# Random polygons, Grassmannians, and a problem of Lewis Carroll. 

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## The Pillow Problem

In 1895, Charles Dodgson (better known by his pen name Lewis Carroll) published a book of 72 mathematical problems designed to be solved "while lying in bed"


14 PILLOW-PROBLEMS.

## 57. $(25,80)$

In a given Triangle describe three Squares, whose bases shall lie along the sides of the Triangle, and whose upper edges shall form a Triangle ;
(1) geometrically; (2) trigonometrically. [27/1/9r
58. $(25,83)$

Three Points are taken at random on an infinite Plane. Find the chance of their being the vertices of an obtuseangled Triangle.
$\left[20 / \mathrm{r} / 8_{4}\right.$

## Carroll's problem is ill-posed

## Question

What does it mean to choose a random triangle?
The issue of choosing a "random triangle" is indeed problematic. I believe the difficulty is explained in large measure by the fact that there seems to be no natural group of transitive transformations acting on the set of triangles.
-Stephen Portnoy, 1994
(Editor, J. American Statistical Association)
There have been many approaches which solve the problem of defining a random triangle in different ways [Guy, Kendall, Portnoy, Edelman/Strang, ... ].

## Choosing a random triangle

Let $a, b$, and $c$ be the sidelengths of the triangle. The space of triangles is parametrized by choices of $a, b$ and $c$ satisfying a collection of conditions:

$$
a \geq 0, b \geq 0, c \geq 0, \quad \text { and } \quad a+b+c=2
$$

and the triangle inequalities

$$
\begin{aligned}
& b+c \geq a \\
& a+c \geq b \\
& a+b \geq c
\end{aligned}
$$



## Choosing a random triangle (2)

We can rewrite the triangle inequalities as

$$
\begin{aligned}
-a+b+c & \geq 0 \\
a-b+c & \geq 0 \\
a+b-c & \geq 0
\end{aligned}
$$

If $s=\frac{a+b+c}{2}$ (the semiperimeter) this suggests new variables:

$$
\begin{aligned}
& s_{a}:=s-a=\frac{-a+b+c}{2} \geq 0 \\
& s_{b}:=s-b=\frac{a-b+c}{2} \geq 0 \\
& s_{c}:=s-c=\frac{a+b-c}{2} \geq 0
\end{aligned}
$$

Note that $s_{a}+s_{b}+s_{c}=s=1$.

## Square roots

This triangle of triangles is covered 8 -fold by the sphere.


We will use $x, y$, and $z$ as coordinates on triangle space. The measure will be surface area on the sphere.

## Triangle Geometry



Proposition
$|x|,|y|$, and $|z|$ are (pairwise) geometric means of the exradii.

## Transitive transformations on the sphere

Rotations of $S^{2}$ can exchange any two triangles.


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Rotations around an axis fix one edge and move the opposing vertex around an ellipse.


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Open Question: is there an elegant triangle-theoretic description of this motion?


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## Carroll's problem

The Pythagorean theorem implies that right triangles have

$$
x^{2} y^{2}=z^{2}, \quad y^{2} z^{2}=x^{2}, \quad \text { or } \quad z^{2} x^{2}=y^{2}
$$



Theorem (with Needham, Shonkwiler, Stewart)
The fraction of obtuse triangles is $\frac{3}{2}-\frac{3 \ln 2}{\pi} \approx 0.838093$

## Generalizing to $n$-gons: start over

Suppose the edges are complex numbers $e_{1}, \ldots, e_{n} \in \mathbb{C}$. The polygon is closed, so

$$
e_{1}+\cdots+e_{n}=0
$$

Let $e_{i}=z_{i}^{2}$ and $z_{i}=u_{i}+\mathbf{i} v_{i}$.

