THE FOCUSING ENERGY–CRITICAL WAVE EQUATION

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Abstract. We survey recent results related to soliton resolution.

Keywords: Ground state, solitary wave, soliton resolution
Classification: 35L15

Since the 1970's there has been a widely held belief that "coherent structures" describe the long–time asymptotic behavior of general solutions to nonlinear hyperbolic/dispersive equations.

This belief has come to be known as the soliton resolution conjecture. This is one of the grand challenges in partial differential equations. Loosely speaking, this conjecture says that the long–time evolution of a general solution of most hyperbolic/dispersive equations, asymptotically in time decouples into a sum of modulated solitons (traveling wave solutions) and a free radiation term (linear solution) which disperses to 0.

This is a beautiful, remarkable conjecture which postulates a "simplification" of the very complicated dynamics into a superposition of simple "nonlinear objects," namely traveling waves solutions, and radiation, a linear object.

Until recently, the only cases in which these asymptotics had been proved was for integrable equations (which reduce the nonlinear problem to a collection of linear ones) and in perturbative regimes.

In 2012–13, Duyckaerts–Kenig–Merle [13], [14] broke the impasse by establishing the desired asymptotic decomposition for radial solutions of the energy critical wave equation in 3 space dimensions, first for a well–chosen sequence of times, and then for general times.

This is the equation

$$\text{(NLW)} \qquad \begin{cases} \partial_t^2 u - \Delta u - |u|^{\frac{4}{N-2}} u = 0, & (x,t) \in \mathbb{R}^N \times I \\ u|_{t=0} = u_0 \in \dot{H}^1, & \partial_t u|_{t=0} = u_1 \in L^2, \end{cases}$$

$N = 3, 4, 5, 6 \dots$ Here, I is an interval, $0 \in I$.

In this problem, small data yield global solutions which "scatter," while for large data, we have solutions $u \in C(I; \dot{H}^1 \times L^2)$, with a maximal interval of existence $(T_-(u), T_+(u))$ and $u \in L^{\frac{2(N+1)}{N-2}}(\mathbb{R}^N \times I')$ for each $I' \Subset I$.

The energy norm is "critical" since for all $\lambda > 0$, $u_\lambda(x,t) := \lambda^{-\frac{N-2}{2}} u(x/\lambda, t/\lambda)$ is also a solution and $\|(u_{0,\lambda}, u_{1,\lambda})\|_{\dot{H}^1 \times L^2} = \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$. The equation is focusing, the conserved energy is

$$E(u_0, u_1) = \frac{1}{2} \int |\nabla u_0|^2 + |u_1|^2 dx - \frac{N-2}{2N} \int |u_0|^{\frac{2N}{N-2}} dx.$$

It is easy to construct solutions which blow–up in finite time say at $T = 1$, by considering the ODE. For instance, when $N = 3$, $u(x,t) = \left(\frac{3}{4}\right)^{1/4} (1-t)^{-1/2}$ is a solution, and using finite speed of propagation it is then easy to construct solutions with $T_+ = 1$, such that $\lim_{t \uparrow T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} = \infty$. This is called type I blow–up. There exist also type II blow–up solutions, i.e. solutions for which $T_+ < \infty$, and

*Supported in part by NSF grants DMS–1265429 and DMS–1463746.

$\sup_{0<t<T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$. Here the break–down occurs by "concentration." The existence of such solutions is a typical feature of energy critical problems.

The first example of such solutions (radial) were constructed for $N = 3$ by Krieger–Schlag–Tataru (2009) [25], then for $N = 4$ by Hillairet–Raphael (2012) [17], and recently by Jendrej (2015) [18] for $N = 5$.

For this equation one expects soliton resolution for type II solutions, i.e. solutions such that $\sup_{0<t<T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$, where T_+ may be finite or infinite.

