The Energy Critical Wave Equation

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The theory of nonlinear dispersive equation has seen a spectacular development in the last 35 years. These equations model phenomena of wave propagation coming from physics and engineering. The areas that give rise to these equations are water waves, optics, lasers, ferromagnetism, general relativity, sigma models, nonlinear elasticity, and many others. These equations also have connections to geometric flows and to Kähler and Minkowski geometries.

Examples of such equations are the generalized KdV equations (water waves)

$$\begin{cases} \partial_t u - \partial_x^3 u + u^k \partial_x u = 0, & x \in \mathbb{R}, t \in \mathbb{R} \\ u\big|_{t=0} = u_0, \end{cases}$$

the nonlinear Schrödinger equations (optics, lasers, ferromagnetism)

$$\begin{cases} i\partial_t u + \Delta u \pm |u|^{p-1} u = 0, & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u\big|_{t=0} = u_0, \end{cases}$$

and the nonlinear wave equation (sigma models, nonlinear elasticity, general relativity)

$$\begin{cases} \partial_t^2 u - \Delta u \pm |u|^{p-1} u = 0, & x \in \mathbb{R}^N, t \in \mathbb{R} \\ u\big|_{t=0} = u_0, \quad \partial_t u\big|_{t=0} = u_1. \end{cases}$$

The initial works studied the behavior of special solutions such as solitons/traveling waves. Then, there was a systematic study of the local well–posedness theory, using extensively tools from harmonic analysis.

The last 25 years have seen a lot of interest in the study of the long–time behavior of solutions, for large data. Issues like blow–up, global existence, scattering and long–time asymptotic behavior have come to the forefront, especially in critical problems.

We will concentrate the discussion on the energy critical wave equation in the focusing case (– sign above). In the defocusing case it was shown (1990–2000) that all data in the energy space yield global solutions which scatter.

The focusing case is very different, since one can have finite–time blow–up, or solutions which exist for all time that do not scatter.

The ultimate goal in this enterprise is to prove soliton resolution for all solutions of the focusing energy critical wave equation which remain bounded in the energy space.

I will describe the progress towards this, obtained in the last 10 years. The hope is that the results that we will describe will be a model for what to strive for in the study of other nonlinear dispersive equations.

Since the 1970's there has been a widely held belief that "coherent structures" describe the long–time asymptotic behavior of general solutions to nonlinear hyperbolic/dispersive equations.

This belief has come to be known as the soliton resolution conjecture.

This is one of the grand challenges in partial differential equations. Loosely speaking, this conjecture says that the long–time evolution of a general solution of most hyperbolic/dispersive equations, asymptotically in time decouples into a sum of modulated solitons (traveling wave solutions) and a free radiation term (linear solution) which disperses to 0.

This is a beautiful, remarkable conjecture which postulates a "simplification" of the very complicated dynamics into a superposition of simple "nonlinear objects," namely traveling waves solutions, and radiation, a linear object.

Until recently, the only cases in which these asymptotics had been proved was for integrable equations (which reduce the nonlinear problem to a collection of linear ones) and in perturbative regimes.

In 2012–13, Duyckaerts–K–Merle broke the impasse by establishing the desired asymptotic decomposition for radial solutions of the energy critical wave equation in 3 space dimensions, first for a well–chosen sequence of times, and then for general times.

This is the equation

$$\begin{cases} \partial_t^2 u - \Delta u - |u|^{4/(N-2)} u = 0, \quad (x,t) \in \mathbb{R}^N \times I \\ u|_{t=0} = u_0 \in \dot{H}^1, \quad \partial_t u|_{t=0} = u_1 \in L^2, \end{cases} \qquad \text{(NLW)}$$

$N = 3, 4, 5, 6 \ldots$ Here, I is an interval, $0 \in I$.

In this problem, small data yield global solutions which "scatter," while for large data, we have solutions $u \in C(I; \dot{H}^1 \times L^2)$, with a maximal interval of existence $(T_-(u), T_+(u))$ and $u \in L^{2(N+1)/(N-2)}(\mathbb{R}^N \times I')$ for each $I' \Subset I$.

