Random polygons, Grassmannians, and a problem of Lewis Carroll.

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The Pillow Problem

In 1895, Charles Dodgson (better known by his pen name Lewis Carroll) published a book of 72 mathematical problems designed to be solved “while lying in bed.”

1 Introduction
Triangles live on a hemisphere and are linked to 2 by 2 matrices. The familiar triangle is seen in a different light. New understanding and new applications come from its connections to the modern developments of random matrix theory. You may never look at a triangle the same way again.

We began with an idle question: Are most random triangles acute or obtuse? While looking for an answer, a note was passed in lecture. (We do not condone our behavior!) The note contained an integral over a region in $\mathbb{R}^6$.

The evaluation of that integral gave us a number – the fraction of obtuse triangles. This paper will present several other ways to reach that number, but our real purpose is to provide a more complete picture of “triangle space.”

Later we learned that in 1884 Lewis Carroll (as Charles Dodgson) asked the same question. His answer for the probability of an obtuse triangle (by his rules) was $\frac{3}{8} \pi \frac{p}{3} \pi \frac{64}{72}$. Variations of interpretation lead to multiple answers (see [11, 33] and their references). Portnoy reports that in the first issue of The Educational Times (1886), Woolhouse reached $\frac{9}{8} \frac{4}{\pi} \frac{2}{\pi} \frac{0.72}{72}$. In every case obtuse triangles are the winners – if our mental image of a typical triangle is acute, we are wrong. Probably a triangle taken randomly from a high school geometry book would indeed be acute. Humans generally think of acute triangles, indeed nearly equilateral triangles or right triangles, in our mental representations of a generic triangle. Carroll’s answer is short of our favorite answer $\frac{3}{4}$, which is more mysterious than it seems. There is no paradox, just different choices of probability measure.

The most developed piece of the subject is humbly known as “Shape Theory.” It was the last interest of the first professor of mathematical statistics at Cambridge University, David Kendall [21, 26]. We rediscovered on our own what the shape theorists knew, that triangles are naturally mapped onto points of the hemisphere. It was a thrill to discover both the result and the history of shape space.

We will add a purely geometrical derivation of the picture of triangle space, delve into the linear algebra point of view, and connect triangles to random matrix theory. We hope to rejuvenate the study of shape theory!

Figure 1: Lewis Carroll’s Pillow Problem 58 (January 20, 1884). 25 and 83 are page numbers for his answer and his method of solution. He specifies the longest side $AB$ and assumes that $C$ falls uniformly in the region where $AC$ and $BC$ are not longer than $AB$.

57. (25, 80)
In a given Triangle describe three Squares, whose bases shall lie along the sides of the Triangle, and whose upper edges shall form a Triangle;

(1) geometrically; (2) trigonometrically. $[27/1/91$

58. (25, 83)

Three Points are taken at random on an infinite Plane. Find the chance of their being the vertices of an obtuse-angled Triangle. $[20/1/84$
Carroll’s problem is ill-posed

**Question**

*What does it mean to choose a random triangle?*

*The issue of choosing a “random triangle” is indeed problematic. I believe the difficulty is explained in large measure by the fact that there seems to be no natural group of transitive transformations acting on the set of triangles.*

—Stephen Portnoy, 1994

*(Editor, J. American Statistical Association)*

There have been many approaches which solve the problem of defining a random triangle in different ways [Guy, Kendall, Portnoy, Edelman/Strang, ...].
Choosing a random triangle

Let $a$, $b$, and $c$ be the sidelengths of the triangle. The space of triangles is parametrized by choices of $a$, $b$ and $c$ satisfying a collection of conditions:

\[
a \geq 0,\ b \geq 0,\ c \geq 0,\ \text{and}\ a + b + c = 2
\]

and the triangle inequalities

\[
\begin{align*}
b + c &\geq a \\
a + c &\geq b \\
a + b &\geq c
\end{align*}
\]
We can rewrite the triangle inequalities as

\[-a + b + c \geq 0\]
\[a - b + c \geq 0\]
\[a + b - c \geq 0\]

If \( s = \frac{a+b+c}{2} \) (the semiperimeter) this suggests new variables:

\[s_a := s - a = \frac{-a + b + c}{2} \geq 0\]
\[s_b := s - b = \frac{a - b + c}{2} \geq 0\]
\[s_c := s - c = \frac{a + b - c}{2} \geq 0\]

Note that \( s_a + s_b + s_c = s = 1 \).
This triangle of triangles is covered 8-fold by the sphere.