$$
\begin{aligned}
0 & =e_{1}+\cdots+e_{n}=z_{1}^{2}+\cdots+z_{n}^{2} \\
& =\left(u_{1}+\mathbf{i} v_{1}\right)^{2}+\cdots+\left(u_{n}+\mathbf{i} v_{n}\right)^{2} \\
& =\left(u_{1}^{2}-v_{1}^{2}\right)+\mathbf{i}\left(2 u_{1} v_{1}\right)+\cdots+\left(u_{n}^{2}-v_{n}^{2}\right)+\mathbf{i}\left(2 u_{n} v_{n}\right) \\
& =\left(u_{1}^{2}+\cdots+u_{n}^{2}-v_{1}^{2}-\cdots-v_{n}^{2}\right)+\mathbf{2} \mathbf{i}\left(u_{1} v_{1}+\cdots+u_{n} v_{n}\right) .
\end{aligned}
$$

or if $\vec{u}=\left(u_{1}, \ldots, u_{n}\right)$ and $\vec{v}=\left(v_{1}, \ldots, v_{n}\right)$

$$
|\vec{u}|^{2}=|\vec{v}|^{2} \quad \text { and } \quad\langle\vec{u}, \vec{v}\rangle=0
$$

## Generalizing to $n$-gons: start over (2)

If we fix the total polygon length to be 2, we have

$$
\begin{aligned}
2 & =\left|e_{1}\right|+\cdots+\left|e_{n}\right|=\left|z_{1}\right|^{2}+\cdots+\left|z_{n}\right|^{2} \\
& =u_{1}^{2}+v_{1}^{2}+\cdots+u_{n}^{2}+v_{n}^{2} \\
& =|\vec{u}|^{2}+|\vec{v}|^{2} .
\end{aligned}
$$

This gives us:

Theorem (Knutson/Hausmann 1997)
If the edges of an $n$-gon are $e_{i}=z_{i}^{2}$ and each $z_{i}=u_{i}+\mathbf{i} v_{i}$, then the polygon is closed and length $2 \Longleftrightarrow \vec{u}$ and $\vec{v}$ are orthonormal.

## Generalizing to $n$-gons: start over (2)

If we fix the total polygon length to be 2, we have

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& =|\vec{u}|^{2}+|\vec{v}|^{2} .
\end{aligned}
$$

This gives us:

## Theorem (Knutson/Hausmann 1997)

The space of closed, length 2 plane polygons is $2^{n}$-fold covered by the "Stiefel manifold" $V_{2}\left(\mathbb{R}^{n}\right)$ of orthonormal 2-frames in $\mathbb{R}^{n}$.

## Rotations

Rotations in the plane of $\vec{u}$ and $\vec{v}$ rotate the $z_{i}$ and rotate the edges $e_{i}=z_{i}^{2}$ twice as fast.


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Theorem (Knutson/Hausmann 1997)
The plane spanned by $\vec{u}$ and $\vec{v}$ determines the polygon up to rotation.

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Theorem (Knutson/Hausmann 1997)
The space of closed, length-2 plane polygons up to translation and rotation is covered by the "Grassmann manifold" $G_{2}\left(\mathbb{R}^{n}\right)$ of 2-planes in $\mathbb{R}^{n}$.

## Bringing the pictures together: $G_{1}\left(\mathbb{R}^{3}\right)=G_{2}\left(\mathbb{R}^{3}\right)$

If $(x, y, z)$ is orthonormal to $\vec{u}$ and $\vec{v}$, then

$$
\left(\begin{array}{lll}
x & u_{1} & v_{1} \\
y & u_{2} & v_{2} \\
z & u_{3} & v_{3}
\end{array}\right) \quad \text { is orthonormal. }
$$

So

$$
x^{2}+u_{1}^{2}+v_{1}^{2}=1, \quad \text { or } \quad x^{2}=1-u_{1}^{2}-v_{1}^{2}
$$

but

$$
\left|z_{1}\right|^{2}=\left|e_{1}\right|=a
$$

so this is our original equation

$$
x^{2}=1-a
$$

## Generalizing $(x, y, z)$ : Plücker Coordinates

## Definition

Any 2-plane $P$ in $\mathbb{R}^{n}$ spanned by $\vec{u}, \vec{v}$ is described by a skew-symmetric $n \times n$ matrix of Plücker coordinates

$$
\Delta(P)_{i j}=\operatorname{det}\left(\begin{array}{ll}
u_{i} & v_{i} \\
u_{j} & v_{j}
\end{array}\right)=\left(u_{i}, v_{i}\right) \times\left(u_{j}, v_{j}\right)
$$

defined up to multiplication by a common scalar. (Changing the basis for $P$ only changes the scalar, so the Plücker coordinates depend only on the plane.)

Our coordinates $(x, y, z)$ are the Plücker coordinates in the upper triangle of the $3 \times 3$ matrix for $G_{2}\left(\mathbb{R}^{3}\right)$.