Some examples of type II solutions when $T_+ = \infty$ are scattering solutions, that is solutions such that $T_+ = \infty$, and $\exists (u_0^+, u_1^+) \in \dot{H}^1 \times L^2$, such that

$$\lim_{t\to\infty} \left\| (u(t), \partial_t u(t)) - \left(S(t)(u_0^+, u_1^+), \partial_t S(t)(u_0^+, u_1^+) \right) \right\|_{\dot{H}^1 \times L^2} = 0,$$

where $S(t)(u_0^+, u_1^+)$ is the solution to the associated linear equation with data (u_0^+, u_1^+). For example, for (u_0, u_1) small in $\dot{H}^1 \times L^2$, we have a scattering solution.

Other examples of type II solutions of (NLW) with $T_+ = \infty$ are the stationary solutions, that is the solutions $Q \neq 0$ of the elliptic equation $\Delta Q + |Q|^{4/(N-2)} Q = 0$, $Q \in \dot{H}^1$ (we say $Q \in \Sigma$).

For example,

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)} \right)^{-(N-2)/2}$$

is such a solution. These stationary solutions do not scatter (if u scatters then $\int_{|x|<1} |\nabla_{x,t} u(x,t)|^2 dx \to 0$ as $t \to \infty$). W has several important characterizations: up to sign and scaling it is the only radial, non–zero solution. Up to translation and scaling it is also the only positive solution.

However, there is a continuum of variable sign, non–radial $Q \in \Sigma$ (Ding 1986 [9], Del Pino–Musso–Pacard–Pistoia 2011 [7], 2013 [8]). W also has a variational characterization as the extremizer for the Sobolev embedding $\|f\|_{L^{\frac{2N}{N-2}}} \leq C_N \|\nabla f\|_{L^2}$. It is referred to as the "ground state."

In 2008, Kenig–Merle [24] established the following "ground state conjecture" for (NLW). For u a solution of (NLW) with $E(u_0, u_1) < E(W, 0)$, the following dichotomy holds: if $\|\nabla u_0\| < \|\nabla W\|$ then $T_+ = \infty$, $T_- = -\infty$, and u scatters in both time directions, while if $\|\nabla u_0\| > \|\nabla W\|$, then $T_+ < \infty$ and $T_- > -\infty$. The case $\|\nabla u_0\| = \|\nabla W\|$ is vacuous because of variational considerations. The threshold case $E(u_0, u_1) = E(W, 0)$ was completely described by Duyckaerts–Merle (2008) [16] in an important work.

The proof of the "ground state conjecture" was obtained through the "concentration–compactness/rigidity theorem" method, introduced by Kenig–Merle for this purpose, which has since become the standard tool to understand the global in time behavior of solutions, below the ground–state threshold, for critical dispersive problems.

Other non–scattering solutions, with $T_+ = \infty$, are the traveling wave solutions. They are obtained as Lorentz transforms of $Q \in \Sigma$. Let $\vec{\ell} \in \mathbb{R}^N$, $|\vec{\ell}| < 1$. Then,

$$\begin{aligned} Q_{\vec{\ell}}(x,t) &= Q_{\vec{\ell}}(x - t\vec{\ell}, 0) \\ &= Q\left(\left[\frac{-t}{\sqrt{1-|\vec{\ell}|^2}} + \frac{1}{|\vec{\ell}|^2} \left(\frac{1}{\sqrt{1-|\vec{\ell}|^2}} - 1 \right) \vec{\ell} \cdot x \right] \vec{\ell} + x \right) \end{aligned}$$

is a traveling wave solution of (NLW).

When Kenig–Merle introduced the "concentration–compactness/rigidity theorem" method to study critical dispersive problems, the ultimate goal was to establish the soliton resolution conjecture.

As I said earlier, for (NLW) one expects to have soliton resolution for type II solutions. Thus, if u is a type II solution, one would want to show that $\exists J \in \mathbb{N} \cup \{0\}$, Q_j, $j = 1, \ldots, J$, $Q_j \in \Sigma$, $\vec{\ell}_j \in \mathbb{R}^N$, $|\vec{\ell}_j| < 1$, $1 \leq j \leq N$, such that, if $t_n \uparrow T_+$ (which may be finite or infinite), there exist $\lambda_{j,n} > 0$, $x_{j,n} \in \mathbb{R}^N$, $j = 1, \ldots, J$, with $\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \to_n \infty$ for $j \neq j'$ (orthogonality of the parameters) and a linear solution $v_L(x,t)$ (the radiation term) such that

$$\begin{aligned}
&(u(t_n), \partial_t u(t_n)) \\
&= \sum_{j=1}^{J} \left(\frac{1}{\lambda_{j,n}^{(N-2)/2}} Q^j_{\vec{\ell}_j} \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, 0 \right), \frac{1}{\lambda_{j,n}^{N/2}} \partial_t Q^j_{\vec{\ell}_j} \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, 0 \right) \right) \\
&\quad + (v_L(x, t_n), \partial_t v_L(x, t_n)) + o_n(1)
\end{aligned}$$

as $n \to \infty$.

