

The Many Faces of Dispersive and Wave Equations

Gigliola Staffilani

MIT

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Outline of the talk

- Introduction
- From a Quantum Many-particles System to NLS
- Strichartz Estimates
- Energy Transfer
- Infinite Dimensional Hamiltonian Systems
- Gibbs Measures
- Non Squeezing theorems
- Open Problems

Tools come from many areas

The extraordinary recent progress in dispersive
and wave equations has involved

- * Harmonic and Fourier Analysis
- * Analytic Number Theory
- * Math Physics
- * Dynamical Systems
- * Probability
- * Symplectic Geometry

A case study: the non-linear Schrödinger equation

To illustrate how these different areas of mathematics come into play we consider

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u = \lambda |u|^2 u & \lambda = \pm 1 \\ u(0, x) = u_0(x) & x \in \mathbb{T}^d \end{cases}$$

This is the periodic initial value problem (IVP).

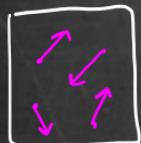
$$\text{Mass} = \int_{\mathbb{T}^d} |u(t, x)|^2 dx$$

$$\text{Hamiltonian} = \int_{\mathbb{T}^d} \frac{1}{2} |\nabla u|^2 + \frac{\lambda}{4} \int_{\mathbb{T}^d} |u|^4 dx$$

Where does the NLS come from?

Bose Einstein Condensate (BEC):

Gas at high temperature



N particles

Gas at low temperature



"Wave Packets"

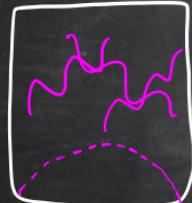
distance = d

wave length = γ

②

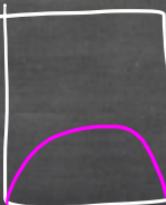
①

Gas at critical T_c



"Matter Wave Overlap"

When $\gamma \approx d$



At zero temperature

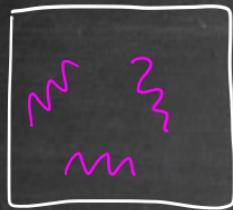
"Giant Matter Wave"

Pure BEC

④

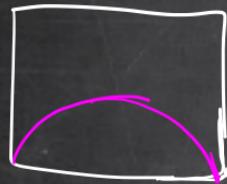
T_c = critical temperature ③

Mathematically



the BBGKY
Hierarchy

"Wave Packets"



↓ take limit
as $N \rightarrow \infty$

The Gross-Pitaevskii
Hierarchy

"Giant Matter Wave"

More Mathematics

If $\underline{x}_k = (x_1, \dots, x_k)$ $x_i \in \mathbb{R}^d$,

$$Y^{(k)}(t, \underline{x}_k, \underline{x}'_n) = \prod_{j=1}^k \mu(t, x_j) \overline{\mu(t, x'_j)} \quad \begin{pmatrix} \text{initial data} \\ \text{of} \\ G-P \end{pmatrix}$$

then

$$Y^{(n)}(t, \underline{x}_k, \underline{x}'_n) = \prod_{j=1}^k \mu(t, x_j) \overline{\mu(t, x'_j)} \quad \begin{pmatrix} \text{solution to} \\ G-P \end{pmatrix}$$

where

$$(NLS) \left\{ \begin{array}{l} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \end{array} \right.$$

Spohn
 Erolös - Schlein - Yau
 Chen - Pavlović

Example of an Integrable System

$$i\partial_t u + \partial_x^2 u = \pm |u|^2 u$$

in \mathbb{R} or \mathbb{T}

is an Integrable system

Lax Pairs, Inverse Scattering,
and infinitely many
Conservation laws:

$$I_s(u) = \int_{\mathbb{R}} \frac{1}{2} |\partial_x^s u|^2 dx + \text{l.o.t.}$$

$$s \in \mathbb{N}.$$

Gross-Pitaevskii Hierarchy
in \mathbb{R} or \mathbb{T}

also admits infinitely
many conserved quantities

Mendelson - Mahmood -
Pavlovic - S.

Well-Posedness

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u(0, x) = u_0(x) \end{cases} \Leftrightarrow u = S(t)u_0 \pm \int_0^t S(t-\tau') |u(\tau')|^2 u(\tau') d\tau'$$

Solution to Cauchy Problem. \Leftrightarrow Fixed point for integral equation.