The energy norm is "critical" since for all $\lambda > 0$, $u_\lambda(x,t) := \lambda^{-(N-2)/2} u(x/\lambda, t/\lambda)$ is also a solution and

$$\|(u_{0,\lambda}, u_{1,\lambda})\|_{\dot{H}^1 \times L^2} = \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}.$$

The equation is focusing, the conserved energy is

$$E(u_0, u_1) = \frac{1}{2} \int |\nabla u_0|^2 + |u_1|^2 dx - \frac{N-2}{2N} \int |u_0|^{2N/(N-2)} dx.$$

It is easy to construct solutions which blow–up in finite time say at $T = 1$, by considering the ODE. For instance, when $N = 3$, $u(x,t) = \left(\frac{3}{4}\right)^{1/4}(1-t)^{-1/2}$ is a solution, and using finite speed of propagation it is then easy to construct solutions with $T_+ = 1$, such that $\lim_{t\uparrow T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1\times L^2} = \infty$. This is called type I blow–up.

There exist also type II blow–up solutions, i.e. solutions for which $T_+ < \infty$, and $\sup_{0<t<T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$. Here the break–down occurs by "concentration." The existence of such solutions is a typical feature of energy critical problems.

The first example of such solutions (radial) were constructed for $N = 3$ by Krieger–Schlag–Tataru (2009), then for $N = 4$ by Hillairet–Raphael (2012), and recently by Jendrej (2015) for $N = 5$.

For this equation one expects soliton resolution for type II solutions, i.e. solutions such that $\sup_{0<t<T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$, where T_+ may be finite or infinite.

Some examples of type II solutions when $T_+ = \infty$ are: scattering solutions, that is solutions such that $T_+ = \infty$, and $\exists (u_0^+, u_1^+) \in \dot{H}^1 \times L^2$, such that

$$\lim_{t\to\infty} \left\| (u(t), \partial u(t)) - \left(S(t)(u_0^+, u_1^+), \partial_t S(t)(u_0^+, u_1^+) \right) \right\|_{\dot{H}^1 \times L^2} = 0,$$

where $S(t)(u_0^+, u_1^+)$ is the solution to the associated linear equation with data (u_0^+, u_1^+). For example, for (u_0, u_1) small in $\dot{H}^1 \times L^2$, we have a scattering solution.

Other examples of type II solutions of (NLW) with $T_+ = \infty$ are the stationary solutions, that is the solutions $Q \neq 0$ of the elliptic equation

$$\Delta Q + |Q|^{4/(N-2)} Q = 0, \quad Q \in \dot{H}^1.$$

We say $Q \in \Sigma$.

For example,

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-(N-2)/2}$$

is such a solution. These stationary solutions do not scatter (if u scatters then $\int_{|x|<1} |\nabla_{x,t} u(x,t)|^2 dx \to 0$ as $t \to \infty$). W has several important characterizations: up to sign and scaling it is the only radial, non–zero solution. Up to translation and scaling it is also the only non–negative solution.

However, there is a continuum of variable sign, non–radial $Q \in \Sigma$ (Ding 1986, Del Pino–Musso–Pacard–Pistoia 2011, 2013). W also has a variational characterization as the extremizer for the Sobolev embedding $\|f\|_{L^{2N/(N-2)}} \leq C_N \|\nabla f\|_{L^2}$. It is referred to as the "ground state."

In 2008, K–Merle established the following "ground state conjecture" for (NLW). For u a solution of (NLW) with $E(u_0, u_1) < E(W, 0)$, the following dichotomy holds: if $\|\nabla u_0\| < \|\nabla W\|$ then $T_+ = \infty$, $T_- = -\infty$, and u scatters in both time directions, while if $\|\nabla u_0\| > \|\nabla W\|$, then $T_+ < \infty$ and $T_- > -\infty$. The case $\|\nabla u_0\| = \|\nabla W\|$ is vacuous because of variational considerations. The threshold case $E(u_0, u_1) = E(W, 0)$ was completely described by Duyckaerts–Merle (2008) in an important work.

The proof of the "ground state conjecture" was obtained through the "concentration–compactness/rigidity theorem" method, introduced by K–Merle for this purpose, which has since become the standard tool to understand the global in time behavior of solutions, below the ground–state threshold, for critical dispersive problems.

