Arrival time

Tobias Holck Colding

JMM, Atlanta, January 7, 2017

Overview

- Modeling of physical phenomena leads to tracking fronts moving with curvature-dependent speed.
- When the speed is the curvature this leads to one of the classical degenerate non-linear differential equations.
- One naturally wonders, "What is the regularity of solutions?"

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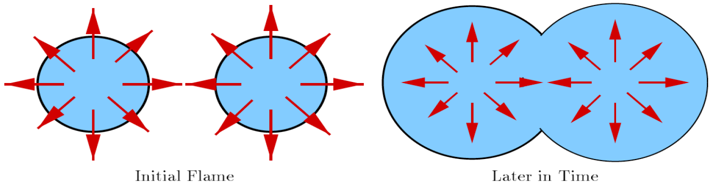
Examples of tracking evolving fronts I

Forest fire.



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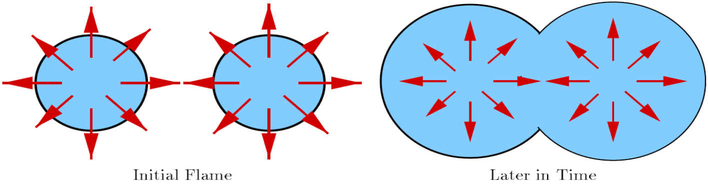

After two fires merge, the evolving front is connected.

Examples of tracking evolving fronts II

Oil droplets in water can be modeled as evolving fronts.

Moral

- We will be interested in optimal regularity of evolution equations for physical phenomena.
- Moral: Without deeply understanding the underlying geometry, it is impossible to prove fine analytical properties.

Joint work with **Bill Minicozzi**.

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Mean curvature I

- Suppose $\Sigma \subset \mathbf{R}^{n+1}$ is a hypersurface.
- $\mathbf{n}$ is the unit normal of Σ.
- $H = \text{div}_\Sigma(\mathbf{n})$ is the mean curvature.
- $\text{div}_\Sigma(\mathbf{n}) = \sum_{i=1}^n \langle \nabla_{e_i} \mathbf{n}, e_i \rangle$; where e_i is an orthonormal basis for the tangent space of Σ.

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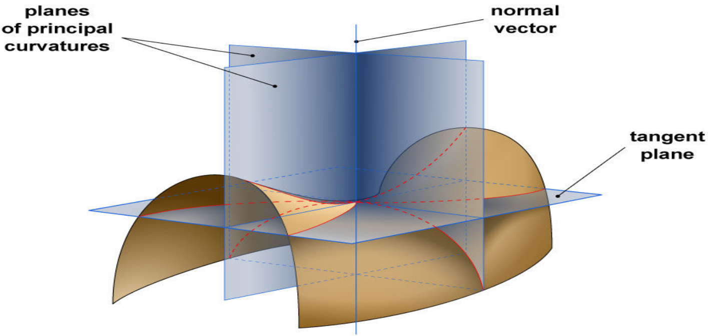

Mean curvature is average of principal curvatures.

Mean curvature III

Physically: mean curvature $=$ surface tension.

Level set

- If $\Sigma = u^{-1}(s)$ for a regular value for the function $u : \mathbf{R}^{n+1} \to \mathbf{R}$.
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Mean curvature flow

Mean curvature flow: Hypersurfaces M_t evolving by

$$\frac{\partial x}{\partial t} = -H\,\mathbf{n}\,.$$

H is the mean curvature, $\mathbf{n}$ the unit normal, of M_t at x.

Origins

"Geometric heat equation".

First studied in mathematics in the 1910s by Birkhoff.

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Simple examples of MCF in $\mathbf{R}^3$:

- 2-spheres of radius $\sqrt{-4t}$ for $t < 0$.
- Cylinders of radius $\sqrt{-2t}$ for $t < 0$.

Both are singular at $t = 0$. “Finite time extinction”.