This has been proven in the radial case, $N = 3$ (DKM 12', 13') [13], [14], and in the general case, $N = 3, 5$ when $T_+ < \infty$ and u is "close" to W, (DKM 12') [12].

Let me discuss now the radial results. In DKM 12' [13], the decomposition was proved for a well–chosen sequence of times $\{t_n\}_n$, while in DKM 13' [14] it was proven for any sequence of times $\{t_n\}_n$.

Let me first quickly describe the proof of the 13' result. The key new idea was the use of the "channel of energy" method introduced by DKM, which was used to quantify the ejection of energy as we approach the final time of existence T_+ ([11]).

The main new fact shown was that if u is a radial, type II, non–scattering solution, $\exists r_0 > 0$, $\eta > 0$, and a small radial global solution $\tilde{u}$, with $u(r,t) = \tilde{u}(r,t)$, for $r \geq r_0 + |t|$, $t \in I_{max}(u)$, such that $\forall t \geq 0$ or $\forall t \leq 0$,

$$\int_{|x| \geq |t| + r_0} |\nabla_{x,t} \tilde{u}(x,t)|^2 dx \geq \eta.$$

The key tool for proving this is what I like to call "outer energy lower bounds," which are valid for solutions of the linear wave equation. Let $N = 3$, for $r_0 > 0$, $P_{r_0} = \left\{ (ar^{-1}, 0) : a \in \mathbb{R}, r \geq r_0 \right\} \subset \dot{H}^1 \times L^2(r \geq r_0)$. Let $\pi^{\perp}_{r_0}$ be the orthogonal projection onto the orthogonal complement of P_{r_0}.

Then: for v a radial solution of the linear wave equation, $\forall t \geq 0$ or $\forall t \leq 0$, we have

$$\int_{|x| \geq |t| + r_0} |\nabla_{x,t} v|^2 \geq c \left\| \pi^{\perp}_{r_0} (v_0, v_1) \right\|^2_{\dot{H}^1 \times L^2(r \geq r_0)} \quad ([11]). \tag{1}$$

In the non–radial case, we have for $N = 3, 5, 7, \ldots$ for v a solution of the linear wave equation, $\forall t \geq$ or $\forall t \leq 0$

$$\int_{|x| \geq |t|} |\nabla v_{x,t}|^2 dx \geq c \int |\nabla v_0|^2 + |v_1|^2 dx \quad ([12]). \tag{2}$$

When $r_0 = 0$, the two inequalities coincide.

For the proof in DKM 13' [14], say when $T_+ = 1$ (the case $T_+ = \infty$ is similar) we first consider $(v_0, v_1) =$ weak limit of $(u(t), \partial_t u(t))$ as $t \uparrow 1$, in $\dot{H}^1 \times L^2$, which can be shown to exist. Then, v_L is the linear solution with data (v_0, v_1) at time 1, the "radiation" term. We let v be the nonlinear solution with data (v_0, v_1) at time 1, so that, with $\vec{v}(t) = (v(t), \partial_t v(t))$, $\vec{v}_L(t) = (v_L(t), \partial_t v_L(t))$, $\|\vec{v}(t) - \vec{v}_L(t)\|_{\dot{H}^1 \times L^2} \to 0$ as $t \to 1$. It is easy to see, from finite speed of propagation, that for t near 1, $\text{supp}(\vec{u}(x,t) - \vec{v}(x,t)) \subseteq \{|x| \leq 1 - t\}$. We then break up $\vec{u}(t_n) - \vec{v}(t_n)$ into a sum of "blocks" (technically, nonlinear profiles U^j associated to a Bahouri–Gérard profile decomposition) plus a "dispersive" error $\vec{w}_n$ which is small in a weaker "dispersive" norm [1].

