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Contents

1. Lecture 1: Recent results and open questions

1.1. **Introduction.** Over the past thirty years symplectic geometry has developed its own identity, and can now stand alongside traditional Riemannian geometry as a rich and meaningful part of mathematics. The basic definitions are very natural from a mathematical point of view: one studies the geometry of a skew-symmetric bilinear form ω rather than a symmetric one. However, this seemingly innocent change of symmetry has radical effects. For example, one dimensional measurements vanish since $\omega(v, v) = -\omega(v, v)$ by skew-symmetry. Hence symplectic geometry is an essentially 2-dimensional geometry that measures the *area* of a complex curve instead of the *length* of a real curve.

Here are some of the features that distinguish it from more traditional geometries.

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• The theory has **two faces**. There are two kinds of geometric subobjects in a symplectic manifolds, hypersurfaces and Lagrangian submanifolds that appear in dynamical constructions, and even-dimensional symplectic submanifolds that are closely related to Riemannian and complex geometry. (As we shall see, the analog of a geodesic in a symplectic manifold is a two-dimensional surface called a J-holomorphic curve.)

• Symplectic structures first arose in the Hamiltonian formulation of the theory of classical mechanics and this **tight interconnection with physics** has persisted ever since. A longstanding mystery in mathematics is the extraordinary power of ideas from physics, notably string theory. These have many ramifications in the symplectic world, specially via the deep notion of mirror symmetry which relates objects in symplectic geometry to those in complex algebraic geometry. Though at first this idea seemed completely mysterious, many nontrivial examples of different kinds have now been worked out and fully understood in a mathematical way.

• Although all the basic symplectic concepts are initially expressed in the smooth category (for example, in terms of differential forms), in some intrinsic way that is not yet well understood they do not really depend on derivatives. There is a notion of the **symplectic capacity** of a subset A that is continuous with respect to the Hausdorff distance function on sets and has the property that a diffeomorphism is (anti-) symplectic if and only if it preserves capacity. Thus symplectic geometry is essentially topological in nature. Indeed one often talks about symplectic topology.

• **Darboux's theorem** says that locally all symplectic manifolds are the same, which means that the only invariants that distinguish one from another are global. On the other hand, the lack of local invariants makes it possible for there to be many automorphisms of the local structure. Indeed any smooth function on a symplectic manifold generates a flow on the manifold that consists of **symplectomorphisms** (i.e. diffeomorphisms that preserve the symplectic structure).

• There is a fascinating mix of **flexibility** and **rigidity** in the symplectic world. Some situations and constructions are governed by only homotopy data (flexibility) while others are constrained by various invariants (rigidity). As one example, one can contrast symplectic with Kähler geometry. A symplectic structure is a significant weakening of a Kähler structure, to such an extent that in some situations all extra structure is lost, while in others the structure remains, perhaps in different form. For instance, the fundamental group of a closed Kähler manifold satisfies various subtle constraints, while that of a symplectic manifold can be any finitely presented group. On the other hand, while deformations of the complex structure of a Kähler manifold are not seen directly in terms of the underlying symplectic manifold, they can sometimes be seen in the structure of its group of symplectomorphisms. Much recent work has been devoted to understanding where rigidity stops and flexibility takes over.

• Another interesting phenomenon can be thought of as a **local to global principle**: statements that hold locally often have analogs that are valid globally. This can be interpreted in various ways. The local statement might be linear, or something valid for short periods of time, or something valid in a small open subset of a manifold on which the geometry is standard. Correspondingly, the global statement would be nonlinear, or valid for all time, or valid on the whole manifold. Arnold's conjectures (discussed more below) are one expression of this idea.

Note: Some parts of these notes are taken from the survey article [31]. Also, these notes contain much more information that will be in the lectures themselves.

1.2. **Basic notions.** Let M^{2n} be a smooth manifold without boundary. A **symplectic structure** ω on M is a **closed** ($d\omega = 0$), **nondegenerate** ($\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$) smooth 2-form. Thus the intrinsic measurements that one can make on a symplectic manifold are 2-dimensional, i.e. if S is a little piece of 2-dimensional surface then one can measure

$$\int_S \omega = \mathrm{area}_\omega\, S.$$

- By Stokes' theorem, the **closedness** of ω is equivalent to saying that this integral does not change when one deforms S keeping its boundary fixed.
- The **nondegeneracy condition** is equivalent to the fact that ω induces an isomorphism between the vector fields X on a manifold and the space of 1-forms via the correspondence:[1]

$$\begin{array}{ccc} T_xM & \xrightarrow{\cong} & T_x^*M \\ X & \mapsto & \iota_X\omega = \omega(X,\cdot) \\ \text{vector fields} & & \text{1-forms} \end{array} \tag{1.1}$$

These two conditions work together as follows. Recall the Cartan formula for the Lie derivative:

$$\mathcal{L}_X(\omega) = d(\iota_X\omega) + \iota_X(d\omega) = d(\iota_X\omega), \tag{1.2}$$

where we use the fact $d\omega = 0$. It follows that if the 1-form $\omega(X,\cdot)$ is closed then $\mathcal{L}_x(\omega) = 0$; in other words the *flow of X preserves* ω. But because ω is nondegenerate, for each smooth function $H : M \to \mathbb{R}$, we may define a unique vector field X_H by requiring $\iota_X\omega = dH$. This is called either the **Hamiltonian vector field** of H or the **symplectic gradient** of H. Similarly, if H_t depends on time $t \in \mathbb{R}$ the flow of the time dependent vector field X_{H_t} where $\iota_{X_t}\omega = dH_t$, preserves ω. Thus:

[1]Warning: many authors use a different sign here, defining $\iota_X\omega := -\omega(X,\cdot)$.

Basic Fact: *Every time-dependent function $H : M \times \mathbb{R} \to \mathbb{R}$ on a closed symplectic manifold (M, ω) generates a flow $\phi_t, t \in \mathbb{R}$, such that*

$$\phi_t^*(\omega) = \omega, \qquad \forall t.$$

If H is time independent, the flow preserves the level sets $H = \text{const}$.

Proof. We only need to check the last statement. For this, we need X_H to be tangent to the level set $H = \text{const}$. But (at regular points x) this tangent space is just the kernel of the linear map $dH_x : T_xM \to \mathbb{R}$, where $dH_x(v) = \omega(X_H, v)$. Thus $dH_x(X_H) = \omega(X_H, X_H) = 0$ by skew symmetry. □

An ω-preserving diffeomorphism $M \to M$ is called a **symplectomorphism**.

Comparison with other geometries: The above arguments show that the group $\mathrm{Symp}(M, \omega)$ of all symplectomorphisms of M is infinite dimensional. The corresponding group in Riemannian geometry would be the group of isometries, which is always finite dimensional (and may well be finite). Thus symplectic geometry is significantly more flexible than Riemannian geometry. We can also compare with volume geometry, since every symplectic structure ω determines a volume form $\omega^n/n!$, that is, a nonvanishing top-dimensional form. In two dimensions, of course, ω is simply an area (or volume) form, but in higher dimensions we will see that symplectic geometry is significantly more rigid than volume geometry. In particular, symplectomorphisms have many special properties that distinguish them from diffeomorphisms that merely preserve volume.

Examples *The linear form $\omega_0 = dx_1 \wedge dy_1 + \dots dx_n \wedge dy_n$ on* **Euclidean space** $\mathbb{R}^{2n}$. In this case, the correspondence (1.1) between tangent and cotangent vectors is given explicitly by the formulas

$$\begin{array}{rcl} X = \frac{\partial}{\partial x_j} & \longleftrightarrow & \iota_X\omega_0 = dy_j \\ Y = \frac{\partial}{\partial y_j} & \longleftrightarrow & \iota_Y\omega_0 = -dx_j. \end{array} \tag{1.3}$$

In Riemannian geometry one identifies the tangent space $T_x\mathbb{R}^{2n}$ of vectors and the cotangent space $T_x^*\mathbb{R}^{2n}$ of covectors (or 1-forms) by making the following identifications:

$$\tfrac{\partial}{\partial x_j} \equiv dx_j, \qquad \tfrac{\partial}{\partial y_j} \equiv dy_j.$$

The isomorphism in (1.3) differs from this by a quarter turn.[2] Correspondingly the **symplectic gradient vector** X_H of a function $H : \mathbb{R}^{2n} \to \mathbb{R}$ differs from usual gradient ∇H by a quarter turn, i.e.

$$\nabla H = J_0 X_H.$$

[2]If one pairs the coordinates x_j, y_j thinking of them as the real and imaginary parts of a complex coordinate $z_j = x_j + iy_j$, then this quarter turn corresponds to multiplication by i. Below, we call this operator J_0.

Explicitly, we can calculate

$$X_H = \sum_j \frac{\partial H}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j}\frac{\partial}{\partial y_j}.$$

For example, the flow of the function $H := \pi \sum |z_j|^2$ on $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ is the clockwise rotation

$$(z_1, \dots, z_n) \mapsto (e^{2\pi it} z_1, \dots, e^{2\pi it} z_n), \tag{1.4}$$

which preserves the spheres $H = \text{const}$ and is periodic of period 1.

In general, the symplectic gradient flow is given by solutions $(x(t), y(t))$ to the Hamiltonian equations

$$\dot{x}(t) = \frac{\partial H}{\partial y_j}, \quad \dot{y}(t) = -\frac{\partial H}{\partial x_j}$$

(where $\dot{}$ denotes the time derivative.) In classical mechanics, the function H is the energy and the flow ϕ_t describes the time evolution of the system. Thus, the statement that the flow of X_H preserves H means that **energy is conserved** as the system changes with time.

Kähler manifolds: *Every Kähler manifold* (M, J, g) *has a symplectic structure* ω_J*.* Recall that a Kähler manifold M first of all is a complex manifold, i.e. it is made from pieces of complex Euclidean space $\mathbb{C}^n$ that are patched by holomorphic maps. Thus its tangent bundle TM has a complex structure. This is expressed in terms of the automorphism $J : TM \to TM$ with $J^2 = -\text{id}$ induced by multiplication by i. (J is called an **almost complex structure**). One adds a metric g to this complex manifold and then defines the symplectic form ω_J by setting

$$\omega_J(x, y) = g(Jx, y).$$

(For this to work, i.e. for ω to be closed, g must be compatible with J in a rather strong sense: J has to be parallel with respect to the Levi-Civita connection. Not all complex manifolds can be given a Kähler structure.) Kähler manifolds are a very important subclass of symplectic manifolds, but certainly not all of them — it is still not understood exactly how much arbitrary symplectic manifolds can differ from Kähler ones. Note that any complex submanifold S of a Kähler manifold is symplectic, i.e. the restriction of ω to S is nondegenerate.

Cotangent bundles. An opposite kind of example (relating to the dynamical aspects of symplectic geometry) is given by the cotangent bundle T^*X of an arbitrary smooth manifold X. Because T^*X is the set of all 1-forms on X, it carries a universal 1-form λ_{can}. (This can be written as $\sum_i p_i dq_i$, in terms of local coordinates (p_i, q_i) on T^*X, where $q_1, \dots, q_n$ are coordinates on X and the p_i are the corresponding momentum coordinates in the fibers T_q^*X.) The canonical symplectic form Ω_{can} on T^*X is then simply $-d\lambda_{can} = \sum_i dq_i \wedge dp_i$. Notice that it vanishes on the zero section and the fibers of T^*X. □

Basic properties.

We now state three elementary theorems about symplectic geometry, that together say that locally one cannot distinguish one structure from another. They also can be thought of as instances of the linear (local) to nonlinear (global) principle mentioned earlier. The proofs all involve constructing diffeomorphisms from a given symplectic form ω to a model (often a linear) form ω_0, which is accomplished via *Moser's homotopy method*, an elementary but clever use of Cartan's formula (1.2). (For proofs see for example [35, Chapter 3].)

The most basic of these results is Darboux's theorem. It is the analog of the fact that any two skew-symmetric forms on a vector space of dimension $2n$ are isomorphic.

Theorem 1.1 (Darboux)**.** *Every symplectic form ω on a smooth manifold is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.*

In a smooth manifold all k-dimensional smooth submanifolds S are locally diffeomorphic. However, in the symplectic world things are very different since the rank of the restriction of the symplectic form to S can vary widely. The most important kinds of submanifolds are:

- **symplectic**: the restriction $\omega|_S$ has maximal rank, i.e. is nondegenerate (e.g. complex submanifolds of a Kähler manifold);
- **Lagrangian**: S has half the dimension of M and $\omega|_S \equiv 0$ (e.g. the zero section and fibers of a cotangent bundle, or the real part of an algebraic Kähler manifold viz. $\mathbb{R}P^n \subset \mathbb{C}P^n$);
- **a hypersurface**: in this case $\omega|_S$ always has a one-dimensional kernel $\{v : \omega(v, w) = 0,\ \forall w \in TS\}$. Thus such a surface carries a 1-dimensional foliation called the **characteristic foliation**, which is given by the flow of X_H whenever $S = \{H = \text{const}\}$. In particular if $\omega|_S$ is exact and this flow always twists positively, then by McDuff [29] there is an associated contact structure on S, [3] and locally near S we can write $\omega = d(t\alpha)$ for a suitable choice of a contact 1-form α on S and a transverse coordinate $t \in (-\varepsilon, \varepsilon)$. In this case S is said to have **contact type** or to be **convex**.

Example 1.2. As we saw above, the characteristic foliation on the boundary of the ball B^{2n} is given by the orbits of the diagonal action of the circle on $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ as in (1.4). In this case one can choose the contact form α so that its Reeb flow is also this circle action.

The next theorem says that a symplectic structure has no interesting deformations within its cohomology class.

[3]Contact geometry is the odd dimensional twin of symplectic geometry; cf. [18, 35]. A contact structure is a maximally non integrable field of hyperplanes ξ, which (if cooriented) can be describes as $\ker\alpha$ for some 1-form α satisfying $\alpha\wedge\alpha^{n-1} \neq 0$. The nonintegrability condition implies that in dimension $2n-1$ the maximum dimension of a submanifold that is everywhere tangent to ξ in $n-1$. Such manifolds are called **Legendrian** and are the contact analogs of Lagrangian submanifolds.

Theorem 1.3 (Moser's stability theorem)**.** *Given $\omega_t, t \in [0,1]$, any family of cohomologous symplectic forms on a closed manifold M, there is a family of diffeomorphisms $\phi_t, t \in [0,1]$, of M with $\phi_0 = \mathrm{id}$ and such that $\phi_t^*(\omega_t) = \omega$.*

For example, in dimension two, the set of area forms with the same total area is convex. Hence any pair can be joined by such a path and so are diffeomorphic. However, in dimensions ≥ 4, the space of symplectic forms in a given cohomology class is never convex, and can have interesting topology; in particular it need not be connected.

Since a path of diffeomorphisms from the identity must act trivially on cohomology the above conclusion cannot hold if the cohomology class $[\omega_t]$ varies. However, one might expect that a path of noncohomologous forms ω_t with $[\omega_0] = [\omega_1]$ would at least have diffeomorphic endpoints. But this need not be so; cf. McDuff [28].

There are also so-called **neighborhood theorems** stating that the symplectic structure of a neighborhood of a symplectic or Lagrangian manifold S is determined by (S, ω_S) together with some normal bundle data along S. In particular we have:

Theorem 1.4 (Weinstein)**.** *Every Lagrangian submanfold $L \subset (M, \omega)$ has a neighborhood symplectomorphic to a neighborhood of the zero section in (T^*L, Ω_{can}).*

Lagrangian submanifolds are crucial elements of symplectic geometry. For example, it is easy to see that a diffeomorphism $\phi : M \to M$ preserves ω if and only if $\operatorname{graph}\phi = \{(x, \phi(x)) : x \in M\}$ is a Lagrangian submanifold of the product $(M \times M, -\omega \times \omega)$. One can use this to expand the kinds of maps one looks at when trying to make functorial constructions. The problem is that every smooth ω-preserving map must have Jacobean determinant 1 everywhere in order to preserve the volume form and so must be a diffeomorphism. This is very limiting. However it is possible to define a consistent and interesting notion of **Lagrangian correspondence** from (M_0, ω_0) to (M_1, ω_1) by looking at Lagrangian submanifolds in the product $(M_0 \times M_1, -\omega_0 \times \omega_1)$; cf. Wehrheim–Woodward [44], Weinstein [45].

Main tools

There is an important difference between Kähler manifolds and symplectic manifolds. A Kähler manifold M has a fixed complex structure built into its points. One adds a metric g to this complex manifold and then defines the symplectic form ω_J by setting

$$\omega_J(v, w) = g(Jv, w), \quad \forall v, w \in T_xM,$$

where J is the associated almost complex structure. On the other hand, a symplectic manifold *first* has the form ω, and *then* there is a compatible family of almost complex structures imposed at the tangent space level (not on the points). More precisely, we say that almost complex structure J :

$TM \to TM$ is an ω**-compatible** on (M, ω) if for all $v, w \in TM$ we have

$$\omega(Jv, Jw) = \omega(v, w), \quad v \neq 0 \Longrightarrow \omega(v, Jv) > 0. \tag{1.5}$$

One can prove (cf. [35, Chapter 2]) that each symplectic manifold (M, ω) has a contractible (in particular, nonempty) family of such J; moreover the associated form $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ is symmetric and positive definite.

It was the great insight of Gromov [19] to realise that in symplectic geometry the correct replacement for geodesics are *J-holomorphic curves*. These are maps $u : (\Sigma, j) \to (M, J)$ of a Riemann surface Σ into M that satisfy the generalized Cauchy–Riemann equation:

$$du \circ j = J \circ du.$$

(Here j is the complex structure on the Riemann surface.) In fact, the image $u(\Sigma)$ is a minimal surface in M when it is given the metric g_J, so the analogy with geodesics is not far fetched. Because the Cauchy–Riemann equation is elliptic, there is a very nice theory of these curves; for details and more references see [36]. If the domain is closed then the space of J-holomorphic maps is the solution space of a Fredholm operator and so is finite dimensional. Moreover, in many situations one can understand its compactification, thereby deriving some invariants that are independent of the particular choice of J. If the domain is open (for example a cylinder $\mathbb{R} \times S^1$) or if it has boundary (for example, a disk D^2), then it is possible to define a corresponding Fredholm operator by imposing suitable conditions at infinity or along the boundary. (For example, in the case of disk the boundary must map to a Lagrangian submanifold.)

Variations of this approach form the primary way of constructing symplectic invariants. For example,

- in **quantum cohomology** one counts closed genus 0 curves to define a deformation of the cup product on $H^*(M)$ called the **quantum cup product**;
- in **Hamiltonian Floer theory** one deforms the J-holomorphic curve equation for a cylinder via the flow of a periodic Hamiltonian $H_t, t \in S^1$, to build a theory that relates Hamiltonian dynamics and geometry;
- **Lagrangian Floer theory** measures the "entanglement", or generalized intersection, between two Lagrangian submanifolds L_0, L_1 by counting J-holomorphic strips $u(\mathbb{R} \times [0, 1])$ with $u(\mathbb{R} \times \{i\}) \in L_i$ whose ends converge to points in $L_0 \cap L_1$, and leads to the definition of the Fukaya category;
- **symplectic homology** is a more topological reformulation of Hamiltonian Floer theory that gives invariants of noncompact symplectic manifolds that are "convex at infinity" in a sense explained below.

All the above approaches give tools for measuring symplectic rigidity. For example, if the Floer homology of a pair of Lagrangians L_0, L_1 does not vanish then it is impossible to move L_0 by a Hamiltonian isotopy to be disjoint from L_1. Also, McLean [38] recently used symplectic homology to detect infinitely many different symplectic structures on $\mathbb{R}^{2n}, n > 2$.

A second circle of ideas, pioneered by Gromov and Eliashberg, has developed powerful ways to construct symplectic forms via surgery. There is a general method due to Gromov that constructs a symplectic form on any open manifold, but this form is completely uncontrolled at infinity and cannot be used as a building block to understand closed manifolds. However, for closed $2n$-dimensional manifolds with integral symplectic form (i.e. $[\omega] \in H^2(M;\mathbb{Z})$), there is a beautiful construction due to Donaldson that provides a symplectic codimension 2 submanifold $D_k \subset M$ Poincaré dual to $k[\omega]$, for sufficiently large k. Moreover, just as in the Lefschetz hyperplane theorem, the complement $M \setminus D_k$ has the homotopy type of a finite n-dimensional cell complex, and is "convex at infinity" in the sense of Eliashberg–Gromov.

This **convexity condition**, a symplectic version of pseudoconvexity in complex geometry, implies that there is a compact submanifold M_0 of M such that $M \setminus M_0 \cong \partial M_0 \times (0,\infty)$, where the codimension-1 boundary ∂M_0 has contact type with contact form α and the end has the associated symplectic structure $d(r\alpha), r \in (0,\infty)$. Contact manifolds are much more amenable to surgery than symplectic manifolds: for example the connected sum of two closed contact manifolds is contact (which is always false for symplectic 4-manifolds, and quite possibly always false in any dimension). It is expected (but to my knowledge not yet proved) that, just as in the Kähler case, $M \setminus D_k$ satisfies a strong global convexity condition; in Eliashberg's terminology, it should be a **Weinstein manifold**. By Eliashberg [13], this would imply that in dimensions six and above it carries a Stein structure, i.e. a compatible complex structure induced from a proper embedding into some Euclidean space $\mathbb{C}^N$. In other words, in high dimensions a closed integral symplectic manifold should have a "hyperplane decomposition" as $D_k \cup (M \setminus D_k)$, where D_k is a symplectic manifold of dimension $2n-2$ and $M \setminus D_k$ is affine complex.

The book Cieliebak–Eliashberg [5] describes the relation between the two notions, Weinstein and Stein, and explains how to construct manifolds with these structures by surgery. This is always possible in dimensions > 4, though there are obstructions in dimension four. The construction in dimension $2n$ is flexible (i.e. governed by formal homotopy invariants) when one adds k-handles for $k < n$, but *can* be more rigid when adding n-handles. Now, k-handles are attached along $(k-1)$-spheres in the contact boundary that are isotropic in the sense that they are everywhere tangent to ξ. Thus for top dimensional handles they are Legendrian. Recently Emmy Murphy [39] introduced the idea of a **loose Legendrian**, showing that they are plentiful. Handles attached along loose Legendrians are flexible: for example, the symplectic homology of a Weinstein manifold vanishes if it is obtained from a ball by attaching k-handles where either $k < n$ or the attaching sphere is loose. However there are plenty of Weinstein manifolds with nonvanishing symplectic homology. Thus there is a very delicate boundary here between flexibility and rigidity than needs further exploration; for further discussion and references, see Cieliebak–Eliashberg [6].

1.3. **Open Questions.** There is a huge variety of open problems in symplectic geometry, and we will only mention a few. The next two lectures will discuss embedding questions, such as:

- when does one (open) symplectic manifold (U_0, ω_0) embed symplectically in another (U_1, ω_1)?
- what obstructions are there?

The answers dictate what kinds of symplectically invariant measurements can be made and hence illuminate the geometric meaning of a symplectic structure. For now, we will put these considerations to one side, and turn to other questions.

In 2-dimensions the symplectic form is an area form, so that symplectic geometry is just area preserving geometry. Although many dynamical questions remain in this dimension, there are no topological questions: a connected 2-manifold has a symplectic structure if and only if it is orientable, and by Moser's Theorem 1.3 the only invariant is the total area. Hence in these lectures we are interested in dimensions four and above. As in smooth topology, dimension four is special. It turns out that gauge theories (in particular Seiberg–Witten theory) have very special properties on symplectic manifolds (first discovered by Taubes) that for example relate Seiberg–Witten theory to the theory of J-holomorphic curves. This gives a great deal of information in this dimension. It is not clear how much of this special structure remains in higher dimensions.

Question 1.5. *To what extent does symplectic geometry in dimensions > 4 retain the rigid features of the four-dimensional theory?*

We will see that some information/restrictions are retained and some are not. Here is a more specific question. In dimension four, one can do surgery on a closed symplectic manifold (M_0, ω_0) to construct a symplectic manifold (M_1, ω_1) with the same homotopy type as M_0 but different Seiberg–Witten invariants; cf. Fintushel–Stern [15].[4] Then $(M_0, \omega_0), (M_1, \omega_1)$ cannot be symplectomorphic. But they also cannot be diffeomorphic, because Seiberg–Witten invariants depend only on the smooth structure of a manifold. Now consider the products of these manifolds with (S^2, σ). One then gets two symplectic manifolds $(M_0 \times S^2, \omega_0 \times \sigma), (M_1 \times S^2, \omega_1 \times \sigma)$ which *are* diffeomorphic (at least when the manifolds are simply connected) but

[4] In fact they construct infinitely many different symplectic manifolds each with the homotopy type of the $K3$ surface.

are *not* **deformation equivalent** as symplectic manifolds.[5] This holds, because although in the smooth world Seiberg–Witten invariants exist only in dimension four, in the symplectic world they coincide with the so-called Gromov invariants given by counting J-holomorphic curves, and so they survive under product with S^2. Moreover the resulting invariants do not change when the form ω_t varies smoothly.

Question 1.6 (Donaldson's four-six question)**.** *If $(M_0, \omega_0), (M_1, \omega_1)$ are symplectic 4-manifolds such that $(M_0 \times S^2, \omega_0 \times \sigma), (M_1 \times S^2, \omega_1 \times \sigma)$ are deformation equivalent, must M_0, M_1 be diffeomorphic?*

This question can be considered in the framework of the passage from four to six dimensions. But it can also be considered under the rubric: to what extent does symplectic structure capture smooth structure? Here is another variant of this second question.

Question 1.7 (Eliashberg)**.** *Suppose that the two smooth closed manifolds M_0, M_1 are homeomorphic. Are they diffeomorphic precisely when their cotangent bundles T^*M_0, T^*M_1 with the canonical symplectic structure are symplectomorphic?*

The first (and so far only) result in this direction is due to Abouzaid, who showed in [1] that if Σ is an exotic $4k+1$ dimensional sphere that does not bound a parallelizable manifold (and these exist) then $T^*\Sigma$ and T^*S^{4k+1} are *not* symplectomorphic. He does this by showing that every homotopy sphere that embeds as a Lagrangian in T^*S^{4k+1} must bound a parallelizable manifold, which he constructs directly out of certain perturbed J-holomorphic curves. This result also throws light on the following conjecture about exact Lagrangians in T^*M, where L is called **exact** if the restriction to L of the canonical 1-form λ_{can} on T^*M is exact.

Conjecture 1.8 (Arnol'd's Nearby Lagrangian Conjecture)**.** *Every exact Lagrangian L in T^*M can be moved to the zero section by a path of Hamiltonian symplectomorphisms.*

Here there is a quite a bit of known information: in particular the projection $L \to M$ along the fibers of T^*M must be a homotopy equivalence; cf. Abouzaid–Kragh [2] for recent results and references.

Finally, we end with a few words on the **existence question**: which manifolds have a symplectic structure? As mentioned earlier, Gromov proved that every open manifold with an almost complex structure also has a symplectic structure. But the question is wide open in the closed case. Here one must require that that M has an almost complex structure J, and that there there is a cohomology class $a \in H^2(M;\mathbb{R})$ with $a^n > 0$, where M

[5]Two symplectic manifolds $(M_0, \omega_0), (M_1, \omega_1)$ are called deformation equivalent if there is a diffeomorphism $\phi : M_0 \to M_1$ and a path of symplectic structures $\rho_t, t \in [0,1]$ on M_0 such that $\rho_0 = \omega_0, \rho_1 = \phi^*(\omega_1)$.

is assumed oriented compatibly with the orientation on the complex bundle (TM, J). In four dimensions manifolds are known that satisfy these criteria but yet do not support any symplectic structure: the first example (due to Taubes [42] via Seiberg–Witten theory) was the connected sum $\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$. However, although if one imposes suitable additional restrictions on the structures one can show in some cases that forms do not exist, the answer to the following question is still unknown.

Problem 1.9. *Is there a closed almost complex manifold of dimension* ≥ 6 *that has a class* a *with* $a^n > 0$ *but no symplectic structure* ω*?*

The preponderance of evidence at this stage seems to indicate that the answer should be "yes". On the other hand, the construction methods for symplectic structures on open manifolds discussed above are progressing to such point that one might be able to build a symplectic structure on M by picking a suitable divisor D, putting symplectic structures on D and $M \smallsetminus D$, and then matching them up in some way. Since D would be four dimensional (assuming M is six dimensional), it of course might not have a symplectic structure. Therefore one would have to get around that difficulty, as well as deal with the "matching" problem. Clearly there is a lot of future work to be done.

2. Lecture 2: Embedding questions: obstructions and constructions

Embedding questions first came into prominence in Gromov's 1985 paper [19] in which he asked the following question:

Gromov's question: *What can one say about the symplectic images $\phi(B)$ of a ball? How are the symplectic and volume preserving cases different?*

This question is interesting even if we restrict to the case when $B = B^{2n}$ is a closed Euclidean ball in $\mathbb{R}^{2n}$ with its standard structure ω_0 and we consider symplectic embeddings ϕ of B into $(\mathbb{R}^{2n}, \omega_0)$. (For short we will write $\phi : B \stackrel{s}{\hookrightarrow} \mathbb{R}^{2n}$ for such an embedding.

The following result can be easily proven using Moser's homotopy method.

Volume preserving embeddings: *if $V \subset \mathbb{R}^{2n}$ is diffeomorphic to a closed ball B with the same volume then there is a volume preserving diffeomorphism $\psi : B \stackrel{\cong}{\to} V$.*

The analogous statement is *not* true for symplectomorphisms (in dimensions > 2) since the boundary ∂V of V has symplectic invariants given by its characteristic foliation (equivalently by the flow of any Hamiltonian H that is constant on ∂V); cf. Example 1.2. Therefore, instead of hoping to find symplectomorphisms from one region *onto* another, we can look for symplectomorphisms from one region *into* another, i.e. a symplectic embedding. We will see later that the characteristic foliation still has a very strong influence on which embeddings are possible, via the notions of Ekeland–Hofer and ECH capacities.

Notational Conventions: We denote

$$\begin{aligned} B(a) : &= \big\{(x_1, y_1, x_2 \dots, y_n) \in \mathbb{R}^{2n} : \sum_{j=1}^{n} \pi(x_j^2 + y_j^2) \le a\big\}, \\ Z(a) : &= B^2(A) \times \mathbb{R}^{2n-2} = \{(x_1, y_1, x_2 \dots, y_n) \in \mathbb{R}^{2n} : \pi(x_1^2 + y_1^2) \le A\}, \end{aligned}$$

sometimes adding a superscript $2n$ to emphasize the dimension. Thus $B(a)$ is a ball labelled by the area of its intersection with the x_1, y_1-plane, while $Z(a)$ is an infinite cylinder with cross section of area a. Note that its first two coordinates lie in symplectically paired directions. Further, both regions have the property that each leaf of the characteristic foliation on their boundaries is a circle that bounds a disk of area a.

Theorem 2.1 (Gromov's Nonsqueezing Theorem). *There is a symplectic embedding*

$$\phi : B^{2n}(a) \stackrel{s}{\hookrightarrow} Z^{2n}(A)$$

if and only if $a \le A$.

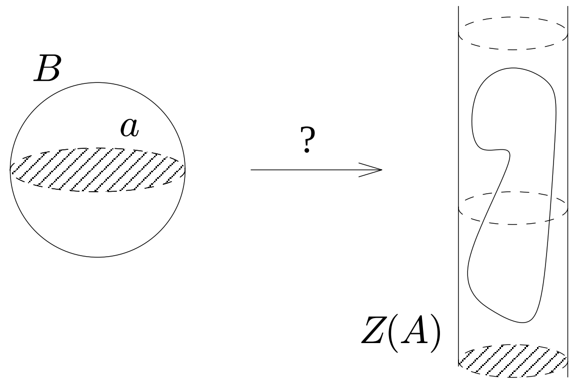

FIGURE 2.1. Does the ball embed symplectically in the cylinder?

This nonsqueezing property is fundamental: we explain why below. Note for now that it implies that symplectomorphisms are very different from volume preserving embeddings. For example, the largest ball that embeds symplectically in the polydisk $B^2(1) \times B^2(1)$ is $B^4(1)$ which has volume just half that of the polydisk.

2.1. **Symplectic capacities.** In their paper [11], Ekeland and Hofer formalized the idea of a *symplectic capacity*, which is a measurement $c(U, \omega)$ of the size of a symplectic manifold (U, ω) (possibly open or with boundary) with the following properties:

(i) (domain of definition) *c is a function with values in $[0, \infty]$ defined on some class of symplectic manifolds of fixed dimension that contains the closures of all open subsets of Euclidean space and is invariant under rescaling;*
(ii) (monotonicity) *if there is a symplectic embedding $(U, \omega) \overset{s}{\hookrightarrow} (U', \omega')$ then* $c(U, \omega) \le c(U', \omega')$;
(iii) (scaling) $c(U, \lambda\omega) = \lambda c(U, \omega)$ *for all* $\lambda > 0$;
(iv) (strong normalization)[6] $a = c\big(B^{2n}(a)\big) = c\big(Z^{2n}(a)\big)$.

The monotonicity property implies that c is a symplectic invariant, that is, it takes the same value on symplectomorphic sets, while the scaling property implies that it scales like a 2-dimensional invariant. One can satisfy the first three properties by considering an appropriate power of the volume; for example one could consider the function $c(U, \omega) := \big(\int_U \omega^n\big)^{1/n}$. However, this function c does not satisfy the second half of the normalization axiom. Indeed, the requirement that a cylinder has finite capacity is what makes this an interesting definition.

The nonsqueezing theorem immediately implies that capacities do exist.

[6]Here I have made the strictest reasonable requirement. One could also simply ask that $1 = c\big(B^{2n}(a)\big) < c\big(Z^{2n}(a)\big) < \infty$.

Theorem 2.2. *The Gromov width* $c_G(U,\omega) := \sup\{a \,|\, B^{2n}(a) \overset{s}{\hookrightarrow} (U,\omega)\}$ *is a capacity on the set of all symplectic* $2n$*-dimensional manifolds.*

The next result holds because the monotonicity and scaling properties imply that if a diffeomorphism ϕ preserves a capacity c then so does its derivative at any point. However, a linear map that preserve a capacity must be symplectic (or antisymplectic). Thus, we have:

Proposition 2.3 (Eliashberg [12], Ekeland–Hofer [11]). *Every diffeomorphism of* (M,ω) *that preserves a capacity is (anti)symplectic, i.e. satisfies* $\phi^*(\omega) = \pm\omega$*. Further, for every symplectic manifold* (M,ω)*, the group of compactly supported symplectomorphisms is* $\mathcal{C}^0$*-closed in the group of all diffeomorphisms, i.e. if* ϕ_k *is a sequence of diffeomorphisms that converge in the uniform topology to a smooth diffeomorphism* ϕ_∞ *then* ϕ_∞ *is also a symplectomorphism.*

Thus one can define what it means for a smooth map f to be symplectic without using the derivative of f. Hence there is a notion of a **symplectic homeomorphism**: namely one that preserves a capacity. Very little is known about this notion. As we will now see, there is a great variety of capacity functions, and for all we know they each might give rise to different classes of symplectic homeomorphisms.

A variety of symplectic capacities. The following examples illustrate just a few of the possible definitions.

(i) The Gromov width. $c_G(U,\omega) = \sup\{\pi a \,|\, B^{2n}(a) \overset{s}{\hookrightarrow} (U,\omega)\}$.

(ii) The Hofer–Zehnder capacity [23]. Recall that every smooth function $H : M \to \mathbb{R}$ on a closed symplectic manifold gives rise to a vector field X_H with $\omega(X_H,\cdot) = dH(\cdot)$, which integrates to a 1-parameter subgroup $\phi_t^H, t \in \mathbb{R}$, of the group of symplectomorphisms called the Hamiltonian flow. A point x is said to be a **nontrivial periodic orbit** of ϕ_t^H of period $T > 0$ if $\phi_T^H(x) = x$ but $\phi_t^H(x) \neq x$ for all $t \in (0,T)$. Let $\mathcal{H}$ be the set of functions $H : M \to \mathbb{R}$ with the following properties:

- $H(x) \geq 0$ for all $x \in M$,
- there is an open subset of M on which $H = 0$;
- H is constant outside a compact subset of the interior of M;
- H has no *fast periodic orbits*, i.e. every nontrivial periodic orbit of ϕ_T^H has period $T \geq 1$.

Then we define the Hofer–Zehnder capacity c_{HZ} as follows:

$$c_{HZ}(U,\omega) = \sup_{H\in\mathcal{H}}\Big(\sup_{x\in M} H(x)\Big).$$

Thus c_{HZ} measures the supremum of the size of the range of a function on U whose derivative is constrained by the fact that it has no fast periodic orbits.

This function c_{HZ} obviously satisfies the first three conditions for capacities. Moreover as λ increases the flow of λH moves faster so that the periods of the periodic orbits decrease. Therefore in order to prove that a set such as B^{2n} has finite capacity one needs a mechanism to prove that periodic orbits for ϕ_t^H must exist under suitable circumstances (for example, if $H = \lambda K$ where K satisfies the first three conditions to be in $\mathcal{H}$ and λ is sufficiently large.) In the original papers this mechanism involved subtle arguments in variational analysis; however one can also prove such results using J-holomorphic methods.

In 2-dimensions it is not hard to see that $c_{HZ}(U,\omega) = \mathrm{area}(U,\omega)$. However, in dimensions four and above this is a very interesting invariant that is very far from being understood. For example, we do not know c_{HZ} for the standard 4-torus $\mathbb{R}^4/\mathbb{Z}^4$.

Sequences of capacities: We now explain the properties of two increasing sequences $\mathcal{C} = (c_0 := 0, c_1, c_2, \dots)$ of capacities that satisfy conditions (i) and (iii) and have the following enhanced monotonicity property:

- (monotonicity) *if there is a symplectic embedding* $(U,\omega) \overset{s}{\hookrightarrow} (U',\omega')$ *then* $\mathcal{C}(U,\omega) \preccurlyeq \mathcal{C}(U',\omega')$ *in the sense that*
$$c_k(U,\omega) \le c_k(U',\omega') \quad \forall\ k \ge 0.$$

Moreover in both cases c_1 is (strongly) normalized.
(iii) Ekeland–Hofer capacities [11]. These form a sequence $\mathcal{C}_{EH} := c_1, c_2, \dots$ o as above. Just as c_{HZ} they are defined by variational methods, picking out the actions[7] of a sequence of "significant" periodic orbits — found by mini-max methods — of the Hamiltonian flow of certain functions $H : \mathbb{R}^{2n} \to \mathbb{R}$ that are constant on the boundary ∂U. For example, denote by $E(a_1, \dots, a_n)$ the $2n$-dimensional ellipsoid[8]

$$E(a_1, \dots, a_n) := \big\{(z_1, \dots, z_n \in \mathbb{C}^n \big| \sum \pi \frac{|z_i|^2}{a_i} \le 1\big\}.$$

Then we have

$$\begin{aligned}
\mathcal{C}_{EH}\big(B^4(1)\big) &= (0,1,1,2,2,3,3,4,4,\dots),\\
\mathcal{C}_{EH}\big(B^2(1) \times B^{2n-2}(A)\big) &= (0,1,2,3,4,5,6,7,\dots),\\
\mathcal{C}_{EH}\big(E(a_1,\dots,a_n)\big) &= \mathrm{Ord}(m_1a_1,\dots,m_na_n \big| m_1,\dots,m_n \ge 0),
\end{aligned} \tag{2.1}$$

where given a collection of nonnegative integers with repetitions, Ord lists them (again with repetitions) in increasing order. Notice that for a generic ellipsoid $E(a_1, \dots, a_n)$ with irrational ratios a_i/a_j the only closed orbits of the Hamiltonian flow on its boundary are those in the n coordinate planes. The orbit in the z_i-plane is bounded by a circle with area a_i and hence its

[7]The action of a periodic orbit γ is the ω_0-area of a disk in $\mathbb{R}^{2n}$ with boundary γ.

[8]One can show that for every quadratic form $Q(x,y)$ the ellipsoid $Q(x,y) \le 1$ in $\mathbb{R}^{2n}$ can be written in this form after a suitable symplectic linear transformation.

m fold-cover has "action" ma_i. The EH capacities are simply this collection of numbers, arranged in increasing order. For example,

$$\mathcal{C}_{EH}\big(E(1,5)\big) = (0,1,2,3,4,5,5,6,7,8,9,10,10,\dots). \tag{2.2}$$

(iv) Embedded contact homology (ECH)-capacities (Hutchings [24]). This is another sequence $\mathcal{C}_{EH} := (c_0, c_1, c_2, \dots)$ as above, that are defined for 4-dimensional symplectic manifolds using a rather sophisticated gauge theory. For subsets of $\mathbb{R}^4$ with contact type boundary (e.g. convex or star-shaped sets) they are an invariant of the contact structure of its boundary, and are defined using the homology of a chain complex whose generators are finite unions of closed Reeb orbits (which in this case coincide with the closed orbits of the characteristic foliation on the boundary; cf. Example 1.2). As with Ekeland–Hofer capacities, ECH capacities measure the actions of certain (homologically) significant unions of these orbits. Because we are now allowed to consider *unions* of orbits this invariant gives more information about ellipsoids. In fact in this case the boundary operator in the ECH chain complex vanishes so that the ECH capacities are just the actions of all possible unions. Thus

$$\mathcal{C}_{ECH}\big(E(a_1,\dots,a_n)\big) = \mathrm{Ord}\big(\sum m_i a_i \big| m_1,\dots,m_n \ge 0\big), \tag{2.3}$$

so that

$$\begin{aligned} \mathcal{C}_{ECH}\big(B^4(1)\big) &= (0,1,1,2,2,2,3,3,3,3,4,4,4,4,4,\dots), \\ \mathcal{C}_{ECH}\big(E(1,4)\big) &= (0,1,2,3,4,4,5,5,6,6,7,7,8,8,8,\dots). \end{aligned} \tag{2.4}$$

It is possible, but harder, to calculate the ECH capacities of products such as $B^2(1) \times B^{2n-2}(A)$.

ECH capacities also behave well under disjoint union. Following Hutchings, given two nondecreasing sequences

$$\mathcal{C} = (c_0 = 0, c_1, \dots), \quad \mathcal{D} = (d_0 = 0, d_1, \dots)$$

let us define their sum $\mathcal{C}\#\mathcal{D}$ by setting

$$(\mathcal{C}\#\mathcal{D})_k := \max_{0 \le i \le k} c_i + d_{k-i}.$$

Then

$$\mathcal{C}_{ECH}(X \sqcup Y) = \mathcal{C}_{ECH}(X) \# \mathcal{C}_{ECH}(Y). \tag{2.5}$$

There are many other capacities, as well as other symplectically invariant measurements that share some of the properties of capacities. (See [7] for an overview.) Rather little is known about the behavior of an arbitrary normalized capacity. For example, as far as we know currently it might be true that all strongly normalized capacities agree on convex subsets of $\mathbb{R}^{2n}$. Here is a somewhat more modest proposal, suggesting that the ball should have the largest capacity among all convex sets with a given volume.

Viterbo's symplectic isoperimetric conjecture. *All strongly normalized capacities c satisfy the inequality*

$$\frac{c(\Sigma)}{\mathrm{Vol}(\Sigma)^{1/n}} \le \frac{c(B)}{\mathrm{Vol}(B)^{1/n}}$$

for every compact convex set $\Sigma \subset \mathbb{R}^{2n}$ *with nonempty interior.*

This is known to be true up to a constant factor that is independent of dimension; [43, 3].

Hofer's question about intermediate capacities; cf [7, §3.9] The notion of capacity as defined above is an essentially 2-dimensional invariant since it is finite on the product $B^2(1) \times \mathbb{R}^{2n-2}$. Hofer asked if there are *intermediate capacities*, i.e. a symplectically invariant measurement that would be infinite on $B^{2d-2}(1) \times \mathbb{R}^{2(n-d)+2}$ but finite on $B^{2d}(1) \times \mathbb{R}^{2(n-d)}$ for some $d > 1$. Alternatively, one could phrase this in terms of compact sets such as **polydisks**:

$$P(a_1, \dots, a_n) := B^2(a_1) \times \cdots \times B^2(a_n), \quad a_1 \le a_2 \le \cdots \le a_n.$$

Thus, when $d = 2$ one might ask:
Is there a function $f(a_1, a_2)$ *that depends nontrivially on* a_2 *and has the property that*

$$P(a_1, \dots, a_n) \overset{s}{\hookrightarrow} P(b_1, \dots, b_n) \implies f(a_1, a_2) \le f(b_1, b_2)?$$

In [20], Guth showed that when $n \ge 3$ the answer to this second question is "NO" by proving the following:

Theorem 2.4 (Guth)**.** *For each* n *there is a constant* $c(n)$ *such that for all* a_i, b_j

$$P(a_1, \dots, a_n) \overset{s}{\hookrightarrow} P(b_1, \dots, b_n) \iff \begin{cases} c(n)a_1 \le b_1, \\ c(n)a_1 \dots a_n \le b_1 \dots b_n. \end{cases}$$

In other words, up to the dimensional constant $c(n)$, the only constraints on embedding polydisks come from Gromov's width and the volume.

Guth's basic idea exploits the fact, first observed by Polterovich, that the 2**-torus**[9] **does not squeeze.** In other words, for all $A > 0$ there are symplectic embeddings $B^{2n}(A) \overset{s}{\hookrightarrow} \mathbb{T}^2(1) \times \mathbb{R}^{2n-2}$, where $\mathbb{T}^2(1)$ denotes the torus of area 1. For example, consider the composite symplectic embedding

$$B^{2n}(A) \xrightarrow{L} \mathbb{R}^2 \times \mathbb{R}^{2n-2} \xrightarrow{\mathrm{pr} \times \mathrm{id}} \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}^{2n-2}.$$

Here L is a linear map that distorts the ball so that its slices by the 2-planes parallel to the z_1-axis are disks of radius $< \frac{1}{2}$ that therefore map injectively

[9]Here we are contrasting the 2-torus (or any higher genus surface) with the 2-sphere. Gromov's nonsqueezing theorem can also be stated as saying that there is no symplectic embedding $B^{2n}(a) \to S^2(A) \times \mathbb{R}^{2n-2}$ when $a > A$.

under the projection pr : $\mathbb{C} \mapsto \mathbb{C}/\mathbb{Z}$. Guth refined this to construct an embedding

$$B^{2n}(A) \overset{s}{\hookrightarrow} \big(\mathbb{T}^2(1) \smallsetminus pt\big) \times \mathbb{R}^{2n-2}.$$

He then used the fact that because $\mathbb{T}^2(1)\smallsetminus pt$ immerses in $B^2(2)$ one can *embed* its thickening $B^2(1) \times \big(\mathbb{T}^2(1)\smallsetminus pt\big)$ into a polydisk such as $P(3,3)$ by using the extra dimensions to separate out the two layers. By this means, he constructs a composite embedding of the form

$$B^2(1) \times P(S,\dots,S) \overset{s}{\hookrightarrow} B^2(1) \times \big(\mathbb{T}^2(1)\smallsetminus pt\big) \times \mathbb{R}^{2n-4} \overset{s}{\hookrightarrow} P(3,3) \times \mathbb{R}^{2n-4}.$$

Pelayo–Ngoc [40] refined this construction to deal with the noncompact sets $B^{2k}\times\mathbb{R}^{2n-2k}$, thereby showing there are no intermediate capacities in Hofer's original sense.

Embedding balls, ellipsoids and polydisks. Guth's theorem gives an excellent qualitative idea for when one polydisk embeds into another. However we are far from being able to answer the following question even in dimension 4.

Problem 2.5. *Find necessary and sufficient conditions for embedding an ellipsoid or polydisk into another ellipsoid or polydisk.*

As we show below, in dimension 4 the question of when one ellipsoid embeds into another is completely solved. Similar methods work for ellipsoids into polydisks; cf Frenkel–Müller [17]. However understanding the problem when the domain is a polydisk is significantly harder. In dimensions 6 and above we do not have a guess as to what the correct answer should be, even for ellipsoids. (This is an interesting special case of Question 1.5.) One easily check, using the above result of Guth, that the obvious analog of the condition in Theorem 2.7 below is incorrect in general. However, judging from recent work of Buse–Hind (cf. [4]) this analog might hold for sufficiently "fat" ellipsoids, i.e. ones for which the ratio a_n/a_1 is suitably small. The most relevant invariants are the Ekeland–Hofer capacities $\mathcal{C}_{EH}$,[10] but, as the following examples show, these are rather weak. We will first discuss the case of polydisks, since rather little is known here, and then deal with ellipsoids. Note that most examples mentioned below are in 4 dimensions.

The Ekeland-Hofer capacities give

$$\mathcal{C}_{EH}\big(P(1,2)\big) = (0,\, 1,\, 2,\, 3,\, 4,\, 5,\, \dots\, ,\, k, \dots),$$
$$\mathcal{C}_{EH}\big(B^4(a)\big) = (0, a,\, a, 2a, 2a, 3a, \dots, a\lceil \tfrac{k}{2}\rceil, \dots).$$

Since $\mathrm{vol} P(1,2) = 2 = \mathrm{vol} B^4(2)$, there is therefore no obstruction from $\mathcal{C}_{EH}$ and volume to the existence of an embedding $P(1,2) \overset{s}{\hookrightarrow} B^4(2)$. One can check that the ECH capacities do no better. On the other hand, a recent triumph with this particular problem is the following sharp bound.

[10] One can also define invariants using symplectic or contact homology, but it is not clear whether one gets anything new this way.

Theorem 2.6 (Hind–Lisi [22]). *$P(1,2)$ embeds symplectically into the ball $B^4(a)$ if and only if $a \geq 3$.*

Since $P(1,2)$ is a subset of $B^4(3)$, the condition is certainly sufficient. The argument for necessity is very delicate, and proceeds by contradiction. Assume that the closed polydisk $P(1,2)$ embeds into a ball $B^4(3-\varepsilon)$ for some $\varepsilon > 0$, and then compactify the ball[11] to $(\mathbb{C}P^2, \omega_a)$, where ω_a integrates over the complex line to $a < 3$. The polydisk $P(1,2)$ has a "corner" $S^1(1) \times S^1(2) = \partial B^2(1) \times \partial B^2(2)$ which is a Lagrangian torus $\mathbb{T}^2$. Pull the manifold apart along this torus ("stretching the neck" in the language of gauge theory) and look at what happens to the holomorphic curves. This is again an essentially 4-dimensional result, since it uses special properties of J-curves in 4-dimensional spaces.

It turns out that in 4 dimensions ellipsoids are easier to understand than polydisks because they have smooth boundaries. In fact one can show that ECH capacities give a sharp obstruction. This was conjectured by Hofer when it became clear that the Ekeland–Hofer capacities did not give necessary and sufficient conditions for an embedding. He was convinced that the obstructions to embedding such a simple shape as an ellipsoid should come from properties of the characteristic flow on the boundary: if it wasn't enough to know the actions of single, perhaps multiply covered, orbits, then the next guess would be that one would get enough information from unions of such orbits. In the case of ellipsoids, these are precisely the ECH capacities since the boundary operator in the ECH chain complex vanishes. Indeed, recall from (2.4) that

$$\mathcal{C}_{ECH}\big(E(a,b)\big) = \mathcal{N}(a,b), \tag{2.6}$$

where $\mathcal{N}(a,b)$ is the set of numbers $\{ma + nb \mid m,n \geq 0\}$ arranged in increasing order, with repetitions. Further, $\mathcal{C} \preccurlyeq \mathcal{D}$ means that $\mathcal{C}_k \leq \mathcal{D}_k$ for all $k \geq 0$. The following result from McDuff [32] solves Hofer's conjecture for ellipsoids.

Theorem 2.7 (Hofer conjecture). *For all $0 < a \leq b, 0 < c \leq d$, we have*

$$\begin{aligned} \mathrm{int}E(a,b) \overset{s}{\hookrightarrow} E(c,d) \quad &\Longleftrightarrow \quad \mathcal{C}_{ECH}\big(E(a,b)\big) \preccurlyeq \mathcal{C}_{ECH}\big(E(c,d)\big) \\ &\Longleftrightarrow \quad \mathcal{N}(a,b) \preccurlyeq \mathcal{N}(c,d). \end{aligned}$$

[11] A nice fact in symplectic geometry is that the complement of a line in $(\mathbb{C}P^2, \omega_a)$ can be identified with the open ball $\mathrm{int}B^4(a)$; equivalently, one can construct $(\mathbb{C}P^2, \omega_a)$ from the closed ball $B^4(a)$ by replacing its boundary 3-sphere $\partial B^4(a)$ by the 2-sphere $S^2(a) = \partial B^4(a)/S^1$ obtained by quotienting its boundary by the diagonal action of S^1 (whose orbits are the leaves of the characteristic foliation). This is closely related to the blow up construction discussed later.

Example 2.8. Consider the problem of embedding int$E(1,5)$ into the smallest possible ball $B^4(a)$. The EH capacities give the inequality

$$\begin{aligned}&(0,\ 1,\ 2,\ 3,\ 4,\ \ 5,\ \ 5,\ \ 6,\ \ 7,\ \ 8,\ \ 9,\dots) \preccurlyeq \\ &(0, a, a, 2a, 2a, 3a, 3a, 4a, 4a, 5a, 5a, \dots)\end{aligned}$$

which gives $a \geq 2$. This does not even get the volume obstruction, which is $5 \leq \sqrt{a}$. On the other hand the ECH capacities give

$$\begin{aligned}&(0,\ 1, 2,\ 3,\ 4,\ \ 5,\ \ 5,\ \ 6,\ \ 6,\ \ 7,\dots) \preccurlyeq \\ &(0, a, a, 2a, 2a, 2a, 3a, 3a, 3a, 3a, \dots)\end{aligned}$$

so that we need $5 \leq 2a$, i.e. $a \geq 5/2$ (which is the correct answer). Note also that the ECH capacities include information about volume: cf. [8].

The general theory of ECH capacities (finally established in Hutchings–Taubes [25]) implies that $\mathcal{N}(a,b) \preccurlyeq \mathcal{N}(c,d)$ is a necessary condition for the existence of an embedding. Therefore the main contribution in [32] was to prove the sufficiency of this condition, i.e. actually to construct embeddings. This is done by a rather indirect method that we will partially explain in the next lecture.

Using the basic existence result for ball embeddings stated in Proposition 3.2 below, Hutchings [24] proved a similar result for embeddings of unions of balls into a ball; see (2.5) for the capacities of such a union.

Theorem 2.9. *For all $a_i, 1 \leq i \leq k$, and $0 < c < d$, we have*

$$\bigsqcup_{i\geq 1} B(a_i) \overset{s}{\hookrightarrow} E(c,d) \iff \mathcal{C}_{ECH}\big(\bigsqcup_{i\geq 1} B(a_i)\big) \preccurlyeq \mathcal{C}_{ECH}\big(E(c,d)\big)$$

We know much less about embedding balls and ellipsoids in higher dimensions. Here are two notable recent results.

Remark 2.10. (i) Buse and Hind [4] manage to prove **packing stability** for balls B^{2n}: namely for any $N \leq \lceil (8\frac{1}{36})^{\frac{n}{2}} \rceil$ and any $\varepsilon > 0$ there is a symplectic embedding of N disjoint equal balls into B^{2n} that **fully fill** the ball in the sense that the image has volume $> (1-\varepsilon)\mathrm{Vol}B^{2n}$.

(ii) By improving Guth's embedding, Hind–Kerman [21] show that if $a > 3$ then for each $S > 0$ there is S' such that $E(1, S, \dots, S) \overset{s}{\hookrightarrow} E(a, a, S', \dots, S')$. Further if $a < 3$ and S is sufficiently large there is no such embedding. (Pelayo–Ngoc [41] sharpen this further, dealing with the case $a = 3$.) Note that the best bound given by $\mathcal{C}_{EH}$ is $a = 2$.

3. Lecture 3: Embedding ellipsoids and Fibonacci numbers

In this third lecture I will try to explain the some parts of proof of Hofer's conjecture Theorem 2.7, restated below, and also describe why Fibonacci numbers are relevant. Note that the methods of proof are ultimately based in gauge theory and hence work only in dimension 4.

Recall that $E(a,b)$ denotes the ellipsoid:

$$E(a,b) = \{(x_1,\dots,x_4) \in \mathbb{R}^4 : \pi\tfrac{x_1^2+x_2^2}{a} + \pi\tfrac{x_3^2+x_4^2}{b} \le 1\}.$$

Here is Hofer's conjecture, where $\mathcal{N}(a,b)$ lists the numbers $ma+nb, m,n \ge 0$ in increasing order with repetitions.

Theorem 3.1 (Hofer conjecture)**.** *For all $0<a\le b,\ 0<c\le d$, we have*

$$\begin{aligned} \mathrm{int}E(a,b) \overset{s}{\hookrightarrow} E(c,d) &\iff \mathcal{C}_{ECH}\big(E(a,b)\big) \preccurlyeq \mathcal{C}_{ECH}\big(E(c,d)\big) \\ &\iff \mathcal{N}(a,b) \preccurlyeq \mathcal{N}(c,d). \end{aligned}$$

In [24] Hutchings showed that $\mathcal{C}_{ECH}(a,b) = \mathcal{N}(a,b)$, and (modulo the cobordism argument in [25]) that the existence of an embedding implies that $\mathcal{N}(a,b) \preccurlyeq \mathcal{N}(c,d)$. On the other hand, [30] develops[12] a necessary and sufficient condition for one 4-dimensional ellipsoid to embed in another by the following two-step process:

- reduce the question to the question of embedding a certain disjoint union of balls into the fixed ball $B(d) = E(d,d)$ (where $d \ge c$), and then
- solve this ball embedding problem.

We will start with the second problem, since this was understood first.

3.1. Existence of ball embeddings.

The question of when a given union of k balls embeds into another ball had in fact been solved in the mid 1990s. However the terms of its solution involved understanding which cohomology classes in a k-fold blow-up[13] of $\mathbb{C}P^2$ are represented by symplectic forms, and hence are rather different from the considerations in ECH. Blowups are relevant here because in the symplectic world one can obtain a symplectic form on the one point blow up of a manifold by cutting out the interior of an embedded ball and then collapsing its boundary along the circle orbits of the characteristic flow to an exceptional divisor. In 4 dimensions this exceptional divisor is a symplectically embedded 2-sphere of self-intersection -1. Because its Gromov invariant is nonzero, it can be controlled by the theory of J-holomorphic curves. Conversely, given a form on the k-fold

[12] The proof of sufficiency had a gap that is now filled in [33, 34].

[13] In algebraic geometry the blow up at a point replaces that point by the set of all complex lines through that point. Thus in complex dimension n the set of all these lines forms a copy of $\mathbb{C}P^{n-1}$ called the exceptional divisor. In the symplectic world one essentially replaces the interior of a ball by the set of all closed orbits of the characteristic flow on tis boundary. As we explained in Lecture 1, each such orbit is the intersection with the ball boundary of a complex line.

blow up, one can remove k disjoint exceptional divisors inserting balls in their stead; cf. [35, Chapter 7].

This has been a very fruitful way to understand both embedding problems and some of the easier topological properties of symplectic 4-manifolds. In particular, in order to show that the union of k balls of sizes $w_1, \dots, w_m$ embeds in a symplectic manifold M it suffices to construct a suitable symplectic form on the k-point blow up of M whose integral over the ith exceptional divisor is w_i. It is not known how to do this unless M has dimension 4 and $b^+(M) = 1$.[14] In this case, one can embed k small balls are then increase their size by "symplectic inflation", a process that deforms the cohomology class of the symplectic form ω by adding appropriate forms that are supported near codimension 2 symplectic submanifolds of M. In 4 dimensions these submanifolds can be constructed as embedded J-holomorphic curves; the condition $b^+(M) = 1$ is needed in order to ensure that there are enough such curves. (Some 4-manifolds (M, J, ω), such as certain tori, have no closed J-holomorphic curves.) For further details and references, see [34].

We next explain a necessary and sufficient condition for a union of balls to embed in a single ball. Given k write $\mathbf{m} := (m_1, \dots, m_k) \in \mathbb{N}^k$, $m_i \geq m_{i+1}$. Let $\mathcal{E} := \cup_{k\geq 1}\mathcal{E}_k$ where

$$\mathcal{E}_k : \quad = \quad \big\{(d; \mathbf{m}) : d^2 + 1 = \sum_i m_i^2, \;\; 3d - 1 = \sum_i m_i, \;\; \text{and } (*)\big\}.$$

We shall refer to the first two conditions above as Diophantine conditions. The third condition $(*)$ is more algebraic, requiring that tuple $(d; \mathbf{m})$ can be reduced to $(0; 1)$ by repeated Cremona moves. Such a move takes $(d, \mathbf{m})$ to $(d'; \mathbf{m}')$, where $(d'; \mathbf{m}')$ is obtained by first transforming $(d; \mathbf{m})$ to

$$(2d - m_1 - m_2 - m_3; d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2, m_4, m_5 \dots),$$

and then reordering the new m_i (discarding zeros) so that they do not increase.

It was shown in Li–Li [27] (using Taubes–Seiberg–Witten theory for closed curves) that $\mathcal{E}_k$ is precisely the set of homology classes represented by symplectic exceptional divisors in the k-fold blow up of $\mathbb{C}P^2$, where we interpret $(d, \mathbf{m})$ as the class $dL - \sum m_i E_i$.[15] It follows that the intersection number $(d; \mathbf{m}) \cdot (d'; \mathbf{m}') := dd' - \sum m_i m_i'$ of any two elements in $\mathcal{E}$ is nonnegative. Presumably this could be proved by some algebraic argument. However, the only way I know to prove it is to use the "positivity of intersections" of J-holomorphic curves, which follows from Taubes–Seiberg–Witten theory.

[14]This means that the intersection form on $H^2(M)$ diagonalizes to $1, -1, -1, \dots, -1$.

[15]Here we denote by L the class of a line $[\mathbb{C}P^1]$ and by E_i the class of the ith exceptional divisor. Observe also that this is a place where symplectic geometry shows how flexible it is in comparison with algebraic geometry. In algebraic (or complex) geometry, one would want to describe the homology classes that can be represented by holomorphically embedded exceptional divisors for a "generic" complex structure on the blow up. This question is far from being understood.

The following criterion was developed by McDuff–Polterovich and Biran (for precise references, see [30]).

Proposition 3.2.
$$\bigsqcup_{i\le k} B(w_i) \overset{s}{\hookrightarrow} B(\mu) \iff a<\mu^2 \text{ and } \mu d \ge \sum m_i w_i,\ \forall (d;\mathbf{m})\in \mathcal{E}_k.$$

Remark 3.3. In [24], Hutchings used ECH capacities to give a potentially more stringent criterion for such a ball embedding, replacing the set $\mathcal{E}_k$ above by the larger set consisting of all classes $E = dL-\sum m_iE_i$ for which $d^2+3d \ge \sum m_i^2+m_i$. However, [32, Proposition 3.2] gives a purely algebraic argument showing that Hutchings' criterion in fact agrees with the one above.

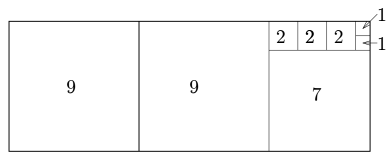

FIGURE 3.1. $\mathbf{w}(25/9) = (1,\ 1,\ 7/9,\ 2/9,\ 2/9,\ 2/9,\ 1/9,\ 1/9)$.

3.2. **From ellipsoids to balls.** We next explain which sets of balls correspond to an ellipsoid. Given mutually prime integers $p>q$ define their **weight expansion** to be the ordered tuple with repetitions obtained by the following process.

Definition 3.4. *Let $a=p/q\in\mathbb{Q}$ written in lowest terms. The* **weight expansion** $\mathbf{w}(a):=(w_1,\dots w_k)$ *of $a\ge 1$ is defined recursively as follows:*

- *$w_1=1$, and $w_n\ge w_{n+1}>0$ for all n;*
- *if $w_i>w_{i+1}=\cdots=w_n$ (where we set $w_0:=a$) then*

$$w_{n+1}=\begin{cases} w_n & \text{if } w_{i+1}+\cdots+w_{n+1}=(n-i+1)w_{i+1}\le w_i\\ w_i-(n-i)w_{i+1} & \text{otherwise;}\end{cases}$$

- *the sequence stops at w_n if the above formula gives $w_{n+1}=0$.*

Thus the weights form a decreasing sequence of numbers, whose *multiplicities* give the coefficients of continued fraction representation of a. Here are some examples.

Example 3.5. $w(8/5)=(1,3/5,2/5,1/5,1/5)$ with multiplicities $1,1,1,2$. This corresponds to the continued fraction expansion $8/5=[1;1,1,2]$ i.e.
$$\frac{8}{5}=\mathbf{1}+\cfrac{1}{\mathbf{1}+\frac{1}{\mathbf{1}+\frac{1}{\mathbf{2}}}}.$$

Similarly in Figure 3.1 the multpliciites for $25/9$ are $2,1,3,2$, and we have
$$\frac{25}{9}=\mathbf{2}+\cfrac{1}{\mathbf{1}+\frac{1}{\mathbf{3}+\frac{1}{\mathbf{2}}}}.$$

Cutting up ellipsoids into balls

Here is the main result that relates ball embeddings to the problem of embedding ellipsoids.

Proposition 3.6. *For each rational $a = p/q \geq 1$, $E(1, a)$ embeds symplectically in the interior of $B^4(\mu)$ if and only if the disjoint union of balls*

$$\bigsqcup_{i \leq M} B(w_i)$$

embeds symplectically in the interior of $B^4(\mu)$, where $\mathbf{w}(a) = (w_1, \ldots, w_M)$ is the weight expansion of a.

The "only if" part of the above statement is easy if we use **toric models**.

Describing embeddings by toric models:

Symplectic toric manifolds are $2n$-manifolds with a Hamiltonian action of an n-torus $(S^1)^n$. This means that there are n functions $H_i : M \to \mathbb{R}$ whose Hamiltonian flows commute. These fit together to form a map $\Phi : M \to \mathbb{R}^n$, called the **moment map**. It turns out that the moment image $\Phi(M)$ is always a convex polytope. Moreover, a celebrated theorem of Delzant says that M is completely determined by this polytope $\Phi(M)$, modulo translations and linear changes of basis in $\mathbb{R}^n$ by matrices in $\mathrm{GL}(n, \mathbb{Z})$. (These basis changes correspond to changing the chosen basic for the torus $\mathbb{T}^2 = S^1 \times S^1$.) Thus the symmetry group here is $\mathrm{Aff}(\mathrm{GL}(n, \mathbb{Z}))$, the affine general linear group over $\mathbb{Z}$.

As we saw in Lecture 1, the rotation action of S^1 on $\mathbb{C}$ is generated by the function $z \mapsto \pi|z|^2$. Similarly, the action of $(S^1)^2$ on $\mathbb{C}^2$ is generated by the map $\Phi : (z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$. Therefore the image $\Phi(E(1, a))$ is the **triangle** with vertices $(0, 0), (0, 1), (a, 0)$. One should be a little careful about what happens to the boundary, but let's ignore this point. Lisa Traynor showed how how to embed the interior of the ball $B^4(1)$ into $\Phi^{-1}(T_0)$ where $T_0 \subset \mathbb{R}^2$ is an *open* triangle with vertices $(0, 0), (0, 1), (1, 0)$. Any triangle equivalent under the action of $\mathrm{Aff}(\mathrm{GL}(n, \mathbb{Z}))$ to a rescaling of this one is called a **standard triangle**.

Thus we can obtain full fillings[16] of $B^4(1)$ by d^2 equal balls by cutting a standard triangle into d^2 standard pieces as in Figure 3.2.

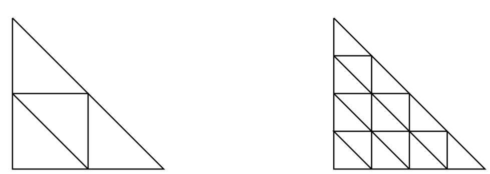

FIGURE 3.2. Embedding d^2 standard triangles into a triangle of size 1.

[16]There are several slightly different notions of full filling. But it suffices here to require that the volume of the domain (which is a disjoint union of open sets) equals the volume of the target.

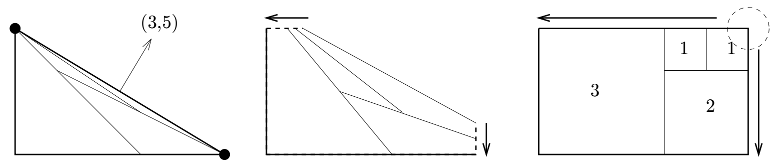

FIGURE 3.3. Cutting triangles into standard triangles as compared to cutting a rectangle into squares. Reading the pictures right to left, one sees that the triangle decomposition is structurally the same as that giving the weights for 5/3: remove the top right vertex of the rectangle and then collapse its top to a point on the y axis and its right side to a point in the x axis.

The geometric definition of a weight expansion involves cutting up a rectangle into squares. But, by Figure 3.3, one can equally well interpret it in terms of cutting up an arbitrary triangle into standard triangles of the appropriate sizes. Thus if the weight expansion of $E(1,a)$ is $\mathbf{w} = (w_1, \dots, w_M)$, one can directly embed M standard triangles of sizes w_i into $\Phi(E(1,a))$, and hence can lift this to an embedding of $\sqcup_i B^4(w_i)$ into $E(1,a)$. As we explain in more detail in [30], it also corresponds to a joint resolution of the two singularities of the toric variety corresponding to the *complement* of the triangle in the positive quadrant. This proves one direction of Proposition 3.6. The other direction is explained in the simple case of $\text{int}E(1,4) \overset{s}{\hookrightarrow} B^4(2)$ in [30]. In general the proof uses relative inflation techniques developed in [34].

3.3. **Calculating the capacity function for ellipsoids into balls.** The connection of the ellipsoidal embedding problem with continued fractions and Fibonacci numbers comes both when when one cuts an ellipsoid up into balls, and when one calculates what the numerical conditions $\mathcal{N}(a,b) \preccurlyeq \mathcal{N}(c,d)$ actually mean in practice.[17]

I will now explain this second point.

Together with Schlenk I calculated the following **embedding capacity** function in [37]:

$$c(a) := \inf\{\mu : E(1,a) \overset{s}{\hookrightarrow} B(\mu)\}.$$

Note that $c(a) \geq \sqrt{a}$ because $\text{vol}\, E(1,a) = \text{vol}\, B(\sqrt{a})$. Clearly the function c is nondecreasing. Moreover, because $E(\lambda a, 1) \subset E(\lambda, \lambda a) =: \lambda(E(1,a))$ for $\lambda \geq 1$, it is easy to see that $c(\lambda a) \leq \lambda c(a)$; in other words the function $a \mapsto \frac{c_{ECH}(a)}{a}$ is nonincreasing. We call this the *scaling property.*

[17] The fact that the sequence $\mathcal{N}(a,b)$ encodes a lot of information about the continued fraction expansion of a/b is not surprising when one sees how they arise in ECH picture. Here they are closely related to the ECH index which involves counting lattice points in triangles.

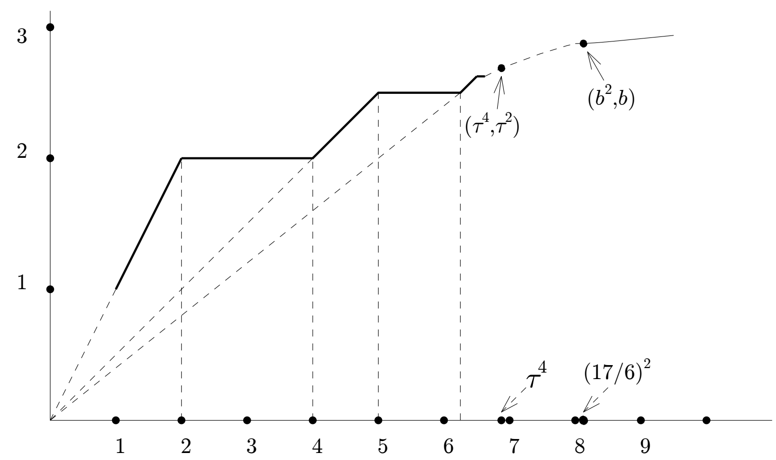

FIGURE 3.4. The approximate graph of $c(a)$. Here $b := \frac{17}{6}$.

Theorem 3.7 (McDuff–Schlenk)**.** *Let* $\tau = \frac{1+\sqrt{5}}{2}$*. The graph of* $c(a)$ *divides into three parts:*

- *if* $1 \le a < \tau^4$ *the graph is piecewise linear – an infinite Fibonacci staircase converging to* (τ^4, τ^2)*;*
- $\tau^4 \le a < 8\frac{1}{36}$ *is a transitional region;* $c(a) = \sqrt{a}$ *except on a finite number of short intervals; further* $c(a) = \frac{a+1}{3}$ *for* $\tau^4 \le a \le 7$*;*
- *if* $a \ge 8\frac{1}{36} = \left(\frac{17}{6}\right)^2$ *then* $c(a) = \sqrt{a}$*.*

Description of the Fibonacci stairs: Let

$$g_1 = 1, \;\; g_2 = 2, \;\; g_3 = 5, \;\; g_4 = 13, \;\; g_5 = 34, \;\; g_6 = 89, \ldots,$$

be the odd terms in the sequence of Fibonacci numbers; set

$$a_n := \big(g_{n+1}/g_n\big)^2, \;\; b_n := g_{n+2}/g_n$$

so that $a_n < b_n < a_{n+1}$, and $a_n \to \tau^4$. Here we set $g_0 := 1$ for convenience, so that $a_0 = 1, b_0 = 2$. Then the claim is that

$$c(x) = x/\sqrt{a_n} \text{ on } [a_n, b_n], \quad \text{and} \quad c(x) = \sqrt{a_{n+1}} \text{ on } [b_n, a_{n+1}].$$

Since $\frac{b_n}{\sqrt{a_n}} = \sqrt{a_{n+1}}$ this gives a continuous graph on the interval $1 \le a < \tau^4$. For example,

$$c(2) = c(b_0) = \sqrt{a_1} = 2 = c(4), \quad c(5) = c(b_1) = \tfrac{5}{2} = c(6\tfrac{1}{4}).$$

It is not hard to check that there is a *unique* continuous function c on $[1, \tau^4]$ that satisfies the scaling property and has the above values at the points a_n, b_n. Hence, if an alternate way could be found to calculate these values, one would know c on $[1, \tau^4]$. This is precisely the approach used by Christofaro–Gardiner and Kleinman in [9]. They recently found a purely combinatorial calculation of $c(a_n), c(b_n)$ using the relation (explained in [37, 9]) between the numbers $\mathcal{N}(a,b)$ and the problem of counting lattice points in triangles.[18]

Fibonacci numbers came up in the calculations in [37] for a completely different, and rather unexpected reason that we now explain.

Notice first that Propositions 3.2 and 3.6 have the following immediate corollary.

Lemma 3.8. $c(a) = \sup\Big\{\sqrt{a}, \sup\limits_{(d,\mathbf{m})\in\mathcal{E}_k} \frac{\sum m_i w_i(a)}{d}\Big\}$, *where* $\mathcal{E}_k$ *is the set of classes in the* k *fold blow up* $\mathbb{C}P^2$ *represented by exceptional spheres.*

Proof. This holds because we need the inner product

$$(\mu, \mathbf{w}(a)) \cdot (d, \mathbf{m}) = \mu d - \sum a_i m_i \geq 0, \quad \forall (d, \mathbf{m}) \in \mathcal{E},$$

where the inner product has type $(1, -k)$. Thus we need $\mu \geq \frac{\sum m_i w_i(a)}{d}$ for all $(d, \mathbf{m}) \in \mathcal{E}$. □

Example 3.9. $\mathcal{E}_4$ has the single element $(1;1,1)$ (corresponding to $L - E_1 - E_2$) and $\mathbf{w}(4) = (1,\,1,\,1,\,1) = (1^{\times 4})$. Therefore $c(4) = 2$. However $\mathcal{E}_5$ also contains $(2;1,\dots,1) = (2;1^{\times 5})$ corresponding to $2L - \sum_{i=1}^5 E_i$. Thus $c(5) = \sup\{\sqrt{5}, 2, 5/2\} = 5/2$. Compare with Example 2.8

Therefore, the calculation in [37] of the capacity function c is based on analyzing which elements $E \in \mathcal{E}$ give obstructions. We need to find classes $E \in \mathcal{E}_k$ such that $\frac{\sum m_i w_i(a)}{d}$ is large when $(d, \mathbf{m})$ is arbitrary and $\mathbf{w}$ is the weight expansion of a. In particular, the existence of the Fibonacci staircase in Theorem 3.7 is based on the somewhat surprising discovery that there are elements of $\mathcal{E} \subset H_2(\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ related to the weight expansions of the numbers $a = \frac{g_{n+2}}{g_n}$ that give the corners of the steps.[19] More precisely, we have:

Proposition 3.10. *Denote the odd Fibonacci numbers by* $g_n, n \geq 1$*, as above, and define* $a_n := (g_{n+1}/g_n)^2$ *and* $b_n := g_{n+2}/g_n$*. Then:*
(i) $E(b_n) := \big(g_{n+1}; g_n \mathbf{w}(b_n)\big) \in \mathcal{E}$*. Moreover, numbers of the form* b_n *are the unique numbers* p_n/q_n *for which there is some* d *with this property, i.e. so that* $\big(d; q_n \mathbf{w}(p_n/q_n)\big) \in \mathcal{E}$. (ii) $E(a_n) := \big(g_n g_{n+1}; g_n^2(\mathbf{w}(a_n)), 1\big) \in \mathcal{E}$.

Idea of proof. It is not hard to show that the elements $E(a_n)$ and $E(b_n)$ satisfy the Diophantine conditions needed to be in $\mathcal{E}$. Moreover, there is an easy inductive proof that $E(b_n)$ satisfies the third condition since $E(b_n)$ reduces to $E(b_{n-2})$ under five Cremona moves. However, the corresponding behavior of $E(a_n)$ is much more complicated. □

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[18] In fact it turns out that triangles whose vertices are the rational points $(0,0), (\frac{g_{n+1}}{g_n}, 0)$, and $(0, \frac{g_n}{g_{n+1}})$ are very interesting from this point of view, because they provide new examples of period collapse for Erhart polynomials.

[19] Since the structure of the lattice $H_2(\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ when $k > 9$ is still rather mysterious, this fact might have relevance to other problems. Note also, that in contrast to our method, in [9] it was the numbers $a = \big(\frac{g_{n+1}}{g_n}\big)^2$ at the inside of the steps that were interesting.

In fact, as we pointed out in Remark 3.3 one does *not* in fact need to check that E satisfies $(*)$ in order to use it to bound $c(a)$. On the other hand, the calculation of $c(a)$ for $a \in (\tau^4, 7]$ is very delicate, relying on a study of auxiliary elements of $\mathcal{E}$, ones that do not themselves provide obstructions but which prevent the existence of other obstructions because of positivity of intersections, i.e. the fact that $E \cdot E' \geq 0$ for $E, E' \in \mathcal{E}$. The situation becomes easier to analyze as $a - \tau^4$ gets larger. Indeed, when $a \geq (\frac{17}{6})^2$ a fairly simple analytic argument shows that no elements of $\mathcal{E}$ can give obstructions more stringent than the volume obstruction.

References

[1] M. Abouzaid, Framed bordism and Lagrangian embeddings of exotic spheres, arXiv:0812.4781.

[2] M. Abouzaid and T. Kragh, On immersion classes of nearby Lagrangians, arXiv:1305.6810.

[3] K. Artstein–Avidan, R. Karasev, and Ya. Ostrover, From symplectic measurements to the Mahler conjecture, arXiv:1303.4197.

[4] O. Buse and R. Hind, Ellipsoidal embeddings and symplectic packing stability, *Compos. Math.* (2013) **149**, 889–902.

[5] K. Cieliebak and Ya. Eliashberg, *From Stein to Weinstein and back: Symplectic geometry of affine couple manifolds*, Colloquium Publications, Volume **59**, American Mathematical Society, Providence, RI. (2012)

[6] K. Cieliebak and Ya. Eliashberg, Flexible Weinstein manifolds, arXiv:1305.1635.

[7] K. Cieliebak, H. Hofer, J. Latschev and F. Schlenk, Quantitative symplectic geometry, arXiv:math/0506191, *Dynamics, Ergodic Theory, Geometry MSRI*, **54** (2007), 1–44.

[8] D. Cristofaro-Gardiner, V. Gripp and M. Hutchings, The asymptotics of ECH capacities, arXiv:1210.2167.

[9] D. Cristofaro-Gardiner and A. Kleinman, Ehrhard polynomials and symplectic embeddings of ellipsoids. arXiv:1307.5493.

[10] S.K. Donaldson, Symplectic submanifolds and almost-complex geometry, *J. Diff. Geom.* **44** (1996), 666–705.

[11] I. Ekeland and H. Hofer, Symplectic topology and Hamiltonian dynamics, *Mathematische Zeitschrift*, **200** (1989), 355–78.

[12] Y. Eliashberg, Rigidity of symplectic and contact structures. Abstracts of reports to the 7th Leningrad International Topology Conference, (1982).

[13] Ya. Eliashberg, Topological characterization of Stein manifolds of dimension > 2. *International Journal of Mathematics*, (1990). **1**, 29–47.

[14] Ya. Eliashberg and N. Mishachev, *An introduction to the h-principle*, Graduate Texts # 48, A.M.S. 2002.

[15] R. Fintushel and R. Stern, Knots, links and 4-manifolds, *Invent. Math.* **134** (1998), 363–400.

[16] A. Floer, H. Hofer and K. Wysocki, Applications of symplectic homology I. *Mathematische Zeitschrift* **217** (1994), 577–606.

[17] D. Frenkel and D. Müller, Symplectic embeddings of 4-dimensional ellipsoids into cubes, arXiv:1210.2266

[18] H. Geiges (2008) *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics **109**, CUP, Cambridge, UK.

[19] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, *Inventiones Mathematicae*, **82** (1985), 307–47.

[20] L. Guth, Symplectic embeddings of polydisks, *Invent. Math.* **172** (2008), 477–489.

[21] R. Hind and E. Kerman, New obstructions to symplectic embeddings, arXiv:0906:4296.

[22] R. Hind and S. Lisi, Symplectic embeddings of polydisks, arXiv:1304.1452

[23] H. Hofer and E. Zehnder, A new capacity for symplectic manifolds, in *Analysis et cetera* (ed. P.H. Rabinowitz and E. Zehnder), pp. 405–29. Academic Press, New York, (1990).

[24] M. Hutchings, Quantitative embedded contact homology, arXiv:1005.2260, to appear in *J. Differential Geometry*.

[25] M. Hutchings and C. H. Taubes, Proof of the Arnold Chord conjecture in three dimensions, II, to appear in *Geom. and Topology* (2012)

[26] Bang-He Li and T.-J. Li, Symplectic genus, minimal genus and diffeomorphisms, *Asian J. Math.* **6** (2002), 123-44. MR1902650, Zbl 1008.57024

[27] Tian-Jun Li and A. K. Liu, Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4- manifolds with $b^+ = 1$, *J. Diff. Geom.* **58** (2001), 331–70. MR1913946 (Zbl 1051.57035

[28] D. McDuff, Examples of symplectic structures, *Invent. Math.* **89** (1987), 13–36.

[29] D. McDuff, Applications of convex integration to symplectic and contact geometry, *Ann. Inst. Fourier, Grenoble* **37** vol 1, (1987), 107–133

[30] D. McDuff, Symplectic embeddings of 4-dimensional ellipsoids, *J. Topol.* **2** (2009), 1–22. MR2499436, Zbl 1166.53051

[31] D. McDuff, Symplectic embeddings and continued fractions: a survey, arxiv:0908.4387, *Journ. Jap. Math. Soc.* **4** (2009), 121–139. MR2576029, Zbl 05675980

[32] D. McDuff, The Hofer conjecture on embedding symplectic ellipsoids. *Journal of Differential Geometry*, (2011). **88**, (2011), 519–532.

[33] D. McDuff, Symplectic embeddings of 4-dimensional ellipsoids: erratum, arXiv:1305.0230.

[34] D. McDuff and E. Opshtein, Nongeneric J-curves and singular inflation, arXiv:1309.6425

[35] D. McDuff and D.A. Salamon, *Introduction to Symplectic Topology*, 2nd edition, Oxford University Press, 1998. (3rd edition in progress)

[36] D. McDuff, D. and D. A. Salamon, *J-holomorphic Curves and Symplectic Topology, Second Edition*. Colloquium Publications, Volume **52**, (2012), American Mathematical Society, Providence, RI.

[37] D. McDuff and F. Schlenk, The embedding capacity of 4-dimensional symplectic ellipsoids, *Annals of Math.*,

[38] M. McLean, Lefschetz fibrations and symplectic homology, *Geom. Topol.* bf 13 (2009), no.4, 1877-1944.

[39] E. Murphy, Loose Legendrian embeddings in high dimensional contact manifolds, arXiv: 1201.2245.

[40] A. Pelayo and S. V. Ngoc, The Hofer question on intermediate capacities, arXiv:1210.1537.

[41] A. Pelayo and S. V. Ngoc, Sharp symplectic embeddings of cylinders, arXiv:1304.5250.

[42] C.H. Taubes, The Seiberg–Witten invariants and symplectic forms. *Mathematical Research Letters*, (1994) **1**, 809–822.

[43] C. Viterbo, Metric and isoperimetric problems in symplectic geometry, *J. Amer. Math. Soc.* **13** (2000), 411-431.

[44] K. Wehrheim and C. Woodward, Floer Cohomology and Geometric Composition of Lagrangian Correspondences, Preprint, 2009. arXiv:0905.1368

[45] A. Weinstein, Symplectic categories, arXiv:0911.4133

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