Spectral Theory Sum Rules, Meromorphic Herglotz Functions and Large Deviations

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Szegő Recursion and Verblunsky Coefficients

We start with orthogonal polynomials on the unit circle, aka OPUC. Let $d\mu$ be a probability measure on $\partial\mathbb{D}$. Then, there are monic orthogonal polynomials, Φ_n, and recursion relations due to Szegő in 1939

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z)$$

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Szegő's Theorem: Toeplitz version

Szegő's Theorem concerns probability measures on $\partial\mathbb{D}$ of the form

$$d\mu(\theta) = w(\theta)\frac{d\theta}{2\pi} + d\mu_s(\theta)$$

where $d\mu_s$ is singular w.r.t. $d\theta$.

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$$c_{k\ell} \equiv \int e^{i(k-\ell)\theta} d\mu(\theta) = \langle e^{-ik\cdot}, e^{-i\ell\cdot}\rangle_{L^2(d\mu)}$$

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While this is true in general, Szegő only proved it when $d\mu_s = 0$.

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In 1920, Szegő realized that, because a Toeplitz matrix is just the Gram matrix of $\{z^j\}_{j=0}^{n-1}$, it is also the Gram matrix of $\{\Phi_j\}_{j=0}^{n-1}$ which is diagonal so

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so using that $\|\Phi_j\|$ is monotone decreasing (by a variational argument), one has an equivalent form of his theorem, namely

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But the recursion relation was only published by Szegő in 1939, so he didn't have a form in term's of α_n and ρ_n.

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$$\sum_{j=0}^{\infty} \log(1 - |\alpha_j|^2) = \int \log(w(\theta)) \frac{d\theta}{2\pi}$$

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In particular, $\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \Rightarrow \Sigma_{ac} = \partial\mathbb{D}$.

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Jacobi Parameters

Next orthogonal polynomials on the real line, aka OPRL. One starts with a probability measure, μ, of compact support in $\mathbb{R}$ and forms the orthonormal polynomials, $\{p_n(x)\}_{n=0}^{\infty}$. They obey recursion relations

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_np_{n-1}(x)$$

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which sets up a one-one correspondence (which we'll call the *Jacobi map*) between such measures (with an infinity of points in their support) and sequences $\{a_n, b_n\}_{n=1}^{\infty}$ of bounded a's in $(0, \infty)$ and b's in $\mathbb{R}$ (called *Jacobi parameters*).

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Jacobi Parameters

There is also a correspondence between point measures with finite support and suitable sets of finitely many Jacobi parameters. If there are n pure points, then P_n is 0 in $L^2(d\mu)$ so $a_n = 0$ and again there are $2n-1$ Jacobi parameters – n b's and $n-1$ a's.

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For later purposes, I need some details on one approach to going from measures to Jacobi parameters. The more usual method than the one I want to discuss just forms the OPs and looks at the recursion parameters. Instead, consider the *once stripped Jacobi parameters*, i.e. $\{a_{j+1}, b_{j+1}\}_{j=1}^{\infty}$ obtained by dropping the first row and column of the Jacobi matrix. For any non–trivial probability measure of compact support, let $m(z) = \int d\mu(x)/(x-z)$ and let m_1 be the spectral measure for the once stripped problem.

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One consequence of this is that the poles of m_1 (i.e. the pure points of μ_1) are precisely the zeros of m.