Other non–scattering solutions, with $T_+ = \infty$, are the traveling wave solutions. They are obtained as Lorentz transforms of $Q \in \Sigma$. Let $\vec{\ell} \in \mathbb{R}^N$, $|\vec{\ell}| < 1$. Then,

$$\begin{aligned} Q_{\vec{\ell}}(x,t) &= Q_{\vec{\ell}}(x - t\vec{\ell}, 0) \\ &= Q\left(\left[\frac{-t}{\sqrt{1-|\vec{\ell}|^2}} + \frac{1}{|\vec{\ell}|^2}\left(\frac{1}{\sqrt{1-|\vec{\ell}|^2}} - 1\right)\vec{\ell}\cdot x\right]\vec{\ell} + x\right) \end{aligned}$$

is a traveling wave solution of (NLW).

These have been shown by Duyckaerts–K–Merle in 2014 to be the only traveling wave solutions.

When K–Merle introduced the "concentration–compactness/rigidity theorem" method to study critical dispersive problems, the ultimate goal was to establish the soliton resolution conjecture.

As I said earlier, for (NLW) one expects to have soliton resolution for type II solutions. Thus, if u is a type II solution, one would want to show that $\exists J \in \mathbb{N} \cup \{0\}$, Q_j, $j = 1, \ldots, J$, $Q_j \in \Sigma$, $\vec{\ell}_j \in \mathbb{R}^N$, $|\vec{\ell}_j| < 1$, $1 \leq j \leq J$, such that, if $t_n \uparrow T_+$ (which may be finite or infinite), there exist $\lambda_{j,n} > 0$, $x_{j,n} \in \mathbb{R}^N$, $j = 1, \ldots, J$, with $\frac{\lambda_{j,n}}{\lambda j',n} + \frac{\lambda_{j',n}}{\lambda j,n} + \frac{|x_{j,n} - x_{j',n}|}{\lambda j,n} \to_n \infty$ for $j \neq j'$ (orthogonality of the parameters) and a linear solution $v_L(x,t)$ (the radiation term) such that

$$
\begin{aligned}
&(u(t_n), \partial_t u(t_n)) \\
&\quad = \sum_{j=1}^{J} \left(\frac{1}{\lambda_{j,n}^{(N-2)/2}} Q^j_{\vec{\ell}_j}\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, 0 \right), \frac{1}{\lambda_{j,n}^{N/2}} \partial_t Q^j_{\vec{\ell}_j}\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, 0 \right) \right) \\
&\qquad + (v_L(x, t_n), \partial_t v_L(x, t_n)) + o_n(1)
\end{aligned}
$$

as $n \to \infty$.

This has been proven in the radial case, $N = 3$ (DKM 12', 13'), and in the general case until recently only when $N = 3, 5$, $T_+ < \infty$ and u is "close" to W, (DKM 11').

Let me discuss now the radial results. In DKM 12', the decomposition was proved for a well–chosen sequence of times $\{t_n\}_n$, while in DKM 13' it was proven for any sequence of times $\{t_n\}_n$.

Let me first quickly describe the proof of the 13' result. The key new idea was the use of the "channel of energy" method introduced by DKM, which was used to quantify the ejection of energy as we approach the final time of existence T_+.

The main new fact shown was that if u is a radial, type II, non–scattering solution, $\exists r_0 > 0$, $\eta > 0$, and a small radial global solution $\tilde{u}$, with $u(r,t) = \tilde{u}(r,t)$, for $r \geq r_0 + |t|$, $t \in I_{max}(u)$, such that $\forall t \geq 0$ or $\forall t \leq 0$,

$$\int_{|x| \geq |t| + r_0} |\nabla_{x,t} \tilde{u}(x,t)|^2 dx \geq \eta.$$

The key tool for proving this is what I like to call "outer energy lower bounds," which are valid for solutions of the linear wave equation. Let $N = 3$, for $r_0 > 0$, $P_{r_0} = \left\{(ar^{-1}, 0) : a \in \mathbb{R}, r \geq r_0\right\} \subset \dot{H}^1 \times L^2(r \geq r_0)$. Let $\pi_{r_0}^{\perp}$ be the orthogonal projection onto the orthogonal complement of P_{r_0}.