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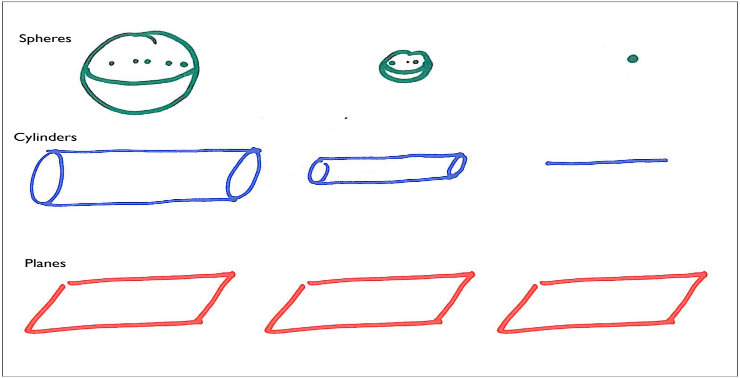

Cylinders, spheres and planes are self-similar solutions of MCF. The shape is preserved, but the scale changes with time.

Two key properties

- H is the gradient of area, so MCF is the negative gradient flow for volume (Vol(M_t) decreases most efficiently).
- Avoidance principle.

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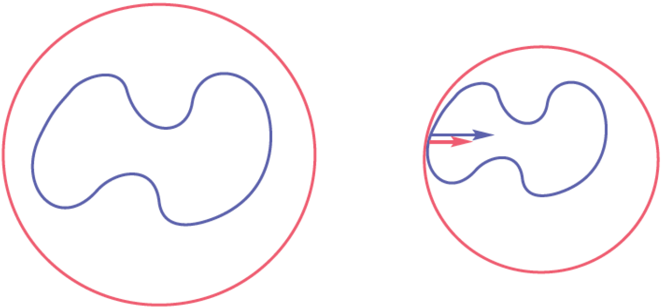
Avoidance property

If M_0 and N_0 are disjoint, then M_t and N_t remain disjoint.



If they touch later, the maximum principle gives a contradiction.

Curve shortening flow

- When $n = 1$ and the hypersurface is a curve, this is the curve shortening flow.
- A (round) circle shrinks through (round) circles to a point in finite time.
- Example of a snake.
- Theorem (Grayson): Any simple closed curve shrinks to a round point in finite time.

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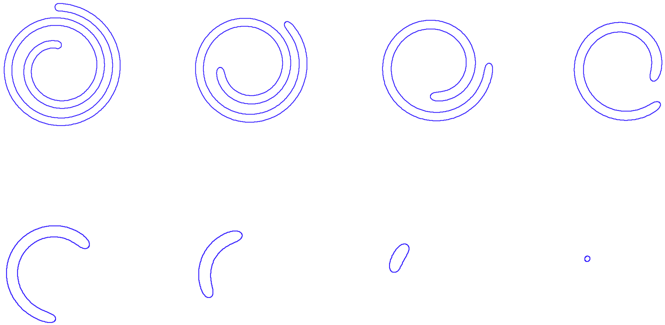
The snake



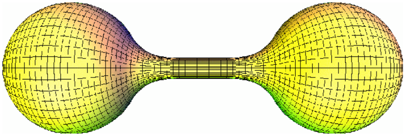
Grayson: even a tightly wound region becomes round under curve shortening flow.

The marriage ring shrinks to a circle

A thin torus of revolution will flow smoothly until it disappears along a circle.

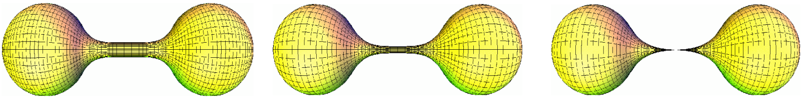
Dumbbell

Grayson's dumbbell, 1989: 2 large bells connected by thin bar.