If one of the "blocks" U^j is not $\pm W$, it will send energy outside the light cone at $t = 1$ (case $t \geq 0$) , a contradiction to the support property of $\vec{u} - \vec{v}$, or arbitrarily close to the boundary of the inverted light cone at $t = 0$ (case $t \leq 0$), also a contradiction. Finally, one uses (1) again to show that the dispersive error has to be small in energy, by a similar argument.

The argument in DKM 12' [13], for a well–chosen sequence of times, was different. The first step, say again in the case $T_+ = 1$, was to show that "no self–similar blow–up" is possible. This means to show, for each $0 < \lambda < 1$, that

$$\lim_{t\uparrow 1} \int_{\lambda(1-t)<|x|<1-t} |\nabla_{x,t} u(x,t)|^2 dx = 0.$$

The proof of this used (1).

One then combines this with virial identities: if $2^* = 2N/(N-2)$, and φ is a suitable cut-off, we have:

$$\partial_t \int \varphi u \partial_t u dx = \int |\partial_t u|^2 dx - \int [|\nabla u|^2 - |u|^{2^*}] dx + \text{error}, \tag{3}$$

$$\partial_t \int \varphi x \cdot \nabla u \partial_t u dx = -\frac{N}{2} \int |\partial_t u|^2 dx + \frac{N-2}{2} \int [|\nabla u|^2 dx - |u|^{2^*}] dx + \text{error}. \tag{4}$$

When $N = 3$, adding $\frac{1}{2}(3) + (4)$, we obtain (using no self–similar blow–up)

$$\partial_t \left(\int \varphi u \partial_t u dx + \int \varphi x \cdot \nabla u \partial_t u dx \right) = -\int |\partial_t u|^2 dx + \text{error}$$

which gives us

$$\lim_{t\uparrow 1} \frac{1}{1-t} \int_t^1 \int_{|x|\le 1-s} |\partial_t u|^2 dx ds = 0.$$

Using this fact, one can show that each nonlinear block U^j is time independent, and hence $\pm W$, and that the dispersive error $\vec{w}_n$ has time derivative going to 0 in L^2, for a well chosen sequence of times. One can then use (2) to show $w_n \to 0$ in $\dot{H}^1$.

We next turn our attention to higher dimensions and the non–radial case. Before doing so, let me mention that the techniques just explained have found important applications to the study of equivariant wave maps and to the defocusing energy critical wave equation with a trapping potential, in works of Côte, Lawrie, Schlag, Liu, Jia, Kenig, etc.

Now we should mention an important fact, proved by Côte–Kenig–Schlag 13' [6]: (1) and (2) fail for all even N, radial solutions. However, (2) holds for $N = 4, 8, 12, \ldots$ for $(v_0, v_1) = (v_0, 0)$ and for $N = 6, 10, 14, \ldots$ for $(v_0, v_1) = (0, v_1)$, but not necessarily otherwise.

Moreover, Kenig–Lawrie–Liu–Schlag [22] have shown that an analogue of (1) holds for all odd N, u radial, and applied this to a stable soliton resolution for exterior wave maps on $\mathbb{R}^3$.

In 14', Casey Rodriguez [26] used this analogue of (1) for all odd N to prove the radial case of soliton resolution along a well–chosen sequence of times for (NLW) in all odd dimensions, following the argument in DKM 12'.

What to do for N even, radial case, non–radial case? We start by discussing the radial case for $N = 4$, which is very close in spirit to co–rotational wave maps from $\mathbb{R}^2$ into the sphere $\mathbb{S}^2$. The first obstacle is that, due to the failure of (1), we did not know that self–similar blow–up is ruled out, which is the first thing to do in order to implement the strategy of DKM 12' for a well–chosen sequence of times.

This was not a difficulty in the work of Côte–Kenig–Lawrie–Schlag 13' [4], [5] on co–rotational wave maps, due to classical results of Christodoulou, Shatah, and Tahvildar–Zadeh from the 90's, who showed it by integration by parts, exploiting the finiteness of the flux, a consequence of the fact that the energy density is non–negative ([2], [27]).