Then: for v a radial solution of the linear wave equation, $\forall t \geq 0$ or $\forall t \leq 0$, we have

$$\int_{|x| \geq |t| + r_0} |\nabla_{x,t} v|^2 \geq \frac{1}{2} \left\| \pi_{r_0}^{\perp}(v_0, v_1) \right\|^2_{\dot{H}^1 \times L^2(r \geq r_0)} \quad \text{(DKM 09').} \qquad (1)$$

In the non–radial case, we have for $N = 3, 5, 7, \ldots$ for v a solution of the linear wave equation, $\forall t \geq 0$ or $\forall t \leq 0$

$$\int_{|x| \geq |t|} |\nabla v_{x,t}|^2 dx \geq \frac{1}{2} \int |\nabla v_0|^2 + |v_1|^2 dx \quad \text{(DKM 11')}. \qquad (2)$$

When $r_0 = 0$, the two inequalities coincide.

The argument in DKM 12 was different. It first showed that "self–similar" blow–up is impossible: $\forall 0 < \lambda < 1$, $(T_+ = 1)$,

$$\lim_{t \uparrow 1} \int_{\lambda(1-t) \leq |x| \leq t} |\nabla_{x,t} u(x,t)|^2 dx = 0.$$

This was shown using (1). Combining this with "virial identities," one then shows that

$$\lim_{t \uparrow 1} \frac{1}{1-t} \int_t^1 \int_{|x|<1-s} |\partial_t u(s)|^2 dx ds = 0,$$

which combined with a "Tauberian argument" and that the only static, radial solution is W (up to scaling and sign) gives that all "blocks" are scalings of $\pm W$, and (2) finishes the proof.

We next turn our attention to higher dimensions and the non–radial case. Before doing so, let me mention that the techniques just explained have found important applications to the study of equivariant wave maps and to the defocusing energy critical wave equation with a trapping potential, in works of Côte, Lawrie, Schlag, Liu, Jia, K, etc.

Now we should mention a fundamental fact, proved by Côte–K–Schlag 13': (1) and (2) fail for all even N, radial solutions. However, (2) holds for $N = 4, 8, 12, \ldots$ for $(v_0, v_1) = (v_0, 0)$ and for $N = 6, 10, 14, \ldots$ for $(v_0, v_1) = (0, v_1)$, but not necessarily otherwise.

Moreover, K–Lawrie–Liu–Schlag 14' have shown that an analogue of (1) holds for all odd N, u radial, and applied this to a stable soliton resolution for exterior wave maps on $\mathbb{R}^3$.

In 14', Casey Rodriguez used this analogue of (1) for all odd N to prove the radial case of soliton resolution along a well–chosen sequence of times for (NLW) in all odd dimensions, following the argument in DKM 12'. The even dimensional case presented new challenges, because of the failure of (1), (2). They were overcome, when $N = 4$, by Côte–K–Lawrie–Schlag, using an analogy with wave maps, and by Jia–K ($N = 6, 8, \ldots$) who introduced a new method, bypassing (1), (2).

I would like to conclude with some recent results in the non–radial setting for all (even and odd) dimensions. In the summer of 2015, Hao Jia was able to extend an analogy with wave maps to the non–radial setting and thus control the flux, when $T_+ < \infty$, i.e. the type II blow–up case. This built on earlier work of Côte–Lawrie–K–Schlag and Jia–K.

This allowed him to obtain a Morawetz type identity (adapted from the wave maps one), to find a well–chosen sequence of times $t_n \to T_+ < \infty$, so that the desired decomposition holds in the non–radial case when $T_+ < \infty$, with an error tending to 0 in the dispersive sense.