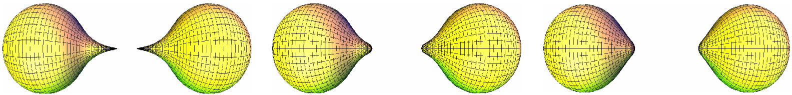


- Under MCF the neck first pinches off, cutting it into two.
- Later, each bell shrinks to a round point.

6 snapshots of the dumbbell MCF



Grayson’s dumbbell; initial surface, step 1 and 2.



Dumbbell; steps 4, 5 and 6.

Revolution: **Altschuler-Angenent-Giga**, **Soner-Souganidis**.

Level set method I

Level set method, **Osher-Sethian**, 1989:

- Choose an initial function on $\mathbf{R}^3$ with M_0 as the level set.
- Simultaneously flow every level set (disjoint by avoidance).

This leads to a degenerate parabolic equation.

Tremendously successful numerically.

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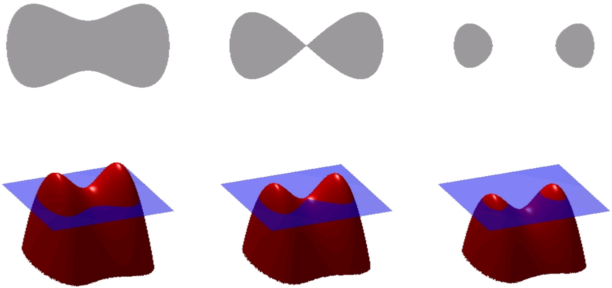
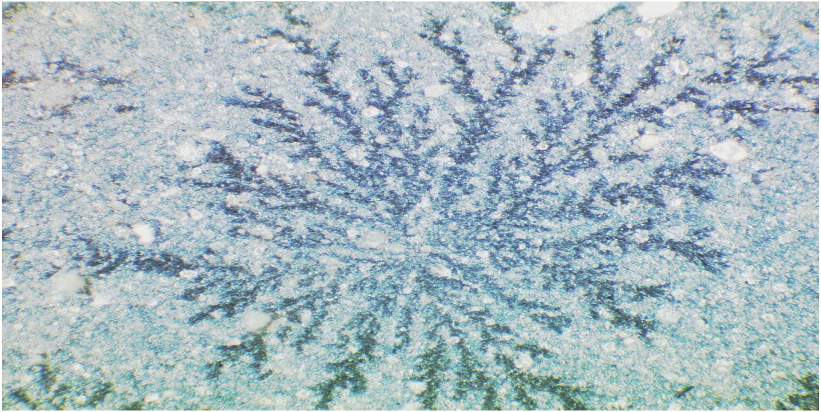


The gray areas represent trees that a forest fire has not yet reached. The fire front is given as a level set of an evolving function in the second line.

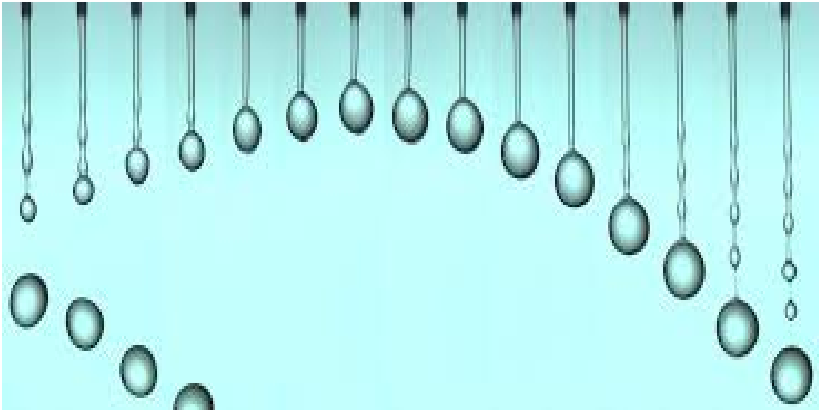
Examples I

Crystal growth can be modeled by the Level Set Method.



Examples II

Droplets can be modeled by the Level Set Method.