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$$\liminf_{n\to\infty} \prod_{j=1}^{n} a_j = \sqrt{2}\exp\left(\int_{-2}^{2} \log|\pi s(x) w(x)| s(x) \frac{dx}{4\pi}\right)$$

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Here is one version of Szegő's Theorem for OPRL. The map $z \mapsto z + z^{-1}$ maps $\partial\mathbb{D}$ to $[-2, 2]$ (via $e^{i\theta} \mapsto 2\cos\theta$) and so measures, μ, on $[-2, 2]$ to measures, ρ, on $\partial\mathbb{D}$ which are symmetric under complex conjugation (since the above map is 2 to 1 except at ± 1). In 1922, Szegő found a relation of the OPRL for μ to the OPUC for ρ and this allowed later authors to prove a version of Szegő's theorem for $d\mu = w(x)\,dx + d\mu_s$ (with $s(x) = (4 - x^2)^{-1/2}$):

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only holds under the a priori condition that μ is supported inside $[-2,2]$ and this is not simply expressible in terms of the Jacobi parameters; for example, it doesn't only depend on the parameters near ∞ and can be changed by modifying a single a or b.

Killip–Simon Theorem

In 2001 (published 2003), Killip and I proved the following gem which we regard as an OPRL analog of the Verblunsky–Szegő gem where $\{E_j^\pm\}_{j=1}^{N_\pm}$ are the eigenvalues outside $[-2, 2]$ (with + above 2 and - below -2):

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if and only if the essential support of μ is $[-2,2]$ and

$$\int_{-2}^{2} \log(w(x))\sqrt{4-x^2}\,dx > -\infty \qquad \sum_{j,\pm} (|E_j^\pm| - 2)^{3/2} < \infty$$

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This result on Jacobi Hilbert-Schmidt perturbations of the free Jacobi matrix should be compared with a celebrated theorem of von-Neumann that any bounded self-adjoint operator has a Hilbert-Schmidt perturbation with only dense point spectrum!

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We called $\int_{-2}^{2} \log(w(x))\sqrt{4-x^2}\, dx > -\infty$ *the quasi-Szegő condition* since the square root appeared to the $+1/2$ power rather than the $-1/2$ in the Szegő condition.

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$$\sum |E_n|^p \leq C \int |V(x)|^{p+d/2}\,dx$$

Killip–Simon Sum Rule

The gem comes from a sum rule. Let

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Then the Killip-Simon sum rule says

$$Q(\mu) + \sum_{j,\pm} F(E_j^{\pm}) = \sum_{n=1}^{\infty} \tfrac{1}{4} b_n^2 + \tfrac{1}{2} G(a_n)$$

As with the Szegő–Verblunsky sum rule, an important point is that it always holds although both sides may be $+\infty$.

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The gem comes from the fact that $F \geq 0$, vanishes exactly at $E = \pm 2$ and is $\mathrm{O}((|E| - 2)^{3/2})$ there and that $G \geq 0$, vanishes exactly at $a = 1$ and is $\mathrm{O}((a-1)^2)$ there.

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As in the OPUC case, this sum rule implies the existence of Hilbert–Schmidt perturbations with mixed spectrum.

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Because the eigenvalues $E_j^{\pm}$ and $E_j^{(1)\pm}$ interlace and F is monotone, the sum is of positive terms and always convergent (interlacing sums).

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Without the L^p–condition, there a singular inner part.

Meromorphic Herglotz Functions

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One can prove that in $\mathbb{D} \cap \mathbb{C}_+$, one has that $|\arg z B(z)| \leq 2\pi$ so that $\arg(f(z)/zB(z))$ is bounded on $\mathbb{D}$.

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Taking $\log$'s, one gets relations between Taylor coefficients of $\log(f(z)/z)$, certain sums involving $\log$s or powers of zeros and poles and integrals $\cos(n\theta) \log|f(e^{i\theta})|$.

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4. What does the function
$$F(E) = \tfrac{1}{4}[\beta^2 - \beta^{-2} - \log \beta^4]; \quad E = \beta + \beta^{-1}$$
mean?

The LD Framework

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$$I(x) = \sup_{\theta} \left[\theta x - \log\left(\mathbb{E}(e^{\theta X})\right)\right]$$

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Notice that $G(a) = I(a^2)$, which we'll see is no coincidence!!!