Here $S(t)u_0$ is the solution to the linear problem

$$\begin{cases} i\partial_t v + \Delta v = 0 \\ v(0, x) = u_0(x) \end{cases}$$

Periodic Strichartz Estimates

The Strichartz estimates we are interested in are

$$\|S(t)u_0\|_{L^q_{[0,1]} L^q_{\mathbb{T}^d}} \leq C \|u_0\|_{H^s(\mathbb{T}^d)}$$

Case: $d=2, q=4$ $S(t)u_0(x) = \sum_{n \in \mathbb{Z}^2} \hat{u}_0(n) e^{i(d_1 n_1^2 + d_2 n_2^2)t} e^{inx}$

for $d_1, d_2 > 0$

- $d_1, d_2 \in \mathbb{Q} \Rightarrow \mathbb{T}^2$ rational torus
- $\exists d_i \notin \mathbb{Q} \Rightarrow \mathbb{T}^2$ irrational torus.

Strichartz Estimates on Rational tori

If π^2 is a rational torus then

Bourgain 90's

$$\|S(t)u_0\|_{L^q_{\pi \times \pi^2}} \leq \|u_0\|_{H^\varepsilon(\pi^2)} \quad \varepsilon > 0$$

Ingredients:

a) π^2 rational $\Rightarrow S(t)u_0(x)$ periodic in time t

b) If $\alpha_1, \alpha_2 \in \mathbb{Q}$ $\#\{n \in \mathbb{Z}^2 / \alpha_1 n_1^2 + \alpha_2 n_2^2 = R\} \sim \exp C \frac{\log R}{\log \log R} < R^\varepsilon$

Analitic Number theory \Rightarrow Harmonic Analysis

Strichartz Estimates on any Torus

$$\|S(t)M\|_{L^4_{[0,1]} \cap L^4_{T^2}} \lesssim \|M\|_{H^\varepsilon} \quad \varepsilon > 0$$

Bourgain - Demeter '14

- Proof is corollary of L^2 -decoupling theorem (also $d > 2$)
- See also Bilinear Strichartz [C. Fan - S - H. Wang - Wilson] and Longer Time Strichartz [Germain - Y. Deng]
- Using Sharp decoupling for curves one can prove a main Conjecture of Vinogradov [Bourgain - Demeter - Guth]

Harmonic Analysis \Rightarrow Analytic Number Theory

Global well-posedness and properties of solutions

The Cauchy problem $\begin{cases} i\partial_t u + \Delta u = |u|^2 u & (\text{defocusing}) \\ u(0, x) = u_0(x) & x \in \mathbb{T}^2 \end{cases}$

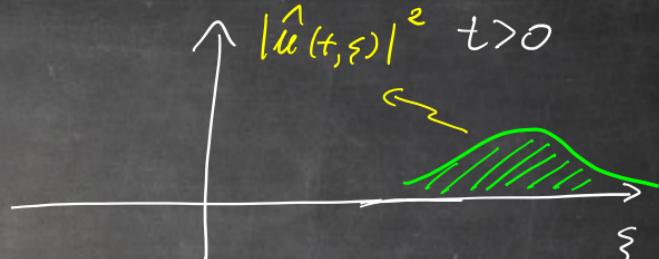
is globally well-posed for $u_0 \in H^s$, $s \geq 1$

The proof uses Strichartz estimates, fixed point arguments, the bounds from the Hamiltonian and iteration.

Question: Can we know more about the behaviour of the solution $u(t, x)$ as $t \rightarrow \infty$?

Transfer of Energy

$t=0$



Question : Does the support of $|\hat{u}(t, \xi)|$ moves to higher frequencies ξ ?

Weak turbulence, forward cascade

To answer one can look at

$$\boxed{\int |\hat{u}(t, \xi)|^2 \langle \xi \rangle^{2s} d\xi = \|u(t)\|_{H^s}^2}$$

and check what happens when $t \rightarrow \infty$.

Growth of Sobolev Norms

Fact 1: Complete Integrability may prevent the growth of Sobolev norms, (only 1D, i.e. cubic NLS, KdV).

Fact 2: Scattering prevents growth of Sobolev norms.

$$\left\{ \begin{array}{l} i\partial_t u + \Delta u = |u|^2 u \\ u(0, x) = u_0(x) \end{array} \right. \quad \left\{ \begin{array}{l} \text{If } x \in \mathbb{T}^2 \text{ then } \|u(t)\|_{H^s} \lesssim |t|^{s+\varepsilon} \\ \text{for } s > 1 \end{array} \right. \quad \boxed{\text{Bourgain, Schringer}} \quad (1)$$

If $x \in \mathbb{T}^2$ rational, given $s > 1$, $\epsilon < c$, $K \gg 1$ \exists a solution u and a time T s.t. $\|u_0\|_{H^s} < \epsilon$ and $\|u(T)\|_{H^s} > K$.

(2)

Colliander - Keel - S - Takaoka - Tao

How do we prove (2)?