Evans-Spruck and **Chen-Giga-Goto**, 1991:

Continuous viscosity solutions, unique, agree with classical solution.

A major success of **Crandall-Lions** viscosity solution theory.

A surface is **mean convex** if $H > 0$.

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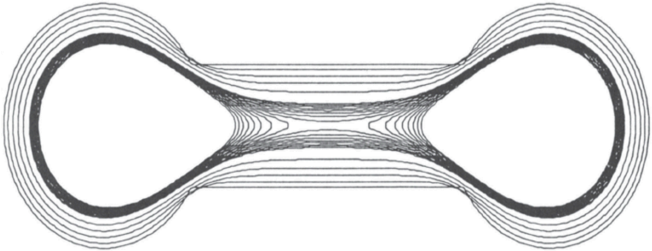
Mean convex MCF II

Define the *arrival time* u on Ω by

$$u(x) = \{t \mid x \in M_t\} .$$



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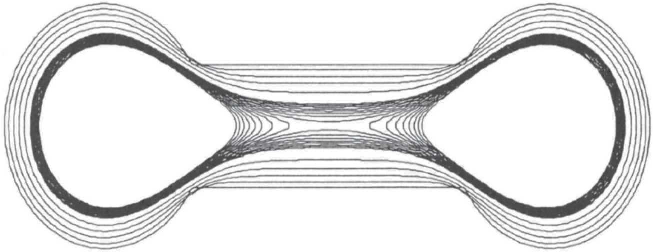

Balacing games

- **Kohn-Serfaty**, 2006: The arrival time has a game theoretic interpretation (to a deterministic game).

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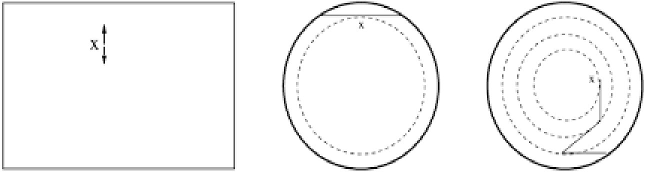

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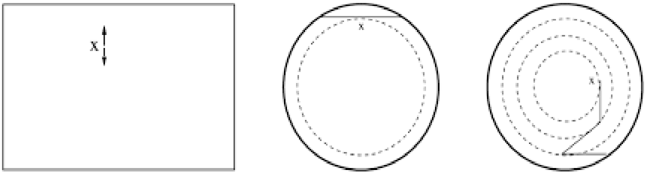

Arrival time for examples

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- u satisfies the degenerate elliptic equation

$$-1 = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \Delta\, u - \mathrm{Hess}_u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right).$$

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Optimal regularity

C-M, 2016:

- u is twice differentiable everywhere.
- At critical points, hessian matches the sphere or cylinder.

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While second derivatives exist, they may not be continuous.

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Differentiability in the convex case

In the **convex case**:

- **Huisken**, 1990: u is C^2.
- **Kohn-Serfaty**, 2006: u is C^3 in $\mathbf{R}^2$.
- **Sesum**, 2008: **convex** M_0 where u is not C^3.

When is u C^2?

C-M, 2016:

C^2 **iff** one critical value and the critical set is either:

1. A single point where Hessian is spherical.
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In (2), the kernel of Hessian is tangent to the curve.

Sphere, cylinder, ring are C^2; dumbbell is not.

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- Twice differentiable is equivalent to that at a critical point the function is up to higher-order terms equal to the quadratic polynomial.
- This second-order approximation is the arrival time of the shrinking round cylinders.
- It suggests that the level sets should be approximately cylinders.

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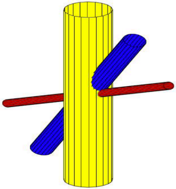

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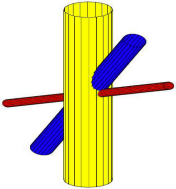

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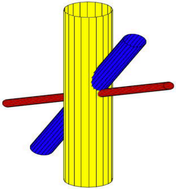

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