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As is well–known, the λ's and w's are independent of each other, the w's are uniformly distributed on the simplex $\{\mathbf{w} | \sum_{j=1}^{n} w_j = 1\}$ and by the Weyl integration formula, the θ's have distribution

$$\frac{1}{n!} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}$$

The first step in the analysis of the measure side is to analyze what probabilists call the *empirical measure* and physicists *the density of states*, namely the random measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$.

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This is easy to understand. The Weyl distribution can be viewed as a discrete two dimensional Coulomb gas in the canonical ensemble (2D because $|x-y|^{-2}$ is the exponential of $-2\log|x-y|$). The $n \to \infty$ limit is a high density limit and due to repulsion, there is a strong tendency towards equal spacing.

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The weights are close to independent (except for the normalization they are) – a slick way to see this is to note if Y_j are positive exponentially distributed iidrv, then $w_j = Y_j / \sum_{j=1}^{n} Y_j$. This allows one (using Cramér's theorem on small blocks) to prove an LDP for the spectral measure with speed n and rate function the Szegő integral $-\int \log(w(\theta))\frac{d\theta}{2\pi}$.

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GUE

The Gaussian Unitary Ensemble, aka GUE, is the probability measure on $n \times n$ self adjoint matrices so that $\{a_{ii}\}_{i=1,\dots,n}, \{\mathrm{Re}(a_{ij})\}_{1\le i<j\le n}$ and $\{\mathrm{Im}(a_{ij})\}_{1\le i<j\le n}$ are independent identically distributed Gaussian random variables of mean zero and suitable, n–dependent variance.

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What results is a new proof of the Killip-Simon sum rule.

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mean? This is the Coulomb potential of the Wigner semi–circle distribution plus a quadratic external field.

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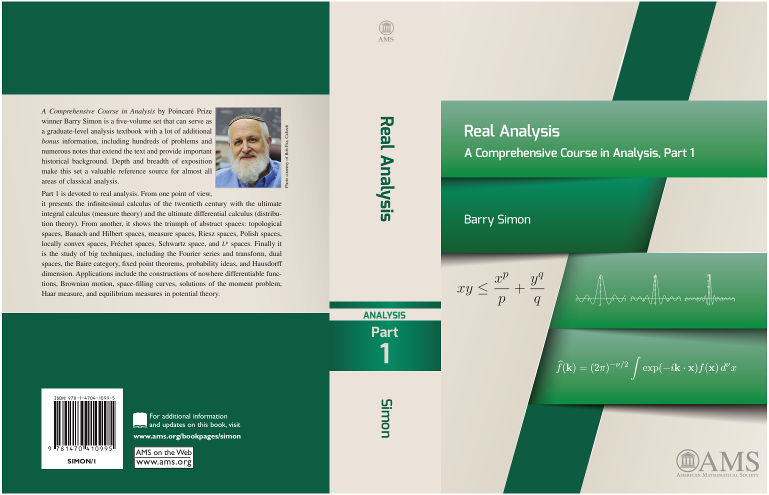

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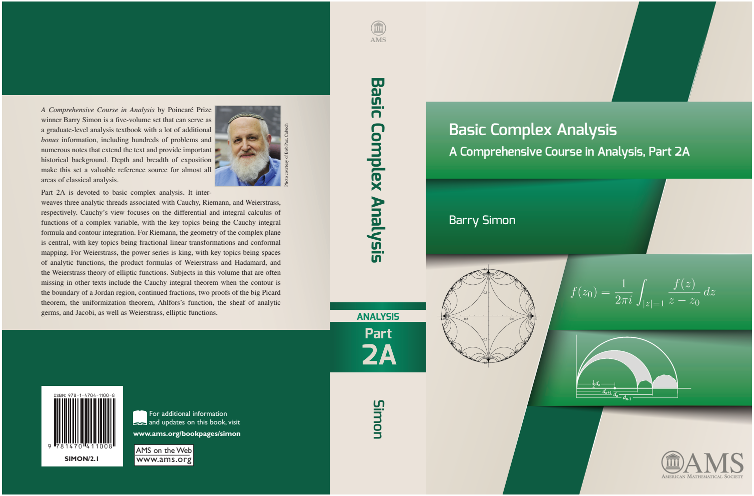