Step 1: look for $v(t, x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(|n|^2 t + x \cdot n)}$

Step 2: restrict the set of active frequencies n :

- * only resonant ones

- * only in a special set Λ of finitely many ones

Step 3: Get to a Toy Model:

$$-i \partial_t b_j(t) = -b_j(t) |b_j(t)|^2 - 2 \overline{b_{j-1}(t)}^2 \overline{b_j(t)} - 2 \overline{b_{j+1}(t)}^2 \overline{b_j(t)}$$

$$j = 0, 1, \dots, M+1$$

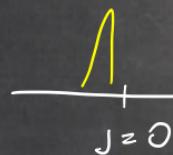
$$b_0(t) = b_{M+1}(t) = 0$$

A dynamical system approach

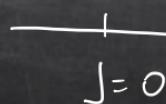
Let $T_i = (0, 0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$ $i = 0, \dots, M+1$

T_i is invariant under the flow of the Toy Model. But we can start near T_i and still move. What we prove is:

$t=0$



$t=T$



there is motion from low to high frequencies!

On This Subject ...

Guadalupe - Kaloshin, Guadalupe - Hou - Procesi $\left(\begin{array}{l} \text{cubic/} \\ \text{quintic NLS} \end{array}\right)$

Gérard - Grellier $(\text{Cubic Zego Equation})$

Hani - Pousader - Tzvetkov - Visaglia $(\text{NLS on } \mathbb{R} \times \mathbb{T}^2)$

Faou - Germain - Hani $(\text{Large box limit of 2D NLS})$

Buckmaster - Germain - Hani - Shatah

(Effective dynamics of the NLS on large domains)

Gibbs Measure (finite dimension)

$$\begin{cases} \dot{a}_n = \frac{\partial H}{\partial b_n} \\ \dot{b}_n = -\frac{\partial H}{\partial a_n} \end{cases}$$

$n = 1, \dots, d$

The Hamiltonian flow $\Phi(t) = (a_n(t), b_n(t))$ preserves the volume:
For any $A \subset \mathbb{R}^{2d}$

$$\text{Vol}(A) = \text{Vol}(\Phi(t)(A)) \quad \forall t.$$

Define now

$$d\mu = \frac{1}{Z} e^{-H(a_n, b_n)} \prod_{n=1}^d da_n db_n$$

time invariant
time invariant

Gibbs Measure

Also the Gibbs measure μ is invariant:

$$\mu(A) = \mu(\Phi(t)(A)) \quad \forall t.$$

Gibbs measure and ∞ dimensional Hamiltonian systems

Consider again

$\mathcal{W}(\mu)$ Wick ordered

$$\begin{cases} i\partial_t u + \Delta u = \mathcal{W}(u) & \text{if } \hat{u}(t, \mathbf{x}) = a_n(t) + i b_n(t) \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) & n \in \mathbb{Z}^2 \\ \mathbf{x} \in \mathbb{T}^2 \end{cases}$$

$$\begin{cases} \dot{a}_n = \frac{\partial H_\Psi}{\partial b_n} \\ \dot{b}_n = - \frac{\partial H_\Psi}{\partial a_n} \end{cases} \quad n \in \mathbb{Z}^2$$

Question: Can one define

" $d\mu = \frac{1}{Z} e^{-H} \prod du$ "? Is it invariant?

- Existence of Gibbs measure

Lebowitz - Rose - Speer 1D
Glimm - Jaffe 2D and 3D

- Invariance of Gibbs measure

Bourgain '96

More on existence and invariance of μ

On Existence: $d\mu = \frac{1}{Z} e^{-\int |u|^q dx} dg$

dg = Gaussian Measure ($\text{supp } dg \text{ on } H^{-\varepsilon}$)

On Invariance: We have the following theorem.

Theorem [Bouguen '96]

Consider the Cauchy problem $\begin{cases} i\partial_t u + \Delta u = d\rho(u) \\ u(0, x) = u_0, x \in \mathbb{T}^2 \end{cases}$. There exist $\Sigma \subseteq H^{-\varepsilon}$ s.t. $\mu(\Sigma) = 1$ and $\forall u_0 \in \Sigma \exists !$

global solution u . Moreover μ is invariant.

Isn't $\Sigma \subset H^{-\varepsilon}$ too rough?

Yes if one expects to see deterministic estimates.

A typical element of Σ is $u_\sigma(w) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(w)}{\langle n \rangle} e^{inx}$, where $g_n(w)$ are i.i.d Gaussian complex random variables and $w \in (\mathcal{S}, \mathcal{P})$, a probability space.