\[ x^2 = s_a = 1 - a, \quad y^2 = s_b = 1 - b, \quad z^2 = s_c = 1 - c \]

We will use \( x, y, \) and \( z \) as coordinates on triangle space. The measure will be surface area on the sphere.
Proposition

\(|x|, |y|, and |z| are (pairwise) geometric means of the exradii.\)
Rotations of $S^2$ can exchange any two triangles.
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Transitive transformations on the sphere

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Off-axis rotations

Open Question: is there an elegant triangle-theoretic description of this motion?
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The Pythagorean theorem implies that right triangles have
\[ x^2 y^2 = z^2, \quad y^2 z^2 = x^2, \quad \text{or} \quad z^2 x^2 = y^2. \]

Theorem (with Needham, Shonkwiler, Stewart)

The fraction of obtuse triangles is
\[ \frac{3}{2} - \frac{3 \ln 2}{\pi} \approx 0.838093 \]
Generalizing to $n$-gons: start over

Suppose the edges are complex numbers $e_1, \ldots, e_n \in \mathbb{C}$. The polygon is closed, so

$$e_1 + \cdots + e_n = 0$$

Let $e_i = z_i^2$ and $z_i = u_i + i v_i$.

$$0 = e_1 + \cdots + e_n = z_1^2 + \cdots + z_n^2$$

$$= (u_1 + i v_1)^2 + \cdots + (u_n + i v_n)^2$$

$$= (u_1^2 - v_1^2) + i(2u_1 v_1) + \cdots + (u_n^2 - v_n^2) + i(2u_n v_n)$$

$$= (u_1^2 + \cdots + u_n^2 - v_1^2 - \cdots - v_n^2) + 2i(u_1 v_1 + \cdots + u_n v_n).$$

or if $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$

$$|\vec{u}|^2 = |\vec{v}|^2 \quad \text{and} \quad \langle \vec{u}, \vec{v} \rangle = 0$$
Generalizing to $n$-gons: start over (2)

If we fix the total polygon length to be 2, we have

$$2 = |e_1| + \cdots + |e_n| = |z_1|^2 + \cdots + |z_n|^2$$

$$= u_1^2 + v_1^2 + \cdots + u_n^2 + v_n^2$$

$$= |\vec{u}|^2 + |\vec{v}|^2.$$

This gives us:

**Theorem (Knutson/Hausmann 1997)**

*If the edges of an $n$-gon are $e_i = z_i^2$ and each $z_i = u_i + i v_i$, then the polygon is closed and length 2 $\iff$ $\vec{u}$ and $\vec{v}$ are orthonormal.*
Generalizing to $n$-gons: start over (2)

If we fix the total polygon length to be 2, we have

$$2 = |e_1| + \cdots + |e_n| = |z_1|^2 + \cdots + |z_n|^2$$
$$= u_1^2 + v_1^2 + \cdots + u_n^2 + v_n^2$$
$$= |\vec{u}|^2 + |\vec{v}|^2.$$

This gives us:

**Theorem (Knutson/Hausmann 1997)**

The space of closed, length 2 plane polygons is $2^n$-fold covered by the “Stiefel manifold” $V_2(\mathbb{R}^n)$ of orthonormal 2-frames in $\mathbb{R}^n$. 
Rotations in the plane of $\vec{u}$ and $\vec{v}$ rotate the $z_i$ and rotate the edges $e_i = z_i^2$ twice as fast.
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**Theorem (Knutson/Hausmann 1997)**

*The plane spanned by $\vec{u}$ and $\vec{v}$ determines the polygon up to rotation.*
Rotations in the plane of $\vec{u}$ and $\vec{v}$ rotate the $z_i$ and rotate the edges $e_i = z_i^2$ twice as fast.

Theorem (Knutson/Hausmann 1997)

The space of closed, length-2 plane polygons up to translation and rotation is covered by the “Grassmann manifold” $G_2(\mathbb{R}^n)$ of 2-planes in $\mathbb{R}^n$. 
Bringing the pictures together: $G_1(\mathbb{R}^3) = G_2(\mathbb{R}^3)$

If $(x, y, z)$ is orthonormal to $\vec{u}$ and $\vec{v}$, then

$$
\begin{pmatrix}
  x & u_1 & v_1 \\
  y & u_2 & v_2 \\
  z & u_3 & v_3 
\end{pmatrix}
$$

is orthonormal.