This obviously does not hold for (NLW) and is a major difficulty. This difficulty was overcome by Côte–Kenig–Lawrie–Schlag 14' [5], by reversing the usual analogy with co-rotational wave maps.

We observed that if u is a radial solution to the energy critical wave equation on $\mathbb{R}^4$, then $\psi(r,t) = ru(r,t)$ solves

$$\partial_t^2\psi - \partial_r^2\psi - \frac{1}{r}\partial_r\psi + \frac{\psi-\psi^3}{r^2} = 0.$$

We let $f(\psi) = \psi - \psi^3$, $F(\psi) = \int_0^\psi f(\alpha)d\alpha = \frac{\psi^2}{2}\left[1 - \frac{\psi^2}{2}\right]$, and the "energy" is

$$\frac{1}{2}\int_0^\infty \left[(\partial_t\psi)^2 + (\partial_r\psi)^2 + \frac{2F(\psi)}{r^2}\right] rdr,$$

which is conserved. Note that if $|\psi| \leq \sqrt{2}$, $F(\psi) \geq 0$.

Now recall that we have the "radiation" v such that $\text{supp}(\vec{u}(t) - \vec{v}(t)) \subset \{0 < r \leq 1-t\}$ and since v is a "regular" solution at $t = 1$, $v \to 0$, for $r = 1-t$, $t \to 1$. Thus, the same holds for u, which shows that, for λ_0 close to 1, $\lambda_0(1-t) < r < (1-t)$ we have the non–negativity of $F(\psi)$, and the classical argument applies, also yielding that now $\psi \to 0$ on $r = \lambda_0(1-t)$.

An iterative argument in λ_0 now gives the lack of self–similar blow–up. Hence we could start the process in DKM 12', and use the fact that on $\mathbb{R}^4$ (2) holds for data of the form $(v_0, 0)$, which is the type that we have for the dispersive error, and everything works.

What do we do when $N = 6$, when the good data in (2) are of the form $(0, v_1)$? This was the same difficulty one encountered for 2–equivariant wave maps into the sphere, and for radial Yang–Mills in $\mathbb{R}^4$.

All of this was overcome in recent work of Hao Jia–Kenig [20], who proved the analog of DKM 12' for a well–chosen sequence of times in all dimensions, and also dealt with all equivariant classes for wave maps and radial Yang–Mills in $\mathbb{R}^4$. This was done by <u>not</u> using the "channels of energy."

The first step is to prove lack of self–similar blow–up. The argument I sketched in $\mathbb{R}^4$ in fact applies to all dimensions, yielding a decomposition with blocks that are $\pm W$ and a dispersive error, for a well–chosen sequence of times $\{t_n^1\}_n$.

We then use again the second virial (4) which now gives

$$\lim_{t\uparrow 1} \frac{1}{1-t}\int_t^1 \int_{|x|\leq 1-s} [|\nabla u|^2 - |u|^{2^*}]dxds = 0.$$

On static solutions, $\int [|\nabla Q|^2 - |Q|^{2^*}]dx = 0$, and thus we obtain, by real variable arguments that, for a possibly different, well–chosen sequence of times $\{t_n^2\}_n$ we have $\overline{\lim} \int [|\nabla w_n|^2 - |w_n|^{2^*}]dx \leq 0$. But, for the dispersive error $\int |w_n|^{2^*} dx \to 0$, which concludes the argument.

I would like to conclude with some recent results in the non–radial setting. Last summer, Hao Jia [19] was able to extend the analogy with wave maps to the non–radial setting and in particular control the flux, when $T_+ < \infty$, i.e. the type II blow–up case.

This allowed him to obtain a Morawetz type identity (adapted from the wave maps one), to find a well–chosen sequence of times $t_n \to T_+ < \infty$, so that the desired decomposition holds in the non–radial case when $T_+ < \infty$, with an error tending to 0 in the dispersive sense. Here he also used the idea of combining virial identities I just explained.

