Real loci in symplectic manifolds

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Background

Let M be a manifold equipped with a nondegenerate closed 2-form ω (the symplectic form).

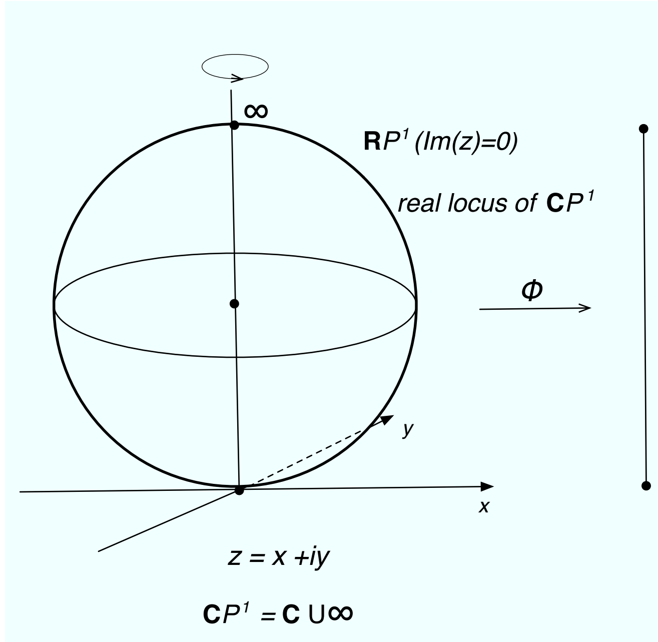
We assume that a Lie group G acts preserving ω, and that the action is obtained from the Hamiltonian flows of a collection of Hamiltonian functions (moment maps $\Phi : M \to Lie(G)^*$).

Example

Examples: the orbits of the (co)adjoint action of G on its Lie algebra are symplectic manifolds, and the moment map is the inclusion into the Lie algebra.

Special case: $S^1 \subset SU(2)$ acts by rotation on S^2, which is the orbit of the adjoint action of $SU(2)$ on its Lie algebra.

Example 1: S^1 rotation action on $S^2 = \mathbb{C}P^1$



Two ingredients:

- rotation action $S^1 \curvearrowright \mathbb{C}P^1$ via $e^{i\theta}[z_1, z_2] = [z_1, e^{i\theta}z_2]$, moment map Φ (height function).

- automorphism τ of $\mathbb{C}P^1$ given by $\tau[z_1, z_2] = [\overline{z_1}, \overline{z_2}]$, fixed points $\mathbb{R}P^1$

Facts:

1. $\boxed{\Phi(\mathbb{C}P^1) = \Phi(\mathbb{R}P^1)}$

2. Cohomology rings:

- *ordinary cohomology*

$H^*(\mathbb{C}P^1; \mathbb{Z}_2) = \mathbb{Z}_2[c]/\langle c^2 \rangle, \deg c = 2$

$H^*(\mathbb{R}P^1; \mathbb{Z}_2) = \mathbb{Z}_2[w]/\langle w^2 \rangle, \deg w = 1$

$\Rightarrow \boxed{H^{2*}(\mathbb{C}P^1; \mathbb{Z}_2) \simeq H^*(\mathbb{R}P^1; \mathbb{Z}_2)}$

General Situation: A *conjugation space* [Hausmann, Holm, Puppe 2005] is a symplectic manifold (M, ω) with

- Hamiltonian torus action $T \curvearrowright M$, moment map $\Phi : M \to \mathfrak{t}^*$
- involution $\tau : M \to M$

$(\tau \circ \tau = \mathrm{id}_M)$

$\tau^* \omega = -\omega$ (τ is anti-symplectic)

τ compatible with T action [Duistermaat 1983]:

$$\boxed{\tau(t.x) = t^{-1}.\tau(x), \ \forall t \in T, x \in M}$$

Denote $M^\tau = \{x \in M \ : \ \tau(x) = x\}$ (*real locus*). The real locus is a Lagrangian submanifold of M (in other words ω restricts to 0 on it, and its dimension is half the dimension of M).

Examples (HHP 2005):

- coadjoint orbits (with the Chevalley involution)
- toric manifolds
- complex Grassmannians
- polygon spaces (for example Klyachko 1994, Kapovich-Millson 1996)
- $\mathbb{C}P^n$ with the standard action of $U(1)^n$ and the involution given by complex conjugation.