During the course of the proof one needs to estimate

$$\| F_3(w) \|_{l^2_n l^2_m}$$

where

$$F_3(w) = \sum_{S_{m,n}} \frac{1}{\langle n_1 \rangle} \frac{1}{\langle n_2 \rangle} \frac{1}{\langle n_3 \rangle} g_{n_1}(w) \overline{g_{n_2}(w)} g_{n_3}(w)$$

$$S_{m,n} = \left\{ (n_1, n_2, n_3) \mid \begin{array}{l} n_1 - n_2 + n_3 = n \\ n_1, n_3 \neq n_2 \end{array} \text{ and } m = |n_1|^2 - |n_2|^2 + |n_3|^2 \right\}$$

Where probability helps

Trivial Estimate: $\|F_3(\omega)\|_{\ell_m^2 \ell_n^2}^2 = \sum_{n,m} \left| \sum_{S_{n,m}} \frac{1}{c_{n_1}} \frac{1}{c_{n_2}} \frac{1}{c_{n_3}} g_{n_1}(\omega) \overline{g_{n_2}(\omega)} g_{n_3}(\omega) \right|^2$

$$C-S \leq \sum_{n,m} |S_{n,m}| \sum_{S_{n,m}} \frac{1}{c_{n_1}^2} \frac{1}{c_{n_2}^2} \frac{1}{c_{n_3}^2}$$

loss! ↙

Smarter Estimate: $\|F_3(\omega)\|_{L^2(S)}^2 = \sum_{n,m} |F_3(\omega)|_{S_{n,m}}^2 \leq \sum_{n,m} \|F_3(\omega)\|_{L^2(S)}^2$

(for ω in a smaller set!)

Large deviation estimate

$$= \sum_{n,m} \sum_{S'_{n,m}} \sum_{S_{n,m}} \frac{1}{c_{n_1}} \frac{1}{c_{n_2}} \frac{1}{c_{n_3}} \frac{1}{c_{n'_1}} \frac{1}{c_{n'_2}} \frac{1}{c_{n'_3}} \times$$

independence +
wick order

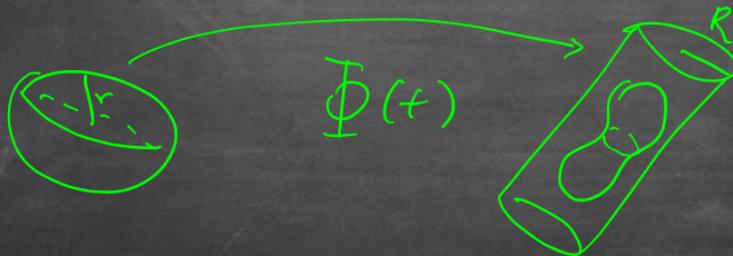
$$\times \int_S g_{n_1}(\omega) \overline{g_{n_2}(\omega)} g_{n_3}(\omega) \overline{g_{n'_1}(\omega)} g_{n'_2}(\omega) \overline{g_{n'_3}(\omega)} \simeq \sum_{n_1,n} \sum_{S_{n_1,n}} \frac{1}{c_{n_1}^2} \frac{1}{c_{n_2}^2} \frac{1}{c_{n_3}^2}$$

no loss! ↗

The Non-Squeezing Theorem

Theorem [Gromov]: Assume $\Phi(t)$ is an Hamiltonian flow in \mathbb{R}^{2d} which is also a symplectomorphism. Let B_r be a ball of radius r in \mathbb{R}^{2d} and C_R be a cylinder of radius R in \mathbb{R}^{2d} . Then

$$\Phi(t)(B_r) \subseteq C_R \Rightarrow R \geq r.$$



Is a non-spreading theorem true in ∞ dimensions?

True, if the flow is a **compact** perturbation of a linear one.

Kuksin

True for the cubic defocusing NLS in \mathbb{T} Boussiges
(Here $L^2(\mathbb{T})$ is the symplectic space)

True for KdV in $H^{-\frac{1}{2}}(\mathbb{T})$ Cocionek-Keel-S-Takaoka-Tao

Partial results for Klein-Gordon almost sure flow
in $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}(\mathbb{T}^3)$ Menakerson

True for the cubic, defocusing NLS in $L^2(\mathbb{R}^2)$

Killip-Visani-Zheng

Some Open Problems

- Understanding Strichartz estimates in a more direct way.
- Improving theorems on weak turbulence. See for example the recent work mentioned earlier.
- Understanding ergodic structures associated to infinite dimensional Hamiltonian flows.
- Using probabilistic approaches to study properties of discrete versions of dispersive PDE, see Chatterjee, Chatterjee - Kirkpatrick.
- Finding more robust arguments to understand the symplectic structures associated to certain dispersive PDE.