In the case $T_+ = \infty$, one new difficulty is the extraction of the linear radiation term. This has been done recently by DKM 16' [15]. Moreover, very recently, in the joint work of D–Jia–K–M 16' [10] we have obtained the soliton resolution for a well–chosen sequence of times, for general type II solutions, both in the case $T_+ < \infty$ and $T_+ = \infty$. The result is:

Theorem 0.1. *Let* $u \in C([0,T_+), \dot{H}^1 \times L^2(\mathbb{R}^N))$, $3 \leq N \leq 6$, *be such that*

$$\sup_{0\leq t<T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1\times L^2} \leq M.$$

Case 1: $T_+ < \infty$. Consider the singular set S, which is a finite set of points, and $x^* \in S$. Then $\exists J^* \in \mathbb{N}, J \geq 1, r^* > 0, v \in \dot{H}^1 \times L^2$ a regular solution near T_+, $t_n \uparrow T_+$, scales $\lambda_n^j, 0 < \lambda_n^j \ll (T_+ - t_n)$, positions $c_n^j \in \mathbb{R}^N$ such that $c_n^j \in B_{\beta(T_+-t_n)}(x^*)$, $\beta \in (0,1)$ with $\vec{\ell}_j = \lim_n (c_n^j - x^*)/(T_+ - t_n)$ well defined and traveling waves $Q^j_{\vec{\ell}_j}$ for $1 \leq j \leq J^*$ such that in the ball $B_{r^*}(x^*)$ we have

$$\begin{aligned}\vec{u}(t_n) &= \vec{v}(t_n)\\ &+ \sum_{j=1}^{J^*} \Big((\lambda_n^j)^{-\frac{N-2}{2}} Q^j_{\vec{\ell}_j}\Big(\frac{x-c_n^j}{\lambda_n^j}, 0\Big), (\lambda_n^j)^{-\frac{N}{2}} \partial_t Q^j_{\vec{\ell}_j}\Big(\frac{x-c_n^j}{\lambda_n^j}, 0\Big)\Big)\\ &+ o_{\dot{H}^1 \times L^2}(1) \quad \text{as } n \to \infty,\end{aligned}$$

and $\lambda_n^j/\lambda_n^{j'} + \lambda_n^{j'}/\lambda_n^j + |c_n^j - c_n^{j'}|/\lambda_n^j \to_n \infty$, $1 \leq j \neq j' \leq J^*$.

Case 2: $T_+ = \infty$. $\exists$ a linear solution u^L such that

$$\lim_{t\to\infty} \int_{|x|\geq t-A} |\nabla(u-u^L)|(x,t)^2 + |\partial_t(u-u^L)|(x,t)^2 dx = 0,$$

for all $A > 0$. Moreover, $\exists J^* \in \mathbb{N}, t_n \uparrow \infty, \lambda_n^j, 0 < \lambda_n^j \ll t_n, c_n^j \in \mathbb{R}^N$ such that $c_n^j \in B_{\beta t_n}(0)$, $\beta \in (0,1)$ with $\vec{\ell}_j = \lim_n c_n^j/t_n$ well defined and traveling waves $Q^j_{\vec{\ell}_j}$ for $1 \leq j \leq J^*$ such that

$$\begin{aligned}\vec{u}(t_n) &= \vec{u}^L(t_n)\\ &+ \sum_{j=1}^{J^*} \Big((\lambda_n^j)^{-\frac{N-2}{2}} Q^j_{\vec{\ell}_j}\Big(\frac{x-c_n^j}{\lambda_n^j}, 0\Big), (\lambda_n^j)^{-\frac{N}{2}} \partial_t Q^j_{\vec{\ell}_j}\Big(\frac{x-c_n^j}{\lambda_n^j}, 0\Big)\Big)\\ &+ o_{\dot{H}^1 \times L^2}(1) \quad \text{as } n \to \infty,\end{aligned}$$

and $\lambda_n^j/\lambda_n^{j'} + \lambda_n^{j'}/\lambda_n^j + |c_n^j - c_n^{j'}|/\lambda_n^j \to_n \infty$, $1 \leq j \neq j' \leq J^*$.

The passage to arbitrary time sequences seems to require substantially different arguments.

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