  The real locus is $\mathbb{R}P^n$.

Assume M is compact.

Theorem 1. (Duistermaat 1983)
$\Phi(M) = \Phi(M^\tau)$.

Theorem 2. (Hausmann-Holm-Puppe 2005) *Under some additional hypotheses (e.g. M^T discrete) we have ring isomorphisms*
$H^{2*}(M;\mathbb{Z}_2) \simeq H^*(M^\tau;\mathbb{Z}_2)$

Goldin-Holm (2004) studied the real locus of a symplectic reduced space under a torus action, and its image under the moment map.

II. Toric manifolds

Delzant's theorem: Toric manifolds: If M is a symplectic manifold of real dimension $2n$ admitting an effective Hamiltonian action of a torus of dimension n (a toric manifold), the image of the moment map is a convex polyhedron (the Delzant polytope). Any two such with the same moment polytope are diffeomorphic via a T-equivariant diffeomorphism that respects the moment maps.

If M is a toric manifold with a compatible antisymplectic involution, then Duistermaat's theorem asserts that the moment map maps the fixed point set of the involution onto the Delzant polytope. This is not necessarily a bijection though.

Example: projective space $\mathbb{C}P^n$ is equipped with the Hamiltonian torus action of $U(1)^n$ and the involution given by complex conjugation. The fixed point set is $\mathbb{R}P^n$. A simple examination of the fixed point set shows that the moment map is not a bijection between the fixed point set and the tetrahedron.

III. THE BASED LOOP GROUP ΩG (Pressley-Segal 1988)

Goal. Extend Theorems 1 and 2 to $M = \Omega G$ (based loops in the compact Lie group G)

Set-up:

- G is a simply connected simple compact Lie group, $T \subset G$ maximal torus
- $\Omega G = \{\gamma : S^1 \to G \ : \ \gamma(1) = e\}$
- T action on ΩG: $(t.\gamma)(z) = t\gamma(z)t^{-1}$
- S^1 action on ΩG: $(e^{i\theta}\gamma)(z) = \gamma(e^{i\theta}z)\left(\gamma(e^{i\theta})\right)^{-1}$ ("rotation" – note we must preserve the condition that $\gamma(1) = e$)

- $T \times S^1 \curvearrowright \Omega G$ is Hamiltonian, moment map $\Phi : \Omega(G) \to \mathrm{Lie}(T) \oplus i\mathbb{R}$
- $\tau : G \to G$ group automorphism, $\tau \circ \tau = \mathrm{id}_G$

Note. Any semisimple compact Lie group G has an involution τ such that $\tau(t) = t^{-1}$ for all t in a *maximal* torus $T \subset G$. This τ is essentially unique (the *Chevalley involution*).

Example: $G = SU(n)$, $\tau(g) = \bar{g}$.

$\Rightarrow \tau : \Omega(G) \to \Omega(G)$, $\tau(\gamma)(e^{i\theta}) = \tau(\gamma(e^{-i\theta}))$

Obviously $\tau \circ \tau = \mathrm{id}_{\Omega G}$

- Symplectic form:

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} < \xi(\theta), \frac{d\eta}{dt}(\theta) > d\theta$$

- Moment map for T action:

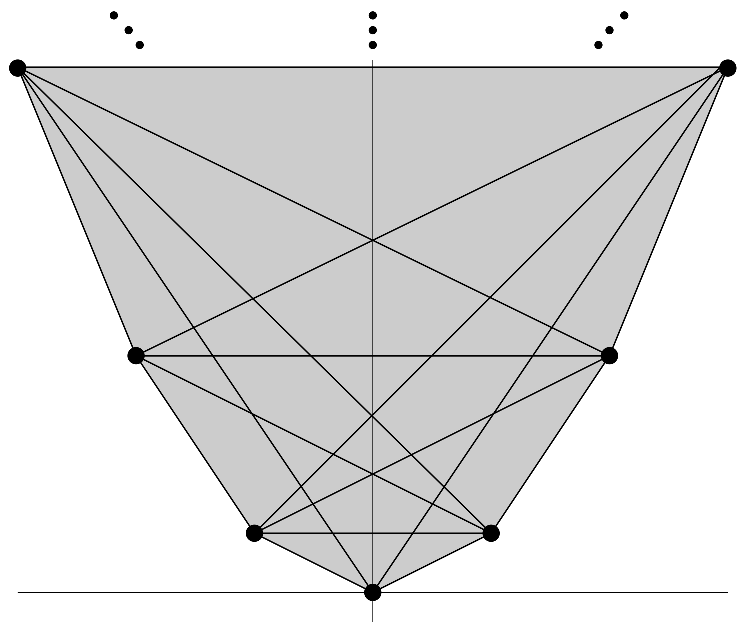
$$p(\gamma) = \frac{1}{2\pi} \int pr_{\mathfrak{t}}(\gamma^{-1} \frac{d\gamma}{dt}) d\theta$$