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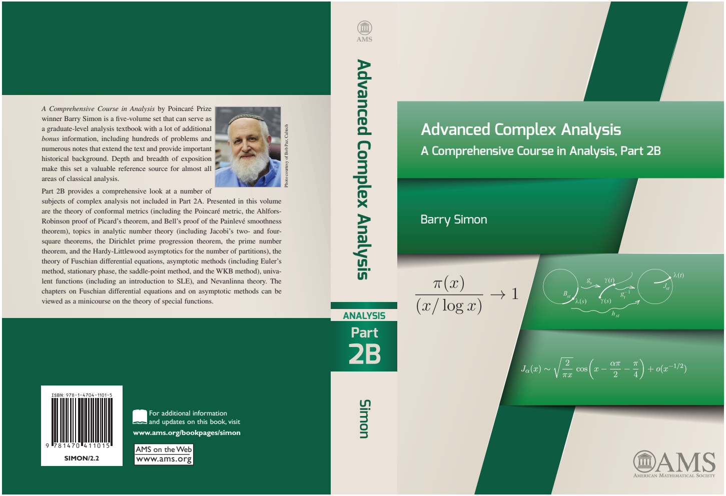

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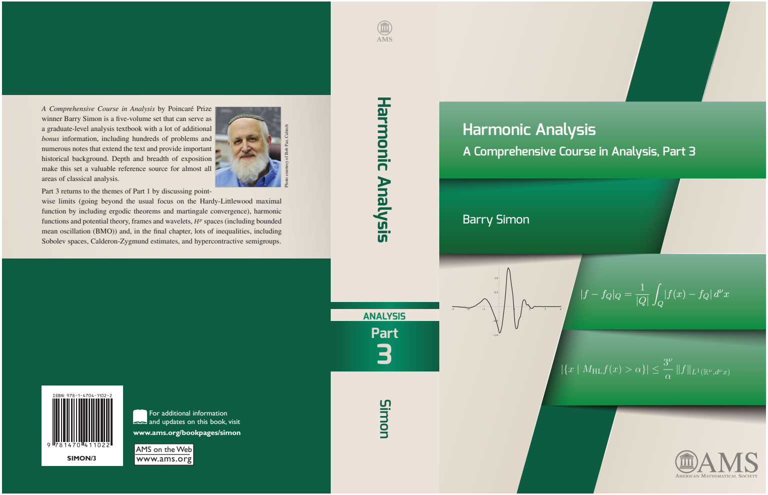

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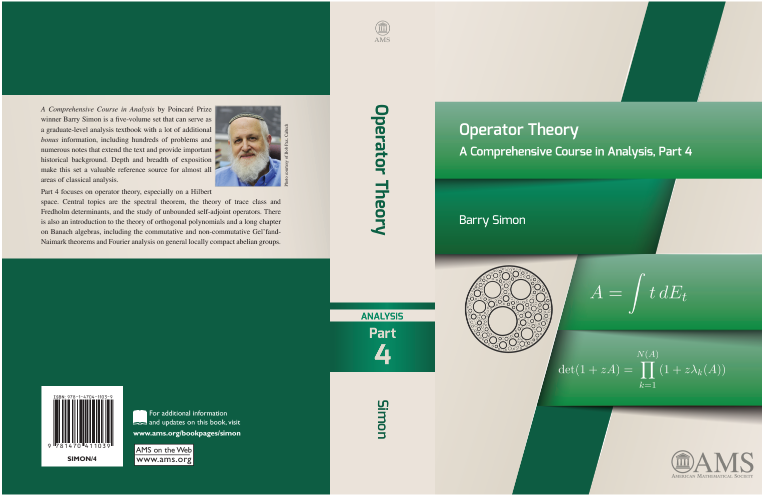

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