FIFTY YEARS OF ALGEBRA IN AMERICA, 1888-1938

BY

E. T. BELL

In spite of all the blandishments of self-love, the facts associated at first with the name of a particular man end by being anonymous, lost for ever in the ocean of Universal Science. The monograph impregnated with individual human quality becomes incorporated, stripped of sentiments, in the abstract doctrine of the general treatise. To the hot sum of actuality will succeed the cold beams of the history of learning.—Santiago Ramón y Cajal.

1. **The general trend.** In connection with the fiftieth anniversary of the American Mathematical Society, it was decided to present brief reports of the progress of mathematics in America during the fifty-year period, 1888–1938, of the Society's existence, and algebra was one of the topics selected for report. As the Latin-American countries appear to have preferred analysis to algebra, America in this report refers only to the United States and Canada.

The output of algebra in America during the past fifty years seems enormous, even disconcerting, to anyone attempting to survey it all in brief compass. A mere bibliography of the more than one thousand research papers on algebra published by American authors during the period under review, with only a line or two indicating the nature of each contribution, would exhaust the space available for this report. Consequently, only the broader aspects of the rapid growth from the age of relative algebraic innocence, when everything was special and detailed, to our present highly sophisticated abstraction, can be considered here. The influences that appear to have been mainly responsible for the evolution of abstract algebra in this country will be kept in view, although frequently they were obscured in a welter of particulars.

It may be said at once that American algebra, contrary to what some social theorists might anticipate, has not been distinctly different from algebra anywhere else during the fifty-year period. According to a popular theory, American algebraists should have shown a preference for the immediately practical, say refinements in the numerical solution of equations occurring in engineering, or perfections of vector analysis useful in physics. But they did not. The same topics—algebraic invariants, linear groups, substitution groups, postulational technique, linear algebra, and some others—were fashionable here when they were elsewhere, and no algebraist seems to have been greatly distressed because he could see no application of his work to science or engineering. If anything, algebra in America showed a tendency to abstractness considerably earlier than elsewhere.

Although algebra in America has not been radically different from any other, there is one respect in which its progress during the past half century differs from that in some other countries, and that is the sudden acceleration imparted to research by the creation of the American Mathematical Society fifty years ago. Two other events also were of the first importance for research in all mathematics, not merely in algebra, on this continent: the opening of the University of Chicago in 1892, and the foundation of the Transactions of the American Mathematical Society in 1900, with E. H. Moore, E. W. Brown, and T. S. Fiske as editors.

When the Society was organized, a small group of American mathematicians highly trained, for the most part in Germany, in what was then modern algebra were already beginning their task of civilizing America algebraically. It was the devoted work of these men, their first-rate competence, and their enthusiasm for modern ideas which started the renaissance of algebra in this country. "Renaissance" is hardly the right word; there was practically nothing to be reborn. This may seem a perverse position to take when it is remembered that in 1888 there were ten volumes of the American Journal of Mathematics, founded by J. J. Sylvester* in 1878, already on the shelf, and that many of those volumes were crammed with projective invariants. For an understanding of what developed after 1888, it is necessary to see the grounds for this position. Sylvester's enthusiasm for algebra during his professorship at the Johns Hopkins University in 1877–1883 was without doubt the first significant influence the United States had experienced in its attempt to lift itself out of the mathematical barbarism it appears to have enjoyed prior to 1878. Elementary instruction was good enough, perhaps better than it is today; research on the European level, with one or two conspicuous exceptions, was nonexistent. For all his enthusiasm, the singularly individual Sylvester could not breathe life into a corpse. His premature opportunity had to await a less inhibited generation. Under Sylvester's personal inspiration, several of his pupils did creditable and even brilliant work; but when they left the warmth of his enthusiastic personality, they either abandoned mathematical research, or rapidly chilled in a deadening round of pedagogical drudgery in colleges and universities administered by mediocrities for the perpetuation of mediocrity.