- Moment map for S^1 action:

$$E(\gamma) = \frac{1}{4\pi} \int |(\gamma^{-1} \frac{d\gamma}{dt})|^2 d\theta$$

Theorem (Atiyah-Pressley 1983) The moment map $(E,p) : \Omega G \to \mathrm{Lie}(T) \oplus i\mathbb{R}$ is convex: in fact

$\Phi(\Omega G)) =$ convex hull $\Phi(\Omega(G)^{T\times S^1})$. Identify fixed point set of $T \times S^1$ action on ΩG: it consists of homomorphisms from S^1 to T (the integer lattice).



$\Phi(\Omega(SU(2))$

IV. DUISTERMAAT TYPE CONVEXITY

Would like Duistermaat convexity:

$$(?)\ \Phi(\Omega G)^\tau) = \Phi(\Omega(G))$$

Define an involution τ on ΩG as above: then τ compatible with the action $T \times S^1 \curvearrowright \Omega G$.

Extension of Duistermaat's convexity theorem to the based loop group:

(*)

$$\tau(s) = s^{-1}$$

for all s in a maximal torus of G

Theorem. (Jeffrey-Mare 2010) If (*) holds, then the moment map $\Phi : \Omega G \to \mathrm{Lie}(T) \times i\mathbb{R}$ satisfies $\Phi(\Omega G) = \Phi\Big((\Omega G)^\tau\Big)$.

Example.
$G = SU(n)$, $\tau(g) = \bar{g}\ \forall g \in SU(n)$.
Standard maximal torus T:
$t = \mathrm{Diag}(z_1, \ldots, z_n)$ where
$|z_1| = \ldots = |z_n| = 1, z_1 \ldots z_n = 1$.
Check $\tau(t) = t^{-1}\ \forall t \in T$.

$$\Rightarrow \Phi(\Omega G) = \Phi\Big((\Omega G)^\tau\Big).$$

0. $\Phi(\Omega G) \subset \mathrm{Lie}(T) \oplus i\mathbb{R}$ is a
convex polytope (Atiyah-Pressley)

1. $\Phi\left((\Omega G)^{\tau}\right) \subset \mathrm{Lie}(T) \oplus i\mathbb{R}$ is convex
(from Chuu-Lian Terng's convexity theorem for isoparametric submanifolds in Hilbert space)

2. If $\gamma \in \Omega G$ is such that $\Phi(\gamma)$ is a vertex of $\Phi(\Omega G)$, then $\Phi(\gamma) = \Phi(\tilde{\gamma})$, for some $\tilde{\gamma} \in \Omega(G)^{\tau}$.

0, 1, 2 $\Rightarrow \Phi(\Omega G)) \subset \Phi\Big((\Omega G)^{\tau}\Big)$

V. Extension of Duistermaat's theorem to $H^*(\Omega G^\tau)$

Observation: if $K := \{g \in G \ : \ \tau(g) = g\}$, then G/K is a Riemannian symmetric space.

Bott and Samelson 1958, 'mysterious application':

$$\dim H^{2q}(\Omega(G);\mathbb{Z}_2) = \dim H^q(\Omega(G/K);\mathbb{Z}_2)\ \forall q \geq 0$$

$$(\Omega G)^\tau = \{\gamma : S^1 \to G | \tau\big(\gamma(e^{i\theta})\big) = \gamma(e^{-i\theta})$$

This identifies $(\Omega G)^\tau$ with

$$\{\gamma : [0,1] \to G | \gamma(0) = e, \gamma(1) \in K\}$$

where $K = \{k \in G | \tau(k) = e\}$.