So

$$x^2 + u_1^2 + v_1^2 = 1, \quad \text{or} \quad x^2 = 1 - u_1^2 - v_1^2$$

but

$$|z_1|^2 = |e_1| = a$$

so this is our original equation

$$x^2 = 1 - a$$
Definition
Any 2-plane $P$ in $\mathbb{R}^n$ spanned by $\vec{u}, \vec{v}$ is described by a skew-symmetric $n \times n$ matrix of Plücker coordinates $\Delta(P)_{ij} = \det \begin{pmatrix} u_i & v_i \\ u_j & v_j \end{pmatrix} = (u_i, v_i) \times (u_j, v_j)$ defined up to multiplication by a common scalar. (Changing the basis for $P$ only changes the scalar, so the Plücker coordinates depend only on the plane.)

Our coordinates $(x, y, z)$ are the Plücker coordinates in the upper triangle of the 3x3 matrix for $G_2(\mathbb{R}^3)$. 
Definition
The Positive Grassmannian is the portion of the Grassmannian where all Plücker coordinates in the upper triangle are positive

\[ \Delta P_{ij} > 0 \iff i < j. \]

It has attracted a lot of interest in string theory and has a beautiful and somewhat mysterious structure.

Theorem (with Needham, Shonkwiler, Stewart)

The positive Grassmannian \( G_2(\mathbb{R}^n)^+ \) consists of planes \( P \) where \((a_i, b_i)\) lie in a common semicircle and the polygon is convex. \( G_2(\mathbb{R}^n) \) is tiled by \( 2^{n-2} \times (n - 1)! \) isometric copies of \( G_2(\mathbb{R}^n)^+ \).

(A comparable interpretation appears in Section 5.3 of Arkani-Hamed, 2012.)
There is a natural way to measure volume in $G_2(\mathbb{R}^n)$ which is $O(n)$ invariant (Haar measure). Using this as a probability measure on polygons:

**Theorem (with Needham, Shonkwiler, Stewart)**

The probability that a random $n$-gon is convex is $2/(n-1)!$.

**Theorem (with Needham, Shonkwiler, Stewart)**

Among random quadrilaterals, 1/3 are convex, 1/3 are reflex, and 1/3 are self-intersecting.
It is easy to sample a random 2-plane in $\mathbb{R}^n$ uniformly: just pick two vectors of $n$ independent Gaussians and take the plane they span. Here’s a random 500-gon:
Theorem (with Deguchi, Shonkwiler)

The edgelength of a random quadrilateral is uniformly distributed on $[0, 1]$. The edgelength of a random $n$-gon is sampled from a Beta distribution with probability density

$$
\phi(y) = \left( \frac{n}{2} - 1 \right) (1 - y)^{\frac{n}{2} - 2}
$$
Definition
The radius of gyration of an $n$-gon $v_1, \ldots, v_n$ is the average (squared) distance between vertices:

$$\frac{1}{n^2} \sum_{i,j \in 1}^{n} |v_i - v_j|^2$$

Theorem (with Deguchi, Shonkwiler)
*The expected radius of gyration of a random planar $n$-gon is*

$$\frac{2}{3} \frac{n + 1}{n(n + 2)}$$
We can use the geometry of the Grassmannian to easily build geodesic (shortest) paths between (lifts of) polygons.
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Space polygons have a similar structure, if we replace $G_2(\mathbb{R}^n)$ with $G_2(\mathbb{C}^n)$, view edges as quaternions rather than complex numbers, and replace squaring with the Hopf map.
All this structure lets you compute some exact probabilities:

**Theorem (with Deguchi, Shonkwiler)**

The expected radius of gyration of a random $n$-gon in $\mathbb{R}^3$ sampled from $G_2(\mathbb{C}^n)$ is

$$\frac{1}{2n}$$

**Theorem (with Grosberg, Kusner, Shonkwiler)**

The expected total curvature of a random $n$-gon in $\mathbb{R}^3$ sampled from $G_2(\mathbb{C}^n)$ is

$$\frac{\pi}{2}n + \frac{\pi}{4} \frac{2n}{2n - 3}$$
From here, you can go in various directions:

- polygons of fixed edgelength (e.g. equilateral polygons) (with Shonkwiler-Duplantier-Uehara)
- polygons of fixed thickness (Chapman, Plunkett)
- linkages and computational geometry
- polygons of fixed bending angle (e.g. molecular models)
- curves instead of polygons (Needham)
- different topologies, such as $\theta$-curves (Deguchi, Uehara)
- shape recognition (Needham, Mumford-Shah-Younes)
- random knots and links (Chapman, Hass, Millett, Rawdon)

...and we invite you to our paper session tomorrow morning!

8:30-11:50am, A602, Atrium Level, Marriott Marquis