In the case $T_+ = \infty$, one new difficulty is the extraction of the linear radiation term. This has been done recently by DKM 16' (arXiv). Moreover, very recently, in the joint work of D–Jia–K–M 16' (arXiv) we have obtained the soliton resolution for a well–chosen sequence of times, for general type II solutions, both in the case $T_+ < \infty$ and $T_+ = \infty$. The result is:

Theorem (DJKM 16')

Let u be a solution of (NLW) *with* $\sup_{0<t<T_+} \|\vec{u}(t)\|_{\dot{H}^1\times L^2} < \infty$.
(i) $T_+ < \infty$. *Let S be the set of singular points. Fix $x_* \in S$. Then* $\exists J_* \in \mathbb{N}$, $J_* \geq 1$, *and* $r_* > 0$, $(v_0, v_1) \in \dot{H}^1 \times L^2$, *a time sequence* $t_n \to T_+$ *(well–chosen), scales* $\lambda_n^j, 0 < \lambda_n^j \ll T_+ - t_n$, *positions* $c_n^j \in \mathbb{R}^3$, $c_n^j \in B_{\beta(T_+-t_n)}(x_*)$, $\beta \in [0,1)$ *with* $\vec{\ell}_j = \lim_n \frac{c_n^j}{T_+-t_n}$ *well defined,* $|\vec{\ell}_j| \leq \beta$ *and traveling waves* $Q^j_{\vec{\ell}_j}$, $1 \leq j \leq J_*$ *such that , inside* $B_{r_*}(x_*)$ *we have*

$$
\begin{aligned}
\vec{u}(t_n) &= (v_0, v_1) \\
&+ \sum_{j=1}^{J_*}\left((\lambda_n^j)^{-1/2} Q^j_{\vec{\ell}_j}\left(\frac{x-c_n^j}{\lambda_n^j}, 0\right), (\lambda_n^j)^{-3/2}\partial_t Q^j_{\vec{\ell}_j}\left(\frac{x-c_n^j}{\lambda_n^j}, 0\right)\right) \\
&+ o_{\dot{H}^1\times L^2}(1) \quad \text{as } n \to \infty,
\end{aligned}
$$

in $\dot{H}^1 \times L^2$ *as* $n \to \infty$. *Moreover,* $\lambda_n^j/\lambda_n^{j'} + \lambda_n^{j'}/\lambda_n^j + |c_n^j - c_n^{j'}|/\lambda_n^j \to_n \infty$, $j \neq j'$.

Theorem (DJKM 16' Contd.)

(ii) $T_+ = \infty$. $\exists ! v_L$ *solving the linear wave equation such that*

$$\lim_{t\to\infty} \int_{|x|\geq t-A} |\nabla_{x,t}(u - v_L)(x,t)|^2 dx = 0, \quad \forall A \geq 0.$$

Also there exists $J_* \in \mathbb{N}$, $0 \leq J_* < \infty$, *a time sequence* $t_n \uparrow \infty$ *(well–chosen) and for* $1 \leq j \leq J_*$ *scales* λ_n^j *with* $0 < \lambda_n^j \ll t_n$, *positions* $c_n^j \in B_{\beta t_n}(0)$, $\beta \in [0,1)$ *with* $\lim_n \frac{c_n^j}{t_n} = \vec{\ell}_j$ *well defined,* $|\vec{\ell}_j| \leq \beta$ *and traveling waves* $Q^j_{\vec{\ell}_j}$ *such that*

$$\begin{aligned} \vec{u}(t_n) &= \vec{v}_L(t_n) \\ &+ \sum_{j=1}^{J_*} \left((\lambda_n^j)^{-1/2} Q^j_{\vec{\ell}_j}\left(\frac{x - c_n^j}{\lambda_n^j}, 0\right), (\lambda_n^j)^{-3/2} \partial_t Q^j_{\vec{\ell}_j}\left(\frac{x - c_n^j}{\lambda_n^j}, 0\right)\right) \\ &+ o_{\dot{H}^1 \times L^2}(1) \quad \text{as } n \to \infty, \end{aligned}$$

in $\dot{H}^1 \times L^2$ *as* $n \to \infty$. *Moreover,* $\lambda_n^j/\lambda_n^{j'} + \lambda_n^{j'}/\lambda_n^j + |c_n^j - c_n^{j'}|/\lambda_n^j \to_n \infty$, $j \neq j'$.

The passage to arbitrary time sequences seems to require substantially different arguments.

Thank you for your attention.