$\Omega(G/K)$ is homotopy equivalent to $(\Omega G)^\tau$

(follows from homotopy theory argument known since introduction of Borel construction, late 1950's)

Example: $G = SU(2)$, τ complex conjugation

$K = SO(2) \cong U(1)$

$G/K = S^2 = \mathbb{C}P^1$

Mitchell (1988) gave an exposition which nicely explains Bott-Samelson's "mystery":

Deduce CW decomposition of $(\Omega_{\text{alg}}G)^\tau = \bigsqcup_j C_j^\tau$.

Conclusion:

$$\dim H^{2q}\Big((\Omega G);\mathbb{Z}_2\Big) = \dim H^{2q}\Big(\Omega_{\text{alg}}G;\mathbb{Z}_2\Big)$$
$$= \#(2q\text{dimensional cells } C_j)$$
$$= \#(q\ -\text{dimensional cells } C_j^\tau) = \dim H^q(\Omega_{\text{alg}}G)^\tau;\mathbb{Z}_2) = \dim H^q(\Omega(G)^\tau;\mathbb{Z}_2)$$

Theorem. (Jeffrey and Mare) *If τ is Chevalley involution of G and $K = G^\tau$, then we have ring isomorphisms*

(*) $H^{2*}\left(\Omega G; \mathbb{Z}_2\right) \simeq H^*(\Omega(G/K); \mathbb{Z}_2)$

Main ideas of the proof.

- Identify $\Omega(G/K) = (\Omega G)^\tau$
- Replace ΩG by $\Omega_{\text{alg}} G$ (see above).

Key point:

- τ leaves each cell of the CW decomposition invariant and acts on it as complex conjugation (see above)

Thus, $(\Omega_{\text{alg}} G, \tau)$ is a *spherical conjugation complex* in the sense of Hausmann, Holm, and Puppe (2005). Then (*) is a direct application of results of [HHP] about spherical conjugation complexes with compatible torus actions.

VI. THE RINGS $H^*(\Omega(G/K);\mathbb{Z}_2)$

Examples:

$$G = SU(2):$$

$$K = SO(2) = S^1$$

$$G/K = S^2$$

$H^*(\Omega G;\mathbb{Z}_2) = \Lambda(\gamma_1, \gamma_2, \gamma_4, \gamma_8, \ldots)$ (where the degree of γ_j is $2j$).

Note: This is not valid without the assumption that the coefficient system is $\mathbb{Z}_2$.

$$H^*(\Omega S^2;\mathbb{Z}_2) = \Lambda(y_1, \delta_1, \delta_2, \delta_4, \delta_8, \ldots)$$

(where y_1 is a cohomology class of degree 1 and δ_j are cohomology classes of degree $2j$). The ring isomorphism

$$\mathcal{I} : H^*(\Omega S^2;\mathbb{Z}_2) \to H^*(\Omega G;\mathbb{Z}_2)$$

sends y_1 to γ_1 and δ_j to γ_{j+1} for all $j \geq 1$.

VII. Torus actions on moduli spaces of flat connections on 2-manifolds

- Moduli spaces of conjugacy classes of representations of the fundamental group of a 2-manifold into a compact Lie group G have a natural system of Hamiltonian flows (Goldman 1986).
- J-Weitsman (1992) observed that these flows are moment maps for a Hamiltonian torus action on an open dense set of moduli space. In the case $G = SU(2)$ the dimension of the torus that acts is half the dimension of the moduli space.
- These moduli spaces of conjugacy classes of representations are ordinarily singular, but for genus 2 and $G - SU(2)$ Narasimhan-Ramanan showed that the moduli space is smooth and isomorphic to $\mathbb{C}P^3$.
- In recent joint work with Nan-Kuo Ho, Khoa Dang Nguyen and Eugene Xia, we have concluded that the Jeffrey-Weitsman torus actions can be used to identify the preimages of the interior of the moment polytope and its faces with the corresponding subsets of $\mathbb{C}P^3$.
- We have also showed that the genus 2 moduli space is a conjugation space, by exhibiting an involution compatible with the torus action.

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