## The Positive Grassmannian

## Definition

The Positive Grassmannian is the portion of the Grassmannian where all Plücker coordinates in the upper triangle are positive

$$
\Delta P_{i j}>0 \Longleftrightarrow i<j .
$$

It has attracted a lot of interest in string theory and has a beautiful and somewhat mysterious structure.

## Theorem (with Needham, Shonkwiler, Stewart)

The positive Grassmannian $G_{2}\left(\mathbb{R}^{n}\right)^{+}$consists of planes $P$ where $\left(a_{i}, b_{i}\right)$ lie in a common semicircle and the polygon is convex. $G_{2}\left(\mathbb{R}^{n}\right)$ is tiled by $2^{n-2} \times(n-1)$ ! isometric copies of $G_{2}\left(\mathbb{R}^{n}\right)^{+}$.
(A comparable interpretation appears in Section 5.3 of Arkani-Hamed, 2012.)

## Random polygons

There is a natural way to measure volume in $G_{2}\left(\mathbb{R}^{n}\right)$ which is $O(n)$ invariant (Haar measure). Using this as a probability measure on polygons:

Theorem (with Needham, Shonkwiler, Stewart) The probability that a random $n$-gon is convex is $2 /(n-1)$ !.


Theorem (with Needham, Shonkwiler, Stewart)
Among random quadriaterals, $1 / 3$ are convex, $1 / 3$ are reflex, and $1 / 3$ are self-intersecting.

## Random $n$-gons

It is easy to sample a random 2-plane in $\mathbb{R}^{n}$ uniformly: just pick two vectors of $n$ independent Gaussians and take the plane they span. Here's a random 500-gon:


## Geometric Probability Calculations

## Theorem (with Deguchi, Shonkwiler)

The edgelength of a random quadrilateral is uniformly distributed on $[0,1]$. The edgelength of a random n-gon is sampled from a Beta distribution with probability density

$$
\phi(y)=\left(\frac{n}{2}-1\right)(1-y)^{\frac{n}{2}-2}
$$



Quadrilaterals


12-gons

## Geometry of random $n$-gons

## Definition

The radius of gyration of an $n$-gon $v_{1}, \ldots, v_{n}$ is the average (squared) distance between vertices:

$$
\frac{1}{n^{2}} \sum_{i, j \in 1}^{n}\left|v_{i}-v_{j}\right|^{2}
$$

Theorem (with Deguchi, Shonkwiler)
The expected radius of gyration of a random planar $n$-gon is

$$
\frac{2}{3} \frac{n+1}{n(n+2)}
$$

## Distances and Geodesics

We can use the geometry of the Grassmannian to easily build geodesic (shortest) paths between (lifts of) polygons.

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## Space Polygons

Space polygons have a similar structure, if we replace $G_{2}\left(\mathbb{R}^{n}\right)$ with $G_{2}\left(\mathbb{C}^{n}\right)$, view edges as quaternions rather than complex numbers, and replace squaring with the Hopf map.


## Geometric Probabilities

All this structure lets you compute some exact probabilities:
Theorem (with Deguchi, Shonkwiler)
The expected radius of gyration of a random n-gon in $\mathbb{R}^{3}$
sampled from $G_{2}\left(\mathbb{C}^{n}\right)$ is

$$
\frac{1}{2 n}
$$

Theorem (with Grosberg, Kusner, Shonkwiler)
The expected total curvature of a random $n$-gon in $\mathbb{R}^{3}$ sampled from $G_{2}\left(\mathbb{C}^{n}\right)$ is

$$
\frac{\pi}{2} n+\frac{\pi}{4} \frac{2 n}{2 n-3}
$$

From here, you can go in various directions:

- polygons of fixed edgelength (e.g. equilateral polygons) (with Shonkwiler-Duplantier-Uehara)
- polygons of fixed thickness (Chapman, Plunkett)
- linkages and computational geometry
- polygons of fixed bending angle (e.g. molecular models)
- curves instead of polygons (Needham)
- different topologies, such as $\theta$-curves (Deguchi, Uehara)
- shape recognition (Needham, Mumford-Shah-Younes)
- random knots and links (Chapman, Hass, Millett, Rawdon)
.... and we invite you to our paper session tomorrow morning!
8:30-11:50am, A602, Atrium Level, Marriott Marquis