From this distance in time it appears that the greatest service which the Johns Hopkins University rendered mathematics in America was not the importation of a distinguished mathematician, nor even the subsidizing of the American Journal of Mathematics, but the impulse which its living example gave to some older American universities and colleges to raise themselves from the mathematically dead. Long before Sylvester came among us as a missionary for what was then the "modern higher algebra," the United States had one great algebraist; but Benjamin Peirce made only

* Unless there is possibility of confusion, initials will be given only the first time that a name appears. Papers cited may be easily located by noting dates and referring to the Jahrbuch über die Fortschritte der Mathematik.

a negligible impression on his American contemporaries in algebra, and his work was not appreciated by their immediate successors until it had received the nod of European condescension. American algebra did not stand on its own feet until long after its most original representative had abandoned the subject.

With the pioneers who began their work about the time the Society was organized, the case was entirely different. American-born and educated, they understood conditions in this country as not even the most intelligent alien could possibly understand them. Their postgraduate training, too, had been much broader and more vital than they could have received anywhere in the English-speaking world at the time. The debt of American algebra to the Germany of the late 1880's and early 1890's is very great, and is no less so because some of those German-trained pioneers were the first to attain independence from their teachers.

Among these first moderns, F. N. Cole was one of the most influential, both through his own work and through that of his many pupils to whom he imparted his quiet, effective passion for the theory of finite groups. Although Cole had studied with Felix Klein, and had an unbounded admiration for Klein's methods, his own rigorously exact work in substitution groups was not reminiscent of Klein. Cole advocated what has been called "pure group-theory" as the proper approach to the subject as it was in the 1880's and 1890's. Of Cole's pupils in groups still active, G. A. Miller is the most prolific, and has been for over forty years. Miller in his turn trained numerous specialists, each of whom made at least one contribution to some phase of substitution groups and, later, to abstract finite groups.

Progress after the first infusion of modern ideas was extraordinarily rapid. In the five years (1888–1893) from the founding of the Society to the Chicago Congress, algebra in this country reached maturity, and was able to hold its own with the average, or perhaps a little better, of what was being done elsewhere.

This is not the occasion for a history of mathematics in the United States; but when such a history is attempted, particularly for the past forty years, the historian will doubtless be impressed by the tremendous influence of one man, E. H. Moore. In the late 1890's and early 1900's, the history of mathematics in this country is largely an echo of Moore's successive enthusiasms at the University of Chicago. Directly through his own work, and indirectly through that of the men he trained, Moore put new life into the theory of groups, the foundations of geometry and of mathematics in general, finite algebra, and certain branches of analysis as they were cultivated in America. Moore's interests frequently changed, and with each change, mathematics in this country advanced. His policy (as he related shortly before his death) in those early years of his great career, was to start some thoroughly competent man well off in a particular field, and then, himself, get out of it. All his work, however, had one constant direction: he strove unceasingly toward the utmost abstractness and generality obtainable. It is to Moore's influence that much of the abstract development of the first two decades of this century in American algebra can be traced. Not the least of Moore's contributions to algebra was the encouragement he gave the University of Chicago's second Ph.D. in mathematics, L. E. Dickson, who took his degree at the age of 22 in 1896. After that, it was no longer necessary for young Americans to go abroad for their training in algebra, and very few have.

The basic training of both Moore and Dickson was received wholly in this country, and to a considerable extent each was self-taught. Not till both were mathematically mature did they visit Europe, Moore to profit by contact with Kronecker and Weierstrass at Berlin, and Dickson by the lectures of Lie at Leipzig. This may serve as a rough measuring stick of mathematical progress in America during the first decade of the Society's existence. But the measure is only rough, or possibly quite inapplicable, for it has not yet been demonstrated that first-rate capacity needs any teacher other than opportunity. Moore appears to have made his own opportunities and to have used the experience thus acquired to give others with the right stuff in them their due.

The early influence of the Chicago group has been emphasized because it gave the first strong impulse, so far as algebra in America is concerned, toward that increasing abstractness which characterizes algebra today, and which first became a universal fashion with the work of Emmy Noether and her pupils, beginning about 1922. This, the latest phase of algebra as it exists today, will be noted later. For the moment, it may be observed that this urge toward the utmost generality attainable has been consistently followed by one school of algebraists in this country for about forty years. In fact, if there is any clue through the tangled jungle of elaborate theories and special theorems for the past fifty years, it is perhaps only this steady progression from the particular to the less particular. Each of several fields of algebra—the theory of forms, groups, hypercomplex number systems—was cultivated intensively at first as a preserve of special problems sufficiently difficult to tax the skill of the solver, each investigated apparently for its own sake and without any deliberate attempt at the generality which frequently accompanies an abstract attack.

The great lesson which David Hilbert taught algebraists in 1888 with his finiteness proofs in algebraic invariants, that a general problem is often more vulnerable than any of its special cases, was received with a certain reluctance. Part, at least, of that fruitful lesson might have been learned even farther back, from Richard Dedekind. But, as it was, it took a full third of a century for the lesson to sink into the general algebraic consciousness. Our only comfort is that our energetic young contemporaries may be overlooking something in their devotion to the general as interesting as what our predecessors missed in their pursuit of the particular.

There are, of course, two sides to this whole question of abstractness, and both are amply illustrated in the past fifty years of American algebra. Valuable hints for an abstract attack were undoubtedly suggested by the vast accumulations of isolated facts; but in the earlier stages the very desirability of a general objective seems to have been doubted. "Abstract" was sometimes mistaken as a synonym for "vague," or "facile," and occasionally the mistake was not (and is not) a foolish one to make. Thus in the earlier evolution it was accounted a work of great merit to dissect out the minute anatomy of a narrow category of algebraic invariants, of linear algebras, or of substitution groups; and to label and classify the results like specimens in a museum of morbid pathology, without once glancing up to catch a glimpse of the whole organism or of its genus.

With sharper insight, the apprentices of the first masters saw that much of this detail painstakingly elaborated by their teachers could be ignored in favor of a simpler, more direct and more abstract formulation, until today, algebra is so excessively abstract that some who have not grown up with the subject are apt, occasionally, to imagine that a discovery of fundamental importance—whatever that may mean—has been made when old things have been merely simplified or rechristened in a more euphonious terminology.

In favor of the details cherished in the earlier period, this may be said: not even the most abstract of abstract algebra can take root and flourish in an absolute vacuum. From one point of view, the abstraction of a particular discipline is akin to the Alexandrian period of criticism and commentary, when Greek geometry kept some semblance of life in itself by feeding on its rich past. Much of that past richness was undoubtedly perfected. But the vigorous daring of invention, the fruitful curiosity about untried things, and the boldness to be unorthodox, were all but completely sterilized. Isolated facts in algebra may be as useless, even for algebra itself, as fragments of shattered rock on a desert. Many believe they are. But unless there are available a few suggestive algebraic phenomena in the beginning, what is to determine the direction of abstractness and generality? Possibly no determination is required; but if such is the fact, there exists no evidence in support of it. Why, in particular, should some algebraist prefer to elaborate one set of postulates rather than another? For at least half a century before the intensive study of groups began, it would have been easy for any logician to state the postulates for a group and to get what superficialities he could out of them; but it was only when a considerable body of detailed information about particular groups and their applications was available that rapid progress was made in the algebra of abstract groups.

In short, neither the special nor the general has been solely responsible for the progress of algebra, in either division, at any time in the past, and there is at present no evidence that one can be cultivated independently of the other if algebra is to continue to grow. The interest here in this interplay between the particular and the general is that in the past third of a century the trend has been increasingly away from the particular. Finally, if the modern abstract algebra of the past two decades is still too close to us for an impartial appraisal, it does appear that many aspects of the algebra of groups, rings, and fields which were hitherto obscured by a multitude of special details, are now seen to be simple consequences of underlying general concepts.

2. **Statistical data.** Although there is no sharp boundary between algebra and other fields, some sort of line must be drawn for the purposes of this survey, and it will be drawn at arithmetic. This seems reasonable enough, as the theory of numbers has not been considered by arithmeticians as a province of algebra since at latest 1801, although parts of the theory are frequently included in treatises on algebra. The reasons for such inclusion are obvious, but they do not reduce the peculiar difficulties of arithmetic to exercises in algebra.

During the past fifty years American algebraists have had eight major interests:

(G) Theory of groups, 43.44; (M) Miscellaneous, 14.35; (L) Linear algebra, 14.10; (P) Postulate systems, 8.46; (J) Modular invariants, 7.14; (E) Theory of equations, 5.61; (I) Algebraic invariants, 5.23; (R) Recent abstract algebra. 2.20.

The order of the topics is that of the number of research papers published by Americans from 1888 to 1938 in the respective topics, in descending order from (G) to (R). Again no absolute separation of the several fields is possible; but it seems probable that a reassignment of certain borderline contributions from one section to another, such as from (G) to (E) or vice versa, would not alter the order (G) to (R). The numbers give the percentages for the respective topics in a total of slightly over one thousand papers, published in research journals, whose character is unequivocally algebraic. Abstracts of papers subsequently published in full, or not published at all, have been excluded, as have also all solutions of problems and all but one or two of the expository articles. No claim for exhaustiveness is made; but again it is probable (from a detailed analysis not reproduced here) that the percentages here given are not significantly in error.

As continuous groups belong to analysis rather than to algebra, they are not included in (G). Parts of Boolean algebra are assigned to (P), which does not include papers of greater philosophical than mathematical interest (such as would not usually be published in a mathematical journal); (M) will be described presently; (E) includes the Galois theory; (J) is classified as arithmetic by Dickson, one of its creators, in his *History of the Theory of Numbers*, but enough of its interest is algebraic to justify its inclusion here; and (R), small in volume, is so prominent at the present time that it merits a separate although somewhat loose compartment. Several of the papers in (P), especially some in the period 1906–1917, could be assigned to (R).

Into (M) are put all contributions to algebra not otherwise classified, for example, the theories of determinants, matrices, quadratic and multilinear forms, aspects of quaternions other than those naturally assigned to (L), linear dependence, reducibility of polynomials, and triad systems.

Another type of algebra (it is certainly neither arithmetic nor analysis) would be included in a detailed report but is omitted here: the numerous algorithms of the finite calculus, such as E. McClintock's calculus of enlargement, the various symbolic methods of the writer and others for use in arithmetic, adaptations of Grassmann's somewhat neglected techniques, and various other algebraic calculi devised for specific purposes which have claimed a considerable share of the attention of American workers during the past fifty years.

The distribution decade by decade of the thousand plus papers mentioned reflects the influences noted in the first section. An indicated decade, as 08–17, in the following table includes both the first year and the last, as 1908, 1917; the third row gives the percentages of the total output appearing in each decade (1)–(5).

(1)	(2)	(3)	(4)	(5)
88–97	98–07	08–17	18–27	28–37½
10.45	21.23	23.49	16.33	28.50

The sudden drop in (4) is due to a decline of 54 per cent in the contributions to the theory of groups in the decade 1918–1927 compared to 1908–1917. The equally sudden rise in (5) is accounted for by an increase of 333 per cent in the output of linear algebra in 1928–1937 over the preceding decade.

In tabular form, with (G), · · · , (R) and (1), · · · , (5) as before, the (smoothed) percentages of the several topics in each decade are the following.*

	(G)	(M)	(L)	(P)	(J)	(E)	(I)	(R)
(1)	9.0	19.2	1.4	0	0	13.6	52.8	4.3
(2)	33.0	7.9	19.7	23.3	4.1	13.6	2.0	0
(3)	27.8	19.8	10.9	11.1	60.7	6.8	25.2	0
(4)	13.0	19.4	12.2	23.5	34.3	20.4	12.8	12.9
(5)	17.2	33.7	55.8	41.1	1.0	45.6	7.2	81.8

From the decade (3) to (4) there was a sudden drop of about 30 per cent in the output of algebra in America; from (4) to (5), a sudden rise

* Thus, about 33 per cent of the total output *in groups* fell in (2); in (*P*), about 23.5 per cent of all contributions *to this division* fell in (4).

again of about 75 per cent. A social statistician might correlate the drop with the entry of the United States into the World War in 1917, and the depression during the post-war decade. An easy analysis, however, shows that on this reasonable hypothesis, only those algebraists who specialized in groups were heavily drafted or deeply depressed; and without further analysis this seems improbable. During the same disastrous decade, postulate systems and the theory of equations even enjoyed a marked boom. These awkward discrepancies between plausible theory and obstinate fact are typical of many in the evolution of modern mathematics.

3. **Groups.** The intense activity of American algebraists in the theory of groups during the past half century was due to a few active men who induced numerous proselytes to contribute at least one paper apiece relating to groups. A statistical analysis of the total output in groups assigns all but a small fraction to Miller, Dickson, and men or women who took their advanced degrees under one of these.

The analysis also shows that the number of contributors who published one paper on groups, usually on some narrowly specialized detail of the theory, "in partial fulfilment of the requirements for the degree of Doctor of Philosophy," and who published little or nothing thereafter, is disproportionately high compared with other fields of algebra. Again, the percentage of those who started in groups and subsequently abandoned the subject to take up others in which they are still active, from differential equations to statistics, is higher than for the rest of algebra. From this several conclusions are evident, only one of which is of immediate interest: the great output in groups was largely the work of about six men.

In any account such as the present, confined to the work of a single continent, it is difficult to avoid leaving the unintentional impression that the work in question sprang full-armed from the heads of the workers concerned. Of course it did nothing of the sort. Without the international background of mathematical knowledge and the constant commerce of ideas between the old world and the new, American mathematics, if it had existed at all, would have been something quite different from what it was during the past fifty years. This can be verified by consulting the frequent references to foreign work in American papers. In particular, without the inspiration of the work of C. Jordan in France, G. Frobenius and O. Hölder in Germany, and W. Burnside in England, to mention only four whose activity partly overlapped the fifty-year period, the history of the theory of groups in America would be much shorter than it is. Similar statements apply to the other fields of algebra cultivated in this country, with the exception of modular invariants.

The earliest American contributions to groups or their applications are almost coincident with the founding of the Society. In the American Journal of Mathematics for 1886 (vol. 8), Cole discusses a problem, suggested to him by Klein at Leipzig, on the general equation of the sixth degree, in which, naturally, groups enter. In 1887 (ibid., vol. 9), he observes in the course of his paper on *Klein's Ikosaeder*, "Of chief interest, however, for the present discussion are the remarkable systems of relations which exist between the theory of *groups of operations*, of which the theory of substitutions constitutes a part, and nearly all other mathematical branches." From this the great importance which Cole attached to exhaustively accurate enumerations of permutation groups of a given order or degree can be anticipated.

Correcting the oversights of his predecessors, Cole in 1893 gave the first accurate determination of the permutation groups of degree seven. This was the beginning of such enumerative work in this country. Cole's methods were those of pure group theory. He had a distaste for long algebraic computations, and frequently expressed his admiration for the skill with which H. Weber explicated intricate theories with a minimum of calculation. His own style reflects this ideal. In the words of G. A. Miller, "during the years 1893–1895, I lived in Ann Arbor, Michigan, at the home of F. N. Cole, who inspired much of this early work." The work referred to is the determination, by Miller, of all permutation groups of degrees eight, nine, and ten.

Although groups were thus well started by 1893 in this country, they were still somewhat beyond the general run of American algebraists. The lucid expository paper of 1891 by O. Bolza (American Journal of Mathematics, vol. 13) *On the theory of substitution groups and its applications to algebraic equations*, did much to popularize the subject and to acquaint young Americans with the outlines of the Galois theory of equations. This paper is a summary of Bolza's lectures at the Johns Hopkins University in January and February, 1889. An editorial footnote states: "This subject being one on which no separate work is found in the English language, Dr. Bolza's development of it here will prove extremely helpful to all students of the subject, especially by supplementing and illustrating the more extended works of Jordan and Netto." To the works mentioned, J. A. Serret's might have been added as a source of inspiration to early American students of groups. In 1892 Cole's translation of E. Netto's *Substitutionentheorie* appeared, including its occasionally unfortunate account of the Galois theory. Nevertheless, the translation did much to spread the gospel of groups, from Baltimore to Berkeley.

The advantages, and indeed the necessity, of abstraction were recognized early. In his paper at the Chicago Congress of 1893 announcing the discovery of the simple group of order 504, Cole remarks: "In an abstract theory like that of groups, too much must not be expected in the way of general development from the accumulation and study of individual examples. No amount of such experimentation could have led to our modern knowledge. Progress is from abstract to abstract." To the same Congress, Moore communicated a memorable instance of the art of abstraction, when he proved incidentally in his paper on a doubly-infinite set of simple groups that every existent finite field is the abstract form of a Galois field of prime-power order. This paper clearly marks the beginning of abstract algebra in America. That the direction of abstract algebra changed at least twice subsequently, once (in 1902) with the introduction of postulational methods, and again (after 1910) as a result of the work of E. Steinitz (*Algebraische Theorie der Körper*, 1910), does not affect the preceding statement. Moore knew exactly what he was doing, and why; and he missed no opportunity to insist to his colleagues and pupils that the existