A tale of rigidity and flexibility: discrete subgroups of higher rank Lie groups

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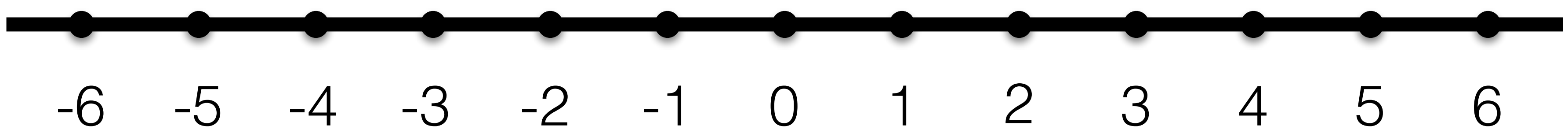
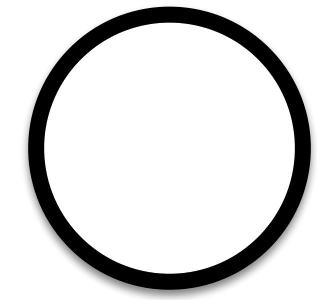
Discrete subgroups

The integers $\mathbb{Z} \subset \mathbb{R}$



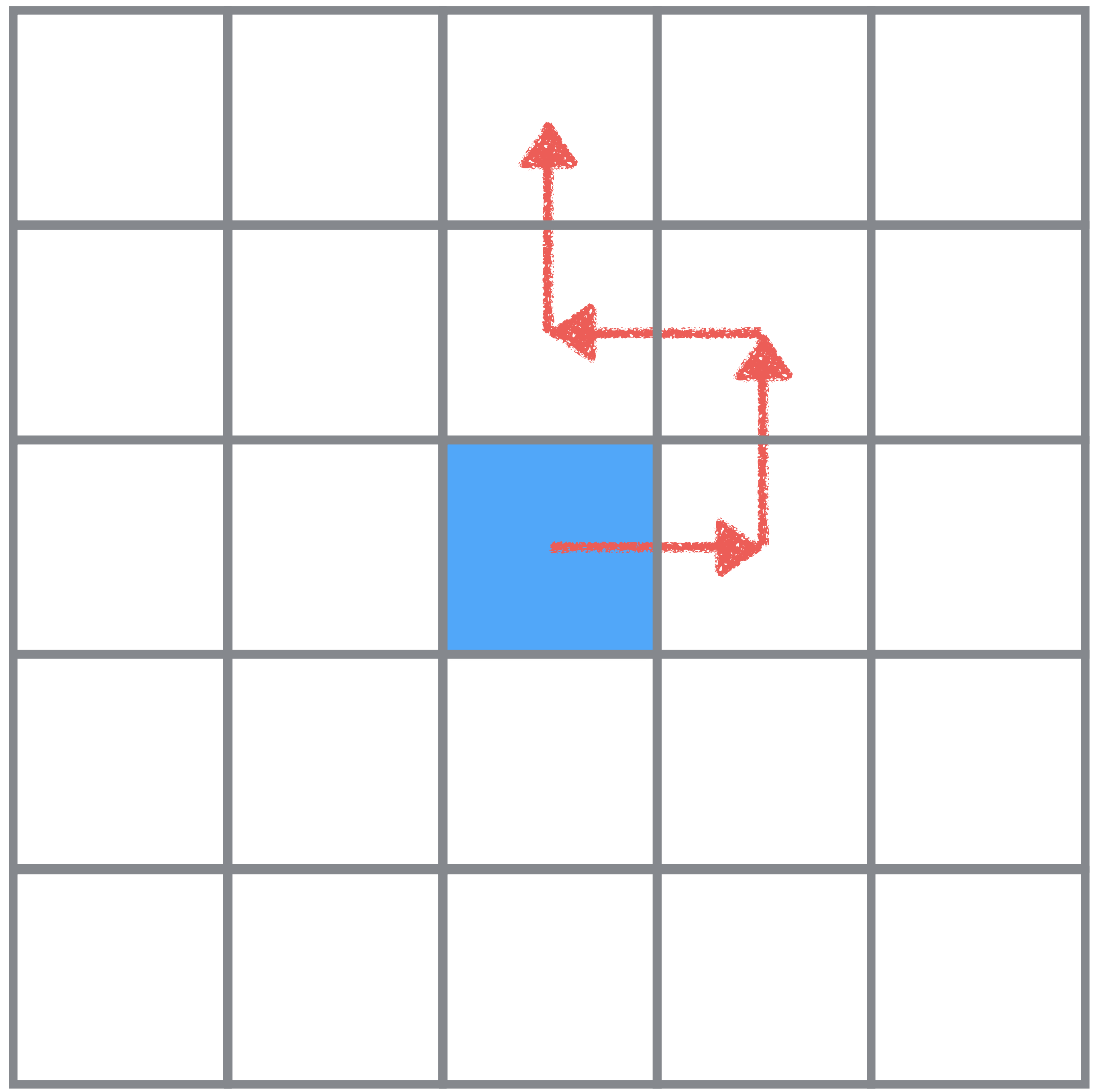


The integer lattice

$\mathbb{Z}^2 \subset \mathbb{R}^2$

$$(x, y) + (z, w) = (x + z, y + w)$$

acts on $\mathbb{R}^2$

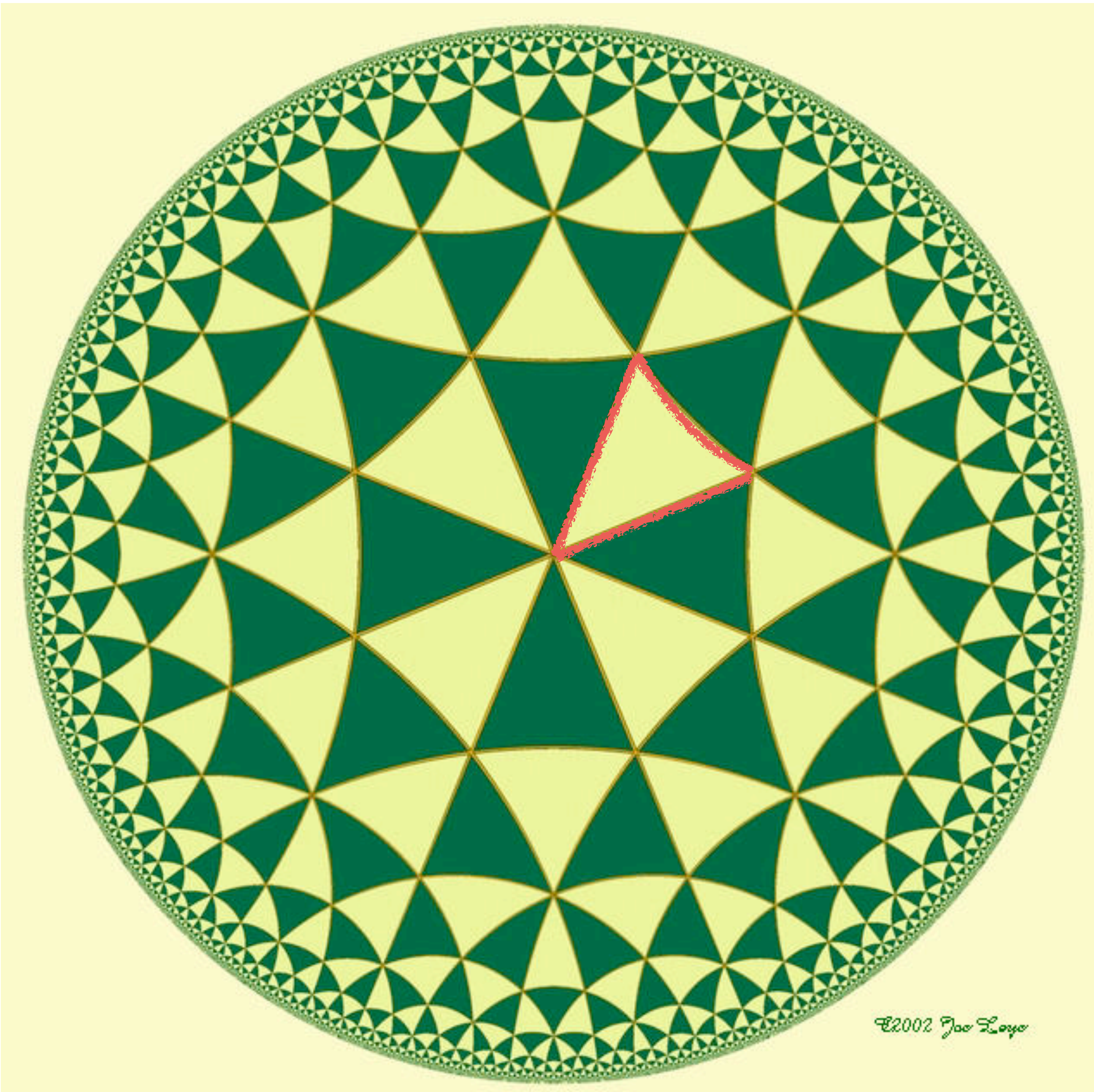


discreteness

=

good
quotient spaces

Symmetries of tilings



Poincaré model of the hyperbolic plane

$$\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$$

Hyperbolic isometries are Möbius transformations that preserve the disk

$$\mathrm{SU}(1,1) \subset \mathrm{SL}(2,\mathbb{C})$$

$$\mathrm{SU}(1,1) = \left\{ \left(\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \right) \,|\, |a|^2 - |b|^2 = 1 \right\}$$

acting by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

The subgroup Γ preserving the tiling

is a discrete subgroup of $\mathrm{SU}(1,1)$

Arithmetic construction

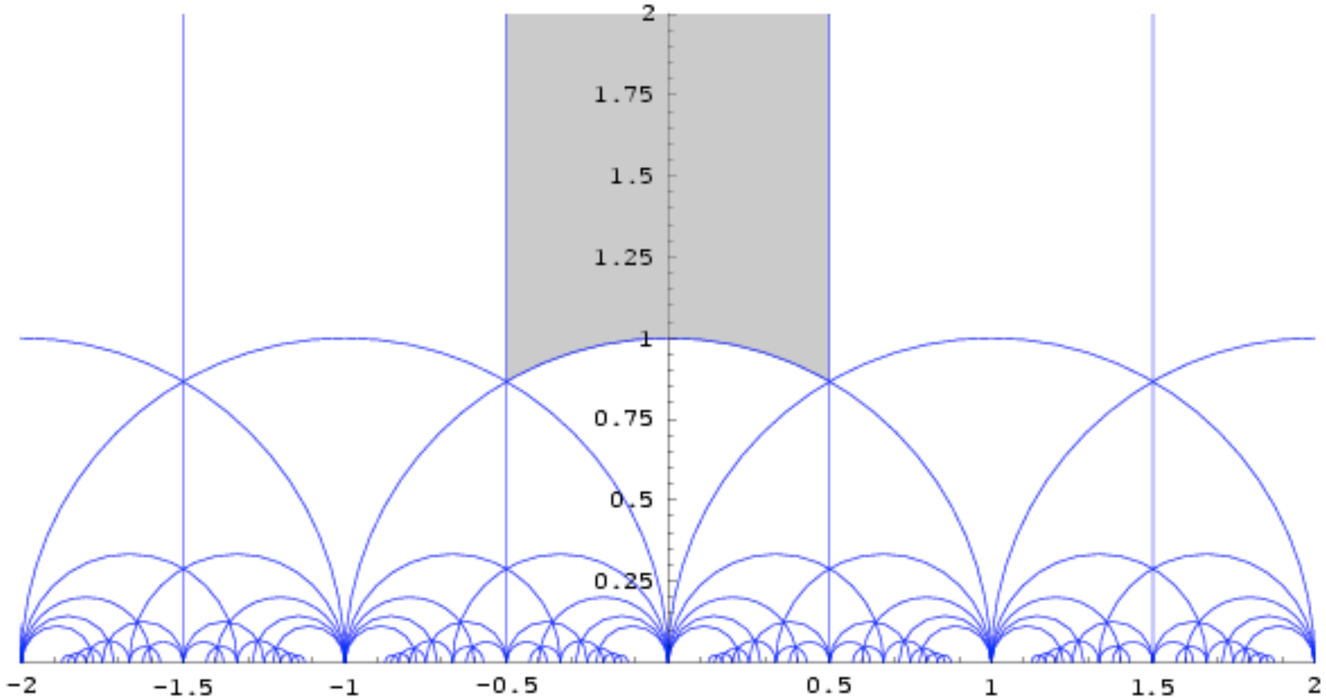
Taking integer points of matrix groups gives rise to discrete subgroups.

An example is $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$, the modular group.

It acts by fractional linear transformations on the upper half plane

$$\mathbf{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$$

The quotient is not compact,
but of finite hyperbolic volume.



For other matrix groups we can consider the following examples:

$\mathrm{SL}(n,\mathbb{Z}) \subset \mathrm{SL}(n,\mathbb{R})$ $\quad$ $\mathrm{Sp}(2n,\mathbb{Z}) \subset \mathrm{Sp}(n,\mathbb{R})$ $\quad$ $\mathrm{SO}(q,\mathbb{Z}) \subset \mathrm{SO}(q)$

$\mathrm{SL}(n,\mathbb{Z}[i]) \subset \mathrm{SL}(n,\mathbb{C})$ $\quad$ $\mathrm{Sp}(2n,\mathbb{Z}[i]) \subset \mathrm{Sp}(n,\mathbb{C})$

Tilings and integer points give rise to **lattices**: discrete subgroups of finite covolume.

The tale about lattices

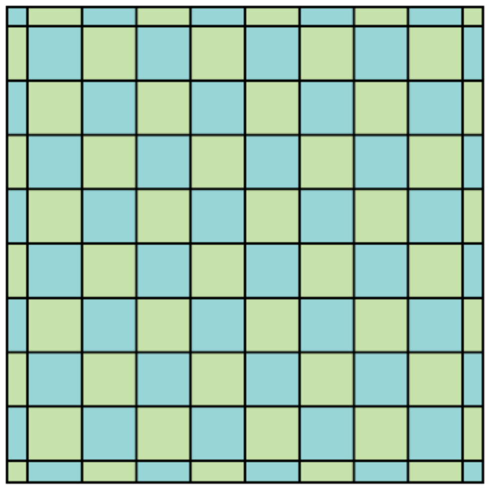
Lattices are “fat” and “tame”.

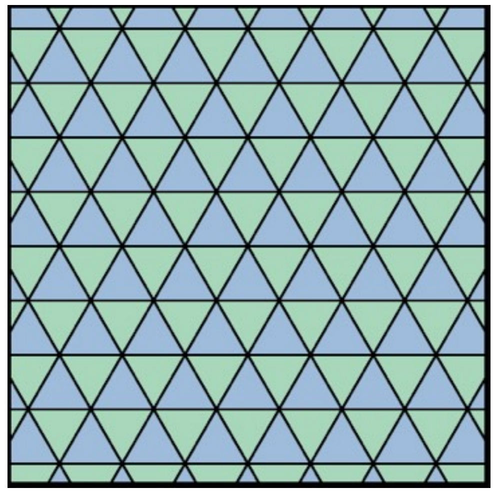
They are of finite covolume. They are finitely generated.
They are fundamental groups of interiors of manifolds with boundary.

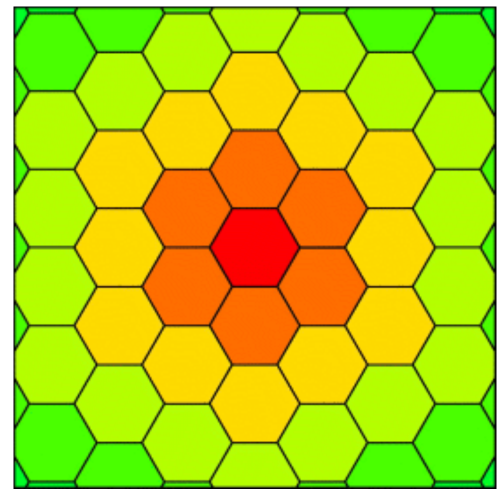
Classification | Moduli spaces | Rigidity

<u>Classification</u>

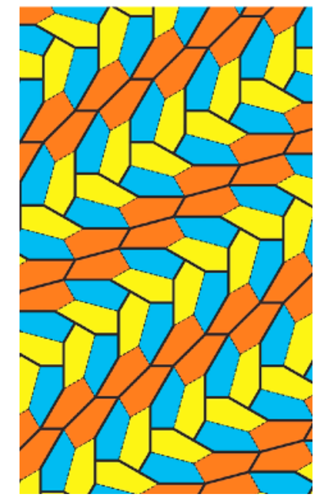
Hilbert: Classify all symmetry groups of periodic tilings of the Euclidean space



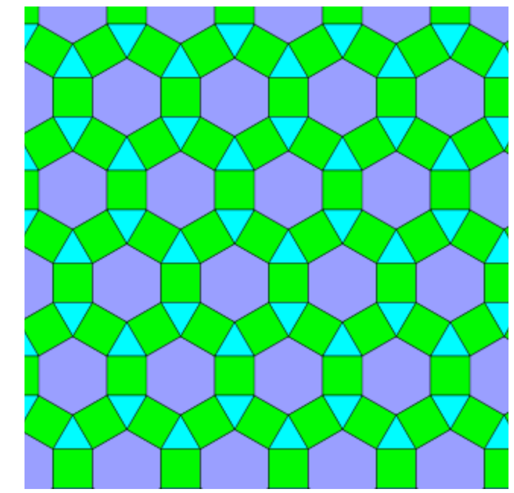




R.A. Nonnenmacher



Casey Mann



Fibonacci

The symmetry group of a periodic tiling of the Euclidean space of dimension n is called a crystallographic group of dimension n. A crystallographic group is a cocompact lattice in the group of isometries of the Euclidean space.

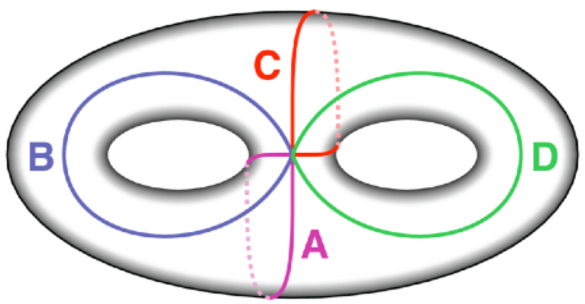
Bieberbach 1: Any crystallographic group of dimension n contains a subgroup of finite index which is isomorphic to $\mathbb{Z}^n$.

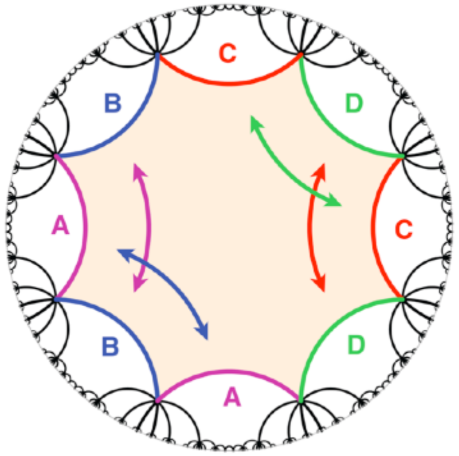
Bieberbach 2: There exists a finite number of isomorphism classes of crystallographic groups of dimension n.

This gives a complete classification of cocompact lattices in $\mathrm{Isom}(\mathbb{R}^n)$.

Moduli space of surfaces

Let S be a surface of genus g > 1. Endow S with a hyperbolic metric.





The fundamental group

$$\pi_1(S_g) = \{A_1, B_1, \cdots, A_g, B_g \mid A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = 1\}$$

acts as a hyperbolic crystallographic group.

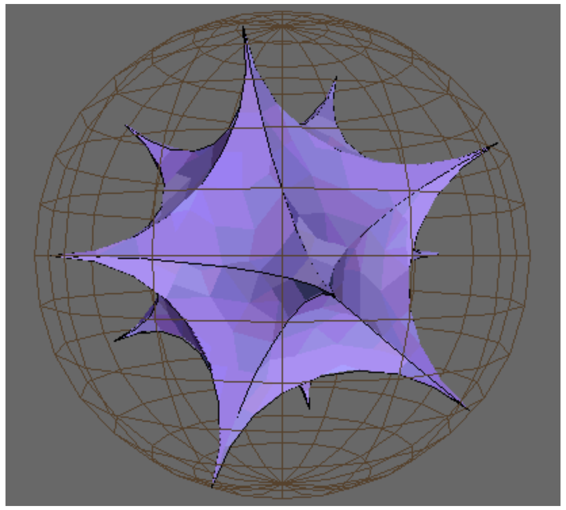
Poincaré: Any torsionfree hyperbolic crystallographic group of dimension 2 is isomorphic to $\pi_1(S_g)$ for some g > 1.
For $g \neq h$, the groups $\pi_1(S_g)$ and $\pi_1(S_h)$ are not isomorphic.

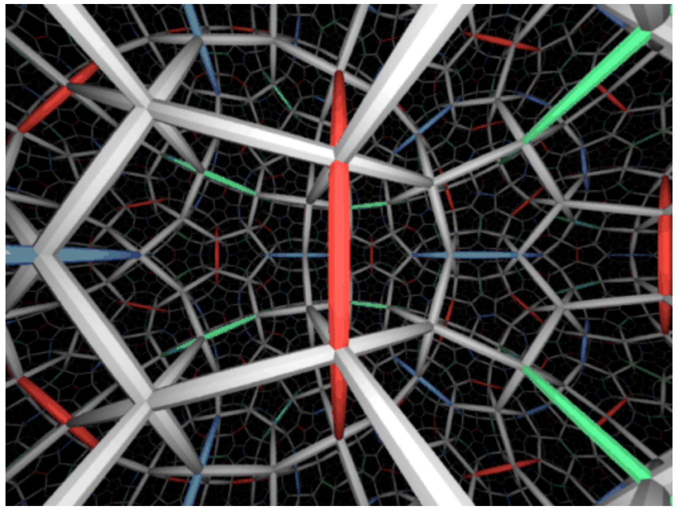
Teichmüller: For any g, the space of isomorphisms of $\pi_1(S_g)$ with a discrete subgroup
Fricke: in $\mathrm{SU}(1,1) \cong \mathrm{SL}(2,\mathbb{R})$ (up to conjugation) is $\mathbb{R}^{6g-6}$.

Fundamental groups of surfaces are flexible. There is a rich moduli space of different hyperbolic structures on a given topological surface!

Geometrizing hyperbolic manifolds

Consider now a compact hyperbolic manifold of dimension n > 2.





The group of isometries of the hyperbolic space of dimension n is $\mathrm{SO}(1,n)$, linear transformation of $\mathbb{R}^{n+1}$ preserving the quadratic form $q_{1,n}(x) = -x_0^2 + \sum_{i=1}^{n} x_i^2$.

Mostow rigidity: Let $\pi_1(M)$ be the fundamental group of a hyperbolic manifold of dim n>2. Then there exists a unique isomorphism of $\pi_1(M)$ with a cocompact lattice in $\mathrm{SO}(1,n)$.

Cocompact lattices in $\mathrm{SO}(1,n)$, n>2, are rigid in $\mathrm{SO}(1,n)$.

The topology determines the geometry!

But, cocompact lattices in $\mathrm{SO}(1,n)$, can be deformed in $\mathrm{SO}(1,m)$ for m>n.

Superrigidity in higher rank

The hyperbolic space is the symmetric space associated to $\mathrm{SO}(1,n)$.
It has negative curvature, which is a feature of the Lie group $\mathrm{SO}(1,n)$ being of rank one.

The Lie groups $\mathrm{SL}(n+1,\mathbb{R})$ or $\mathrm{Sp}(2n,\mathbb{R})$ are of higher rank if n>1.
Their symmetric spaces are non-positively curved, but not negatively curved.
They have totally geodesic and isometrically embedded Euclidean planes.

Example: $X_{n+1} = \mathrm{SL}(n+1,\mathbb{R})/\mathrm{SO}(n+1)$ is the space of scalar products on $\mathbb{R}^{n+1}$.
The subspace of scalar products for which the standard basis vectors of $\mathbb{R}^{n+1}$ stay orthogonal is a subspace of X_{n+1} which is isometric to $\mathbb{R}^n$.

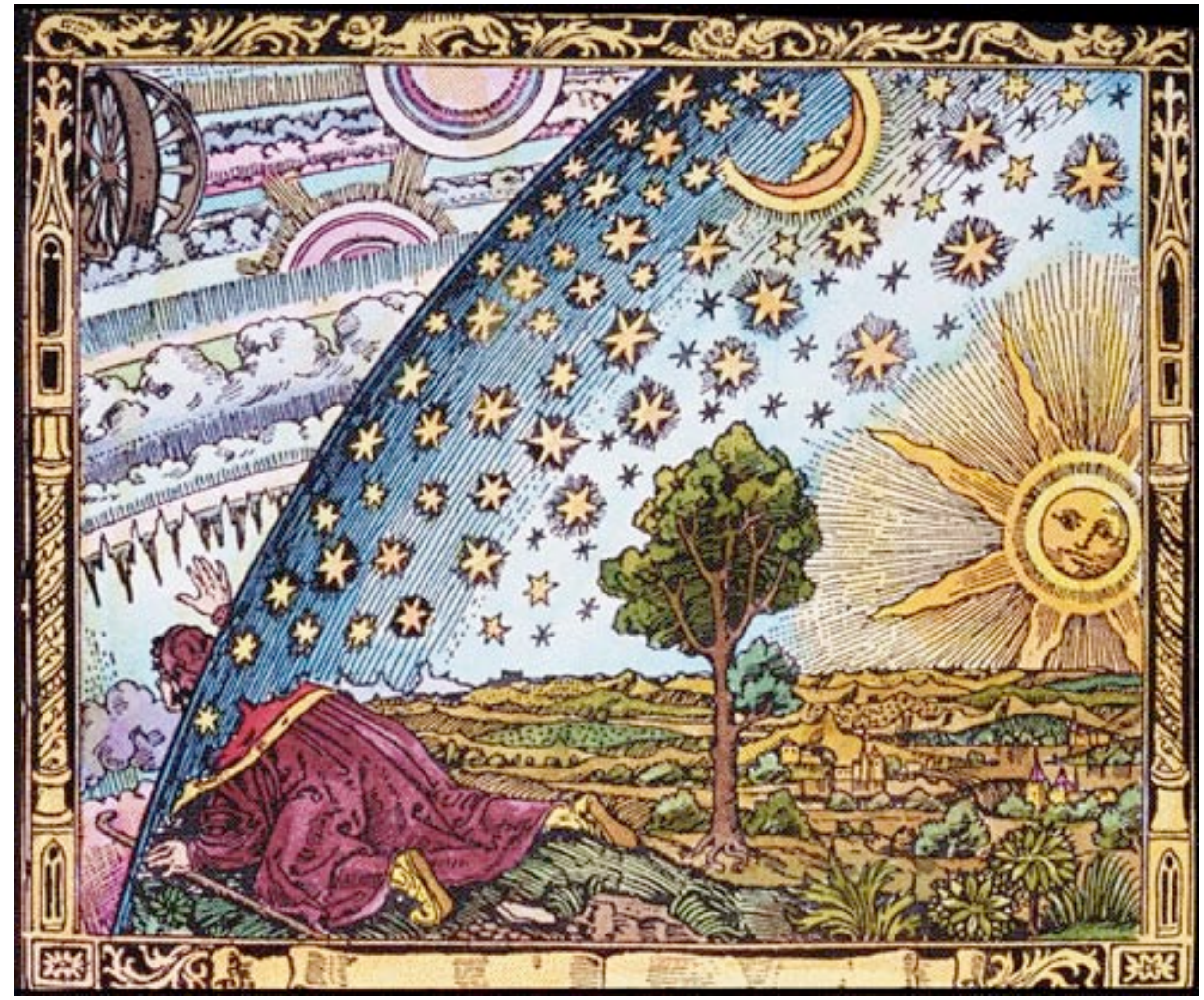
Margulis superrigidity: Let Γ be a lattice in a simple Lie group G of higher rank.
Then any unbounded homomorphism $\rho : \Gamma \to G'$ extends to a homomorphism of Lie groups $\rho_G : G \to G'$.

Margulis arithmeticity: Every lattice in a simple Lie group G of higher rank is arithmetic.

$$\boxed{\text{Lattices in higher rank are superrigid. They are all arithmetic.}}$$

Beyond Lattices

What do we find if we look beyond the world of lattices?



If we consider discrete subgroups which are not lattices…..

….. we find ourselves in the company of strange creatures, which we do not know how to tame.

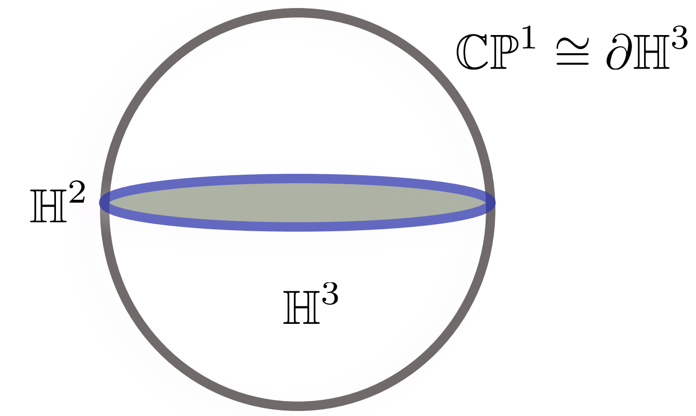
But there are some beautiful examples …. and the beginnings of a new structure theory.

Quasifuchsian deformations

Consider $\rho_0 : \pi_1(S_g) \to \mathrm{SL}(2,\mathbb{R}) \subset \mathrm{SL}(2,\mathbb{C}) = \mathrm{Isom}(\mathbb{H}^3)$

Then $\rho_0(\pi_1(S_g))$ preserves a totally geodesic copy of $\mathbb{H}^2$

The discrete subgroup $\rho_0(\pi_1(S_g)) \subset \mathrm{SL}(2,\mathbb{C})$ is not a lattice.

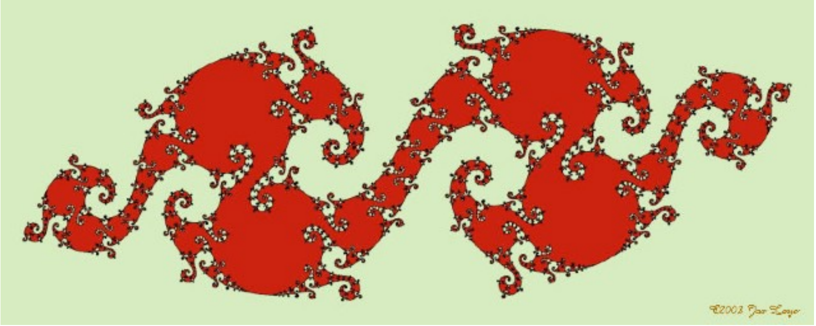


Focus on the sphere at infinity, the equator is preserved.

$\rho_0(\pi_1(S_g))$ acts properly discontinuously
on the upper and lower hemisphere

We can deform it and consider a family of homomorphisms

$$\rho_t : \pi_1(S_g) \to \mathrm{SL}(2,\mathbb{C})$$



There is a fractal Jordan curve
in the sphere at infinity, which is
preserved by $\rho_t(\pi_1(S_g))$.

The convex hull of that curve is a closed convex
subset $C \subset \mathbb{H}^3$ on which $\rho_t(\pi_1(S_g))$ acts properly
discontinously and cocompactly.

Convex cocompact subgroups

Let G be a simple Lie group and X its symmetric space.

A discrete subgroup Γ of G is convex cocompact if there exists a closed convex set $C \subset X$ which is preserved by Γ, and such that $\Gamma \backslash C$ is compact.

If G is of rank one, then

- the space of convex cocompact realizations of Γ is open in $\mathrm{Hom}(\Gamma, G)$.
- any convex cocompact subgroup is undistorted.
- any torsion free convex cocompact subgroup is the fundamental group of the interior of a compact manifold with boundary (tameness).

In rank one Lie groups convex cocompact subgroups provide a rich class of good discrete subgroups.

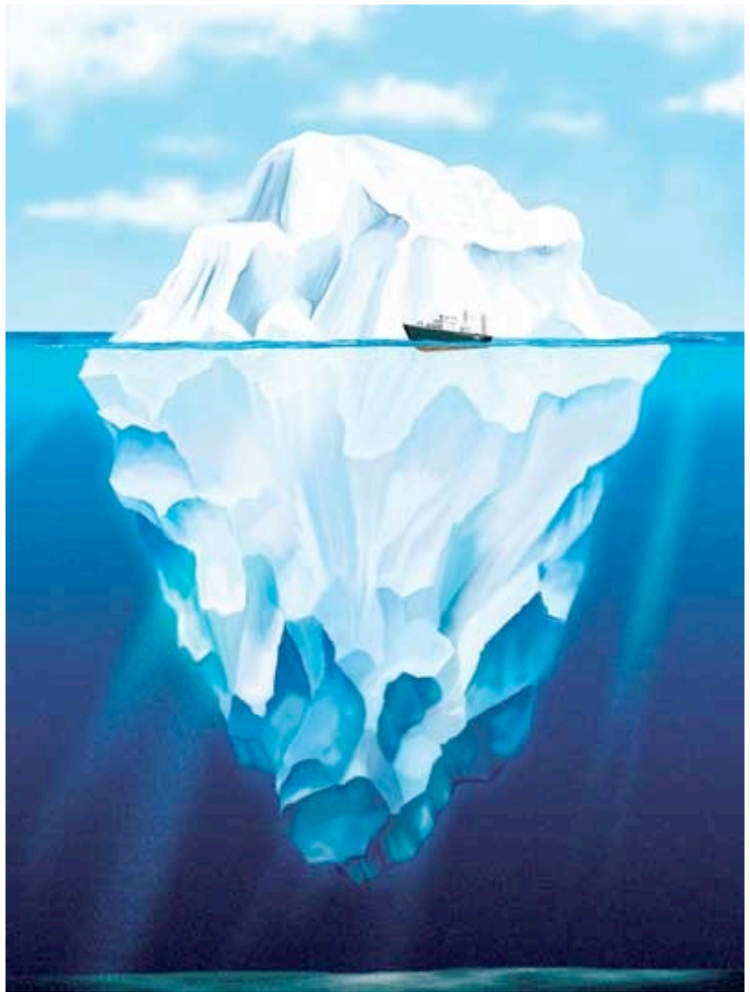
Kleiner-Leeb, Quint:

If G is of higher rank and Γ a convex cocompact subgroup which does not preserve a properly embedded symmetric space, then Γ is a cocompact lattice.

There are no interesting convex cocompact subgroups in higher rank Lie groups

The paradigm in higher rank is rigidity!

Paradigm shift: Flexibility and Rigidity



Combining **flexible** groups with higher rank **rigidity** we get rich classes of discrete subgroups in Lie groups of higher rank with interesting structure theory :

Anosov representation of hyperbolic groups

generalization of convex cocompact subgroups to the setting of higher rank Lie groups.

[Labourie, Guichard-W, Kapovich-Leeb-Porti, Gueritaud-Guichard-Kassel-W, Bridgeman-Canary-Labourie-Sambarino, Bochi-Potrie-Sambarino,…]

Higher Teichmüller spaces

special classes of discrete embeddings of fundamental groups of surface into special Lie groups of higher rank.

We discovered the tip of an iceberg!

We are sailing around the iceberg, seeing more and more aspects - some familiar and some new.

Higher Teichmüller spaces

Fricke-Teichmüller space is identified with a subset, in fact a connected component,

$$\mathrm{Hyp}(S) \subset \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

consisting of discrete embeddings.

Replace $\mathrm{PSL}(2,\mathbb{R})$ by a non-compact simple Lie group G

Higher Teichmüller spaces are subsets $\mathcal{T}(S,G) \subset \mathrm{Hom}(\pi_1(S), G)/G$

which are unions of connected components consisting of discrete embeddings

Hitchin components

G is a split real Lie group

$\mathrm{SL}(n,\mathbb{R})$ $\mathrm{SO}(n,n+1)$ $\mathrm{SO}(n,n)$ $\mathrm{Sp}(2n,\mathbb{R})$

Maximal representations

G is a Hermitian Lie group

$\mathrm{Sp}(2n,\mathbb{R})$ $\mathrm{SU}(n,n)$ $\mathrm{SO}(2,n)$ $\mathrm{SO}^*(2n)$

When $G = \mathrm{PSL}(2,\mathbb{R})$ the Hitchin component and the space of maximal representations agree with the Fricke-Teichmüller space.
For other groups they resemble classical Teichmüller space in many ways.

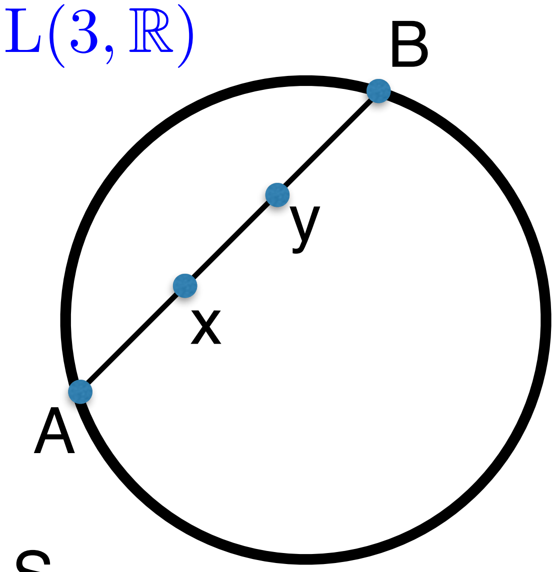
[Goldman, Labourie, Fock-Goncharov, Burger-Iozzi-W, Bonahon-Dreyer, Bridgeman-Canary-Labourie-Sambarino, Li, Gaiotto-Moore-Neitzke, Zhang, Lee-Zhang, Guichard-W, Le, Burger-Pozzetti, Fanoni-Pozzetti …]

Convex real projective structures

The Hitchin component $\mathcal{T}_{Hit}(S, \mathrm{PSL}(3,\mathbb{R})) \subset \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(3,\mathbb{R}))/\mathrm{PSL}(3,\mathbb{R})$ is the connected component containing the homomorphism:

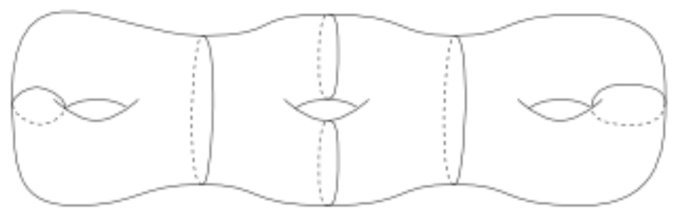
$$\rho_0 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}(1,2) \subset \mathrm{PSL}(3,\mathbb{R})$$

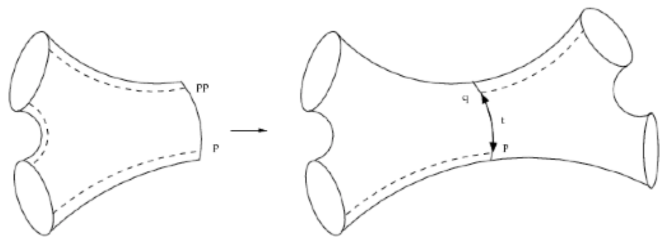
This homorphism preserves the Klein-model of the hyperbolic plane.

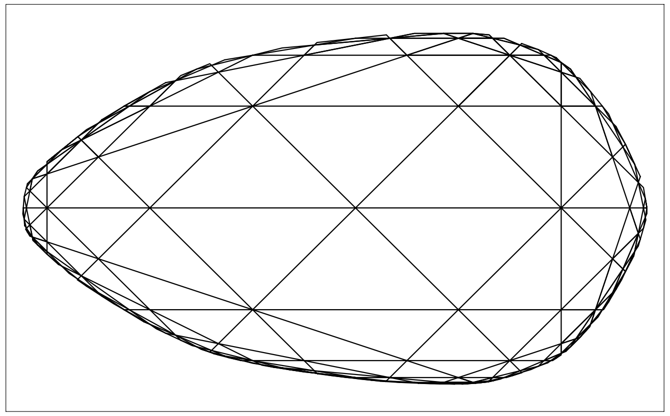


The Hitchin component parametrizes convex real projective structures on S.

[Goldman, Choi-Goldman]







- Fenchel-Nielsen type coordinates but pair of pants have interior parameters [Goldman, Fock-Goncharov, Bonahon-Dreyer, Zhang]
- Infinitely many integer points in $\mathcal{T}_{Hit}(S, \mathrm{PSL}(3,\mathbb{R}))$ [Long-Reid-Thislethwaithe]
- Riemannian metric on $\mathcal{T}_{Hit}(S, \mathrm{PSL}(3,\mathbb{R}))$ [Li, Bridgeman-Canary-Labourie-Sambarino]

What do we see?

What is beneath the surface?

Higher Teichmüller spaces parametrize geometric structures on compact manifolds M.
Are these manifolds M always compact bundles over S? [Guichard-W, Baraglia, Alessandrini-Li]

Parametrizations of Hitchin components, and partially of maximal representations.
Are there natural (Hamiltonian) flows associated to these parameters?
[Fock-Goncharov, Strubel, Bonahon-Dreyer, Zhang, W-Zhang]

Infinitely many (non-equivalent) integer points in higher Teichmüller spaces.
Can we count them or determine their asymptotics? [Long-Reid-Thislethwaite, Burger-Labourie-W]

Riemannian metric on higher Teichmüller spaces, invariant under mapping class group.
How do the quotients look like? Are there other metrics?
[Bridgeman-Canary-Labourie-Sambarino, Li]

Is there a complex analytic theory of higher Teichmüller spaces? [Dumas-Sanders]

Is there a similar theory for fundamental groups of hyperbolic manifolds M?
[Benoist, Barbot-Merigot, Cooper-Tillmann, Ballas-Danciger-Lee]

Structure theory of non-hyperbolic discrete subgroups in Lie groups of higher rank?
[Kapovich-Leeb-Porti, Guichard-Kassel-W, Canary-Lee-Stover, Danciger-Gueritaud-Kassel]

Structure theory for discrete subgroups of affine groups $G \ltimes V$?
[Goldman-Margulis-Minsky-Labourie, Abels-Margulis-Soifer, Choi-Goldman, Danciger-Gueritaud-Kassel, Smilga, Ghosh]

Conclusion

Lattice in Lie groups have been fairly well understood.

General discrete subgroups are much harder to investigate.

For a long time it seemed out of reach to find interesting classes of discrete subgroups of higher rank Lie groups, which are not lattices.

With higher Teichmüller spaces and Anosov representations we see the beginnings of a structure theory of “nice” discrete subgroups of higher rank Lie groups.

But there is a lot waiting to be discovered beneath the surface!

Get on your diving gear and explore!