \mathcal{M}_n(\mathbb{C})$ is our complex Ginibre matrix, then a.s. as $n \to \infty$, $n^{-1/2}G$ converges in $\star$-moments to a circular element.

Discontinuity of the Brown measure. Due to the unboundedness of the logarithm, the Brown measure μ_a depends discontinuously on the $\star$-moments of a [**BL01, Śni02**]. The limiting measures are perturbations by "balayage". A simple counter example is given by the matrices of example 1.2. For random matrices, this discontinuity is circumvented in the Girko Hermitization by requiring a uniform integrability, which turns out to be a.s. satisfied the random matrices $n^{-1/2}X$ in the circular law theorem 2.2.

However, Śniady [**Śni02**, Theorem 4.1] has shown that it is always possible to regularize the Brown measure by adding an additive noise. More precisely, if G is as above and $(A_n)_{n\geq 1}$ is a sequence of matrices where A_n takes its values in $\mathcal{M}_n(\mathbb{C})$, and if the $\star$-moments of A_n converge to the $\star$-moments of $a \in \mathcal{M}$ as $n \to \infty$, then a.s. $n \to \infty$ $\mu_{A_n + tn^{-1/2}G}$ converges to μ_{a+tc}, c is circular element free of a. In particular, by choosing a sequence t_n going to 0 sufficiently slowly, it is possible to regularize the Brown measure: a.s. $\mu_{A_n+t_n n^{-1/2}G}$ converges to μ_a. The Śniady theorem was revisited recently by Guionnet, Wood, and Zeitouni [**GWZ11**].

Outliers. The circular law theorem 2.2 allows the blow up of an arbitrary (asymptotically negligible) fraction of the extremal eigenvalues. Indeed, it was shown by Silverstein [**Sil94**] that if $\mathbb{E}(|X_{11}|^4) < \infty$ and $\mathbb{E}(X_{11}) \neq 0$ then the spectral radius $|\lambda_1(n^{-1/2}X)|$ tends to infinity at speed $\sqrt{n}$ and has a Gaussian fluctuation. This observation of Silverstein is the base of [**Cha10**], see also the ideas of Andrew [**And90**]. More recently, Tao studied in [**Tao11**] the outliers produced by various types of perturbations including general additive perturbations.

Sum and products. The scheme of proof of theorem 2.2 (based on Hermitization, logarithmic potential, and uniform integrability) turns out to be quite robust. It allows for instance to study the limit of the empirical distribution of the eigenvalues of sums and products of random matrices, see [**Bor11**], and also [**GT10b**] in relation with Fuss-Catalan laws. We may also mention [**OS10**]. The crucial step lies in the control of the small singular values.

Cauchy and the sphere. It is well known that the ratio of two independent standard real Gaussian variables is a Cauchy random variable, which has heavy tails. The complex analogue of this phenomenon leads to a complex Cauchy random variable, which is also the image law by the stereographical projection of the uniform law on the sphere. The matrix analogue consists in starting from two independent copies G_1 and G_2 of the Complex Ginibre Ensemble, and to consider the random matrix $Y = G_1^{-1}G_2$. The limit of μ_Y was analyzed by Forrester and Krishnapur [**FK09**]. Note that Y does not have i.i.d. entries.

Random circulant matrices. The eigenvalues of a non-Hermitian circulant matrix are linear functionals of the matrix entries. Meckes [**Mec09**] used this fact together with the central limit theorem in order to show that if the entries are i.i.d. with finite positive variance then the scaled empirical spectral distribution of the eigenvalues tends to a Gaussian law. We can imagine a heavy tailed version of this phenomenon with α-stable limiting laws.

Dependent entries. According to Girko, the circular law theorem 2.2 remains valid for random matrices with independent rows provided some natural hypotheses [**Gir01**]. Indeed, a circular law theorem is available for random Markov matrices including the Dirichlet Markov Ensemble [**BCC11a**], for random matrices with i.i.d. log-concave isotropic rows [**Ada11**], for random doubly stochastic matrices [**Ngu12**], for random matrices with given rows sums [**NV12**], and for random matrices with projected rows [**Tao11**]. Another Markovian model consists in a non-Hermitian random Markov generator with i.i.d. off-diagonal entries, which gives rise to new limiting spectral distributions, possibly not rotationally invariant, which can be interpreted using free probability theory [**BCC12**]. All these models can be analyzed using the Hermitization lemma 4.2. Another kind of dependence comes from truncation of random matrices with depend entries such as Haar unitary matrices. Namely, let U be distributed according to the uniform law on the unitary group $\mathbb{U}_n$ (we say that U is Haar unitary). Dong, Jiang, and Li have shown in [**DJL11**] that the empirical spectral distribution of the diagonal sub-matrix $(U_{ij})_{1\leq i,j\leq m}$ tends to the circular law if $m/n \to 0$, while it tends to the arc law (uniform law on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$) if $m/n \to 1$. Other results of the same flavor can be found in [**Jia09**]. Yet another way to add some dependence consists in considering an infinite array $(X_{ij}, X_{ji})_{1 \leq i < j \leq n}$ of i.i.d. pairs of complex random variables, independent of $(X_{ii})_{i \geq 1}$ an i.i.d. sequence of random variables, and we assume that $\mathrm{Var}(X_{12}) = \mathrm{Var}(X_{21}) = 1$ and $\mathrm{Cor}(X_{12}, X_{21}) = t \in \{z \in \mathbb{C} : |z| \leq 1\}$. It was then shown recently by Naumov [**Nau12**] and by Nguyen and O'Rourke [**NO12**] that $\mu_{n^{-1/2}X}$ converges to a universal limit computed by Girko [**Gir90**] and known as the elliptic law. This model interpolates between Hermitian and non-Hermitian random matrices, and its Gaussian version is well known [**Ben10, Led08, Joh07, KS11**].

Tridiagonal matrices. The limiting spectral distributions of random tridiagonal Hermitian matrices with i.i.d. entries are not universal and depend on the law of the entries, see [**Pop09**] for an approach based on the method of moments. The non-Hermitian version of this model was studied by Goldsheid and Khoruzhenko [**GK05**] by using the logarithmic potential. Indeed, the tridiagonal structure produces a three terms recursion on characteristic polynomials which can be written as a product of random 2×2 matrices, leading to the usage of a multiplicative ergodic theorem to show the convergence of the logarithmic potential (which appears as a Lyapunov exponent). The approach relies directly on remark 4.5, and in particular, neither the Hermitization nor the control the smallest and small singular values are needed here. Despite this apparent simplicity, the structure of the limiting distributions may be incredibly complicated and mathematically mysterious, as shown on the Bernoulli case by the physicists Holz, Orland, and Zee [**HOZ03**].

Single ring theorem. Let $D \in \mathcal{M}_n(\mathbb{R}_+)$ be a random diagonal matrix and $U, V \in \mathcal{M}_n(\mathbb{C})$ be two independent Haar unitary matrices, independent of D. The law of $X := UDV^*$ is unitary invariant by construction, and $\nu_X = \mu_D$ (it is a random SVD). Assume that μ_D tends to some limiting law ν as $n \to \infty$. It was conjectured by Feinberg and Zee [**FZ97b**] that μ_X tends to a limiting law which is supported in a centered ring of the complex plane, i.e. a set of the form $\{z \in \mathbb{C} : r \leq |z| \leq R\}$. Under some additional assumptions, this was proved by Guionnet, Krishnapur, and Zeitouni [**GKZ09**] by using the Hermitization technique and specific aspects such as the Schwinger-Dyson non-commutative integration by parts. Guionnet and Zeitouni have also obtained the convergence of the support in a more recent work [**GZ10**]. The Complex Ginibre Ensemble is a special case of this unitary invariant model. Khoruzhenko discovered a new and relatively simple model (quadratized rectangular Ginibre matrix) which gives rise to a single ring.

Roots of random polynomials. The random matrix X has i.i.d. entries and its eigenvalues are the roots of its characteristic polynomial. The coefficients of this random polynomial are neither independent nor identically distributed. Beyond random matrices, one may consider a random polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ where $a_0, \ldots, a_n$ are independent random variables. By analogy with random matrices, one may ask about the behavior as $n \to \infty$ of the roots $\lambda_1(P), \ldots, \lambda_n(P)$ of P in $\mathbb{C}$ and in particular the behavior of their empirical measure $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(P)}$. The literature on this subject is quite rich and takes its roots in the works of Littlewood and Offord, Rice, and Kac. Several references are given in [**BC12**]. Geometrically, the complex number z is a root of P if and only if the vectors $(1, z, \ldots, z^n)$ and $(\overline{a_0}, \overline{a_1}, \ldots, \overline{a_n})$ are orthogonal in $\mathbb{C}^{n+1}$, and this connects the problem to Littlewood-Offord type problems and small balls probabilities. As for random matrices, the case where the coefficients are real is more subtle due to the presence of real roots. Regarding the complex case, the zeros of Gaussian analytic functions is the subject of a recent monograph [**HKPV09**] in connection with determinantal processes. Various cases are considered in the literature, including Kac polynomials for which $(a_i)_{0\leq i\leq n}$ are i.i.d. and Weyl polynomials for which $a_i = \frac{1}{\sqrt{i!}}b_i$ for all i and $(b_i)_{0\leq i\leq n}$ are i.i.d. The recent works [**KZ12**] makes use of the logarithmic potential in order to study to behavior of the empirical measure of the roots. In particular, Weyl's random polynomials give rise to a circular law.

Sparsity. The circular law theorem 2.2 may remain valid if one allows the entries law to depend on n. This extension contains for instance sparse models in which the law has an atom at 0 with mass $p_n \to 1$ at a certain speed, see [**GT10a, TV08, Woo11**].

Heavy tails. It is quite natural to ask about the analogues of the quarter circular and circular law theorems 2.1-2.2 when X_{11} has an infinite variance (and thus heavy tails), say $\lim_{t\to\infty} t^\alpha \mathbb{P}(|X_{11}| \geq t) = 1$ for some $0 < \alpha < 2$. In this case, and following [**BAG08, BDG09, BCC11b**], there exists an absolutely continuous probability measure ν_α on $\mathbb{R}_+$ such that a.s. $\nu_{n^{-1/\alpha}X} \rightsquigarrow \nu_\alpha$ as $n \to \infty$. This probability measure ν_α depends only on α and inherits the tail behavior of the entries, namely $\lim_{t\to\infty} t^\alpha \nu_\alpha([t,\infty)) = 1$. It can also be proved that ν_α converges weakly to the quarter circular law as α converges to 2. Regarding the eigenvalues, and following [**BCC11b, BC12**], there exists an absolutely continuous probability measure μ_α on $\mathbb{C}$ such that $\mu_{n^{-1/\alpha}X} \rightsquigarrow \mu_\alpha$ as $n \to \infty$. The law μ_α depends only on α, is isotropic, and converges weakly to the circular law as α converges to 2. As $r \to \infty$, the radial tail behavior of its density is up to a multiplicative constant equivalent to $r^{2(\alpha-1)}e^{-\frac{\alpha}{2}r^\alpha}$. This exponential decay is quite surprising and contrasts with the power tail behavior of ν_α. It indicates that X is typically far from being a normal matrix. Also, we see that the eigenvalues limit spectrum is more concentrated than the singular values limit spectrum. Actually, in the finite variance case, the phenomenon is already present since the quarter circular law has support $[0,2]$ while the circular law has support the unit disc.

Following [**BCC11b, BC12**], the convergence of $\mu_{n^{-1/\alpha}X}$ can be deduced from the convergence of $\nu_{n^{-1/\alpha}X-zI}$ for every $z \in \mathbb{C}$ by using the Hermitization lemma as for the standard circular law theorem. Unfortunately, this does not give much information on the limit μ_α. Also, it is more convenient to use the quaternionic Cauchy-Stieltjes transform. Moreover, the convergence of $\nu_{n^{-1/\alpha}X-zI}$ can be addressed using a strategy that differs significantly from the proof of the standard quarter circular law theorem. This strategy is known as the "objective method" and goes back to Aldous and Steele in randomized combinatorial optimization [**AS04**]. Indeed, for a suitable notion of local convergence, $n^{-1/\alpha}X$ converges as $n \to \infty$ to a limiting random operator defined on the Hilbert space $\ell^2(\mathbb{N})$, called the Poisson Weighted Infinite Tree (PWIT). While Poisson statistics arises naturally as in all heavy tailed phenomena, the fact that a tree structure appears in the limit is roughly explained by the observation that non-vanishing entries of the rescaled matrix $n^{-1/\alpha}X$ can be viewed as the adjacency matrix of a sparse random graph which locally looks like a tree. In particular, the convergence to PWIT is a weighted-graph version of familiar results on the local tree structure of Erdős-Rényi random graphs.

For links with free probability, we refer to Aldous and Lyons [**AL07, Lyo10**, Example 9.7 and Sub-Section 5], and to Male [**Mal11**].

Random oriented regular graphs and Kesten-McKay measure. An oriented r-regular graph is a graph on n vertices ($n \geq r \geq 3$) such that all vertices have r incoming and r outgoing oriented edges. Consider the adjacency matrix A of a random oriented r-regular graph sampled from the uniform measure (there exists suitable simulation algorithms using matchings of half edges). It is conjectured that as $n \to \infty$, a.s. μ_A converges to the probability measure $\mu^{(r)}$ given by

$$\frac{1}{\pi}\frac{r^2(r-1)}{(r^2-|z|^2)^2}\mathbf{1}_{\{|z|<\sqrt{r}\}}\,dxdy.$$

The probability measure $\mu^{(r)}$ is also the Brown measure of the free sum of r unitary, see Haagerup and Larsen [**HL00**]. The Hermitian (actually symmetric) version of this measure is known as the Kesten-McKay distribution for random non-oriented r-regular graphs, see [**Kes59, McK81**]. We recover the circular law from $\mu^{(r)}$ when $r \to \infty$ up to rescaling. Recently, it was shown by Basak and Dembo [**BAD12**] that if $U_1, \ldots, U_r$ are i.i.d. Haar distributed $n \times n$ unitary matrices, then the empirical spectral distribution of their sum $U_1 + \cdots + U_r$ tends to $\mu^{(r)}$ as $n \to \infty$. The proof relies on the logarithmic potential and on a recent result on the least singular value by Rudelson and Vershynin [**RV12**]. Towards the conjecture, we may expect the same result for the sum $P_1 + \cdots + P_r$ of i.i.d. Haar distributed $n \times n$ permutation matrices $P_1, \ldots, P_r$, relying on some extension of [**RV12**] to permutation matrices.

Local universality. The circular law theorem 2.2 is a universal statement concerning the global regime of the eigenvalues. It is quite natural to ask about the universality of the local regime, both at the edge of the limiting support and inside the limiting support. Some answers were recently obtained by Bourgade, Yau, and Yin [**BYY12b, BYY12a**] and by Tao and Vu [**TV12**].

References

[Ada11] R. Adamczak, *On the Marchenko-Pastur and circular laws for some classes of random matrices with dependent entries*, Electronic Journal of Probability **16** (2011), 1065–1095.

[AGZ10] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni, *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics, vol. 118, Cambridge University Press, Cambridge, 2010. MR 2760897

[AL07] D. Aldous and R. Lyons, *Processes on unimodular random networks*, Electron. J. Probab. **12** (2007), no. 54, 1454–1508 (electronic).

[And90] A. L. Andrew, *Eigenvalues and singular values of certain random matrices*, J. Comput. Appl. Math. **30** (1990), no. 2, 165–171.

[AS04] David Aldous and J. Michael Steele, *The objective method: probabilistic combinatorial optimization and local weak convergence*, Probability on discrete structures, Encyclopaedia Math. Sci., vol. 110, Springer, Berlin, 2004, pp. 1–72.

[BAD12] A. Basak and A. A Dembo, *Limiting Spectral distribution of sum of Unitary matrices*, arXiv:1208.5100, 2012.

[BAG08] Gérard Ben Arous and Alice Guionnet, *The spectrum of heavy tailed random matrices*, Comm. Math. Phys. **278** (2008), no. 3, 715–751.

[Bai97] Z. D. Bai, *Circular law*, Ann. Probab. **25** (1997), no. 1, 494–529.

[BAZ98] Gérard Ben Arous and Ofer Zeitouni, *Large deviations from the circular law*, ESAIM Probab. Statist. **2** (1998), 123–134 (electronic). MR 1660943 (2000g:60045)

[BC12] Charles Bordenave and Djalil Chafaï, *Around the circular law*, Probab. Surv. **9** (2012), 1–89. MR 2908617

[BCC11a] Charles Bordenave, Pietro Caputo, and Djalil Chafaï, *Circular Law Theorem for Random Markov Matrices*, preprint arXiv:0808.1502 [math.PR] to appear in Probab. Theor. and Related Fields, 2011.

[BCC11b] ———, *Spectrum of non-Hermitian heavy tailed random matrices*, Communications in Mathematical Physics **307** (2011), no. 2, 513–560.

[BCC12] ———, *Spectrum of Markov generators on sparse random graphs*, arXiv:1202.0644, 2012.

[BDG09] Serban Belinschi, Amir Dembo, and Alice Guionnet, *Spectral measure of heavy tailed band and covariance random matrices*, Comm. Math. Phys. **289** (2009), no. 3, 1023–1055.

[Ben10] Martin Bender, *Edge scaling limits for a family of non-Hermitian random matrix ensembles*, Probab. Theory Related Fields **147** (2010), no. 1-2, 241–271.

[BL01] Ph. Biane and F. Lehner, *Computation of some examples of Brown's spectral measure in free probability*, Colloq. Math. **90** (2001), no. 2, 181–211.

[Bor11] Charles Bordenave, *On the spectrum of sum and product of non-Hermitian random matrices*, Electronic Communications in Probability **16** (2011), 104–113.

[Bro86] L. G. Brown, *Lidskiĭ's theorem in the type* II *case*, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser., vol. 123, Longman Sci. Tech., Harlow, 1986, pp. 1–35.

[BS06] Z. D. Bai and J. W. Silverstein, *Spectral Analysis of Large Dimensional Random Matrices*, Mathematics Monograph Series 2, Science Press, Beijing, 2006.

[BS10] Zhidong Bai and Jack W. Silverstein, *Spectral analysis of large dimensional random matrices*, second ed., Springer Series in Statistics, Springer, New York, 2010. MR 2567175 (2011d:60014)

[BYY12a] P. Bourgade, H.-T. Yau, and J. Yin, *Local Circular Law for Random Matrices*, ArXiv e-prints (2012).

[BYY12b] ———, *The local circular law II: the edge case*, ArXiv e-prints (2012).

[CGLP12] Djalil Chafaï, Olivier Guédon, Guillaume Lecué, and Alain Pajor, *Interactions between compressed sensing, random matrices, and high dimensional geometry*, to appear in Panoramas et Synthèses, Société Mathématique de France, 2012.

[Cha10] D. Chafaï, *Circular law for noncentral random matrices*, Journal of Theoretical Probability **23** (2010), no. 4, 945–950.

[CZH08] Zehua Chen, Jin-Ting Zhang, and Feifang Hu (eds.), *Advances in statistics*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008, Including the proceedings of the conference in honor of Professor Zhidong Bai on his 65th birthday, held at the National Uni. of Singapore, Singapore, July 20, 2008. MR 2407790 (2009f:62003)

[DJL11] Zhishan Dong, Tiefeng Jiang, and Danning Li, *Circular Law and Arc Law for Truncation of Random Unitary Matrix*, preprint, personal communication by Tiefeng Jiang, 2011.

[DM78] Claude Dellacherie and Paul-André Meyer, *Probabilities and potential*, North-Holland Mathematics Studies, vol. 29, North-Holland Publishing Co., Amsterdam, 1978.

[DS07] R. B. Dozier and J. W. Silverstein, *On the empirical distribution of eigenvalues of large dimensional information-plus-noise-type matrices*, J. Multiv. Anal. **98** (2007), no. 4, 678–694.

[Ede97] A. Edelman, *The probability that a random real Gaussian matrix has k real eigenvalues, related distributions, and the circular law*, J. Multivariate Anal. **60** (1997), no. 2, 203–232.

[ESY10] László Erdős, Benjamin Schlein, and Horng-Tzer Yau, *Wegner estimate and level repulsion for Wigner random matrices*, Int. Math. Res. Not. IMRN (2010), no. 3, 436–479.

[FK52] B. Fuglede and R. V. Kadison, *Determinant theory in finite factors*, Ann. of Math. (2) **55** (1952), 520–530.

[FK09] Peter J Forrester and Manjunath Krishnapur, *Derivation of an eigenvalue probability density function relating to the poincar disk*, J. of Physics A: Math. and Theor. **42** (2009), no. 38, 385204.

[For10] P. J. Forrester, *Log-gases and random matrices*, London Mathematical Society Monographs Series, vol. 34, Princeton University Press, Princeton, NJ, 2010. MR 2641363

[FZ97a] J. Feinberg and A. Zee, *Non-Hermitian Random Matrix Theory: Method of Hermitian Reduction*, Nucl. Phys. B (1997), no. 3, 579–608.

[FZ97b] Joshua Feinberg and A. Zee, *Non-Gaussian non-Hermitian random matrix theory: phase transition and addition formalism*, Nuclear Phys. B **501** (1997), no. 3, 643–669.

[Gin65] J. Ginibre, *Statistical ensembles of complex, quaternion, and real matrices*, J. Mathematical Phys. **6** (1965), 440–449.

[Gir84] V. L. Girko, *The circular law*, Teor. Veroyatnost. i Primenen. **29** (1984), no. 4, 669–679.

[Gir90] ———, *Theory of random determinants*, Mathematics and its Applications (Soviet Series), vol. 45, Kluwer Academic Publishers Group, Dordrecht, 1990, Translated from the Russian.

[Gir94] ———, *The circular law: ten years later*, Random Oper. Stochastic Equations **2** (1994), no. 3, 235–276.

[Gir01] ———, *Theory of stochastic canonical equations. Vol. I*, Mathematics and its Applications, vol. 535, Kluwer Academic Publishers, Dordrecht, 2001. MR 1887675 (2003f:60048)

[Gir04a] ———, *The strong circular law. Twenty years later. I*, Random Oper. Stochastic Equations **12** (2004), no. 1, 49–104.

[Gir04b] ———, *The strong circular law. Twenty years later. II*, Random Oper. Stochastic Equations **12** (2004), no. 3, 255–312.

[Gir05] ———, *The circular law. Twenty years later. III*, Random Oper. Stochastic Equations **13** (2005), no. 1, 53–109.

[GK00] Ilya Ya. Goldsheid and Boris A. Khoruzhenko, *Eigenvalue curves of asymmetric tridiagonal random matrices*, Electron. J. Probab. **5** (2000), no. 16, 28 pp. (electronic). MR 1800072 (2002j:82061)

[GK05] ———, *The Thouless formula for random non-Hermitian Jacobi matrices*, Israel J. Math. **148** (2005), 331–346, Probability in mathematics. MR 2191234 (2006k:47082)

[GKZ09] A. Guionnet, M. Krishnapur, and O. Zeitouni, *The single ring theorem*, preprint available on arXiv:0909.2214 [math.PR] to appear in the Annals of Mathematics, 2009.

[GNJNP07] E. Gudowska-Nowak, A. Jarosz, M. Nowak, and G. Pappe, *Towards non-Hermitian random lévy matrices*, Acta Physica Polonica B **38** (2007), no. 13, 4089–4104.

[GT10a] F. Götze and A. Tikhomirov, *The circular law for random matrices*, Ann. Probab. **38** (2010), no. 4, 1444–1491. MR 2663633

[GT10b] ———, *On the asymptotic spectrum of products of independent random matrices*, arXiv:1012.2710 [math.PR], 2010.

[GV10] Vyacheslav L. Girko and Anna I. Vladimirova, *L.I.F.E.:* $\prod_{j=1}^m H_n^{(j)} \sim \widetilde{L.I.F.E.} \sim \{H_n^{(1)}\}^m$ *and Halloween Law*, Random Oper. Stoch. Equ. **18** (2010), no. 4, 327–353. MR 2747827

[GWZ11] Alice Guionnet, Philip Matchett Wood, and Ofer Zeitouni, *Convergence of the spectral measure of non normal matrices*, preprint arXiv:1110.2471 [math.PR], 2011.

[GZ10] Alice Guionnet and Ofer Zeitouni, *Support convergence in the single ring theorem*, preprint arXiv:1012.2624 [math.PR], 2010.

[Har12] Adrien Hardy, *A note on large deviations for 2D Coulomb gas with weakly confining potential*, Electron. Commun. Probab. **17** (2012), no. 19, 12. MR 2926763

[HJ94] R. A. Horn and Ch. R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994, Corrected reprint of the 1991 original.

[HKPV09] J. Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág, *Zeros of Gaussian analytic functions and determinantal point processes*, University Lecture Series, vol. 51, AMS, Providence, RI, 2009. MR 2552864 (2011f:60090)

[HL00] U. Haagerup and F. Larsen, *Brown's spectral distribution measure for R-diagonal elements in finite von Neumann algebras*, J. Funct. Anal. **176** (2000), no. 2, 331–367.

[HOZ03] D. E. Holz, H. Orland, and A. Zee, *On the remarkable spectrum of a non-Hermitian random matrix model*, J. Phys. A **36** (2003), no. 12, 3385–3400, Random matrix theory. MR MR1986425 (2004e:82033)

[HS07] Uffe Haagerup and Hanne Schultz, *Brown measures of unbounded operators affiliated with a finite von Neumann algebra*, Math. Scand. **100** (2007), no. 2, 209–263. MR 2339369 (2008m:46139)

[Hwa86] C.-R. Hwang, *A brief survey on the spectral radius and the spectral distribution of large random matrices with i.i.d. entries*, Random matrices and their applications (Brunswick, Maine, 1984), Contemp. Math., vol. 50, Amer. Math. Soc., Providence, RI, 1986, pp. 145–152.

[Jia09] Tiefeng Jiang, *Approximation of Haar distributed matrices and limiting distributions of eigenvalues of Jacobi ensembles*, Probab. Theory Related Fields **144** (2009), no. 1-2, 221–246. MR 2480790 (2010a:15080)

[Joh07] Kurt Johansson, *From Gumbel to Tracy-Widom*, Probab. Theory Related Fields **138** (2007), no. 1-2, 75–112. MR MR2288065 (2008h:60203)

[Kes59] Harry Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354.

[Kos92] E. Kostlan, *On the spectra of Gaussian matrices*, Linear Algebra Appl. **162/164** (1992), 385–388, Directions in matrix theory (Auburn, AL, 1990).

[KS11] B. Khoruzhenko and H.-J. Sommers, *Non-Hermitian ensembles*, The Oxford Handbook of Random Matrix Theory (G. Akermann, J. Baik, and Ph. Di Francesco, eds.), Oxford University Press, 2011.

[KZ12] Z. Kabluchko and D. Zaporozhets, *Universality for zeros of random analytic functions*, arXiv:1205.5355, 2012.

[Led01] M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001. MR MR1849347 (2003k:28019)

[Led08] ———, *Complex Hermite polynomials: from the semi-circular law to the circular law*, Commun. Stoch. Anal. **2** (2008), no. 1, 27–32. MR MR2446909

[Lyo10] Russell Lyons, *Identities and inequalities for tree entropy*, Combin. Probab. Comput. **19** (2010), no. 2, 303–313.

[Mal11] Camille Male, *Free probability on traffics: the limiting distribution of heavy wigner and deterministic matrices*, preprint, personal communication, 2011.

[McD89] Colin McDiarmid, *On the method of bounded differences*, Surveys in combinatorics, 1989 (Norwich, 1989), London Math. Soc. Lecture Note Ser., vol. 141, CUP, Cambridge, 1989, pp. 148–188. MR 1036755 (91e:05077)

[McK81] Brendan D. McKay, *The expected eigenvalue distribution of a large regular graph*, Linear Algebra Appl. **40** (1981), 203–216.

[Mec09] Mark Meckes, *Some results on random circulant matrices*, High Dimensional Probability V: The Luminy Volume (2009), 213–223.

[Meh67] M. L. Mehta, *Random matrices and the statistical theory of energy levels*, Acad. Press, 1967.

[Meh04] ———, *Random matrices*, third ed., Pure and Applied Mathematics, vol. 142, Acad. Press, 2004.

[Nau12] A. Naumov, *Elliptic law for real random matrices*, arXiv:1201.1639, 2012.

[Ngu12] Hoi H. Nguyen, *Random doubly stochastic matrices: the circular law*, arXiv:1205.0843, 2012.

[NO12] Hoi H. Nguyen and S. O'Rourke, *The Elliptic Law*, arXiv:1208.5883, 2012.

[NV12] Hoi H. Nguyen and Van H. Vu, *Circular law for random discrete matrices of given row sum*, arXiv:1203.5941, 2012.

[OS10] S. O'Rourke and A. Soshnikov, *Products of Independent Non-Hermitian Random Matrices*, preprint arXiv:1012.4497 [math.PR], 2010.

[PH98] Dénes Petz and Fumio Hiai, *Logarithmic energy as an entropy functional*, Advances in differential equations and mathematical physics (Atlanta, GA, 1997), Contemp. Math., vol. 217, Amer. Math. Soc., Providence, RI, 1998, pp. 205–221. MR 1606719 (99b:46101)

[Pop09] Ionel Popescu, *General tridiagonal random matrix models, limiting distributions and fluctuations*, Probab. Theory Related Fields **144** (2009), no. 1-2, 179–220. MR MR2480789

[PZ10] G.M. Pan and W. Zhou, *Circular law, extreme singular values and potential theory*, J. Multivar. Anal. **101** (2010), no. 3, 645–656.

[RC09] Tim Rogers and I.P. Castillo, *Cavity approach to the spectral density of non-Hermitian sparse matrices*, Phys. Rev. E **79** (2009), no. 4, 012101.

[Rid03] B. Rider, *A limit theorem at the edge of a non-Hermitian random matrix ensemble*, J. Phys. A **36** (2003), no. 12, 3401–3409, Random matrix theory.

[Rid04] ———, *Deviations from the circular law*, Probab. Theory Related Fields **130** (2004), no. 3, 337–367. MR 2095933 (2005k:82023)

[Rog10] Tim Rogers, *Universal sum and product rules for random matrices*, J. Math. Phys. **51** (2010), no. 9, 093304, 15.

[RS06] B. Rider and Jack W. Silverstein, *Gaussian fluctuations for non-Hermitian random matrix ensembles*, Ann. Probab. **34** (2006), no. 6, 2118–2143. MR 2294978 (2008d:60036)

[RV07] B. Rider and B. Virág, *The noise in the circular law and the Gaussian free field*, Int. Math. Res. Not. IMRN (2007), no. 2, Art. ID rnm006, 33.

[RV08] M. Rudelson and R. Vershynin, *The Littlewood-Offord problem and invertibility of random matrices*, Adv. Math. **218** (2008), no. 2, 600–633.

[RV12] ———, *Invertibility of random matrices: unitary and orthogonal perturbations*, arXiv:1206.5180, 2012.

[Sil94] J. W. Silverstein, *The spectral radii and norms of large-dimensional non-central random matrices*, Comm. Statist. Stochastic Models **10** (1994), no. 3, 525–532.

[Śni02] P. Śniady, *Random regularization of Brown spectral measure*, J. Funct. Anal. **193** (2002), no. 2, 291–313.

[SS12] E. Sandier and S. Serfaty, *2D Coulomb Gases and the Renormalized Energy*, ArXiv e-prints (2012).

[ST97] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Fundamental Principles of Mathematical Sciences, vol. 316, Springer-Verlag, 1997, Appendix B by Thomas Bloom. MR MR1485778 (99h:31001)

[Tal95] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 81, 73–205. MR MR1361756 (97h:60016)

[Tao11] T. Tao, *Outliers in the spectrum of iid matrices with bounded rank perturbations*, preprint arXiv:1012.4818 [math.PR] to appear in Probability Theory and Related Fields, 2011.

[TV08] T. Tao and V. Vu, *Random matrices: the circular law*, Commun. Contemp. Math. **10** (2008), no. 2, 261–307.

[TV09a] ———, *From the Littlewood-Offord problem to the circular law: universality of the spectral distribution of random matrices*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 3, 377–396. MR 2507275 (2010b:15047)

[TV09b] ———, *Inverse Littlewood-Offord theorems and the condition number of random discrete matrices*, Ann. of Math. (2) **169** (2009), no. 2, 595–632. MR MR2480613

[TV10a] ———, *Random matrices: the distribution of the smallest singular values*, Geom. Funct. Anal. **20** (2010), no. 1, 260–297. MR 2647142

[TV10b] ———, *Random matrices: universality of ESDs and the circular law*, Ann. Probab. **38** (2010), no. 5, 2023–2065, With an appendix by Manjunath Krishnapur. MR 2722794

[TV10c] Terence Tao and Van Vu, *Smooth analysis of the condition number and the least singular value*, Math. Comp. **79** (2010), no. 272, 2333–2352. MR 2684367 (2011k:65065)

[TV12] T. Tao and Van H. Vu, *Random matrices: Universality of local spectral statistics of non-Hermitian matrices*, arXiv:1206.1893, 2012.

[VDN92] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free random variables*, CRM Monograph Series, vol. 1, AMS, 1992, A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.

[Voi94] Dan Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory. II*, Invent. Math. **118** (1994), no. 3, 411–440. MR 1296352 (96a:46117)

[Wid94] Harold Widom, *Eigenvalue distribution for nonselfadjoint Toeplitz matrices*, Toeplitz operators and related topics (Santa Cruz, CA, 1992), Oper. Theory Adv. Appl., vol. 71, Birkhäuser, 1994, pp. 1–8. MR 1300209 (95g:47043)

[Woo11] Philip Matchett Wood, *Universality and the circular law for sparse random matrices*, preprint arXiv:1010.1726 [math.PR] to appear in the Annals of Applied Probability, 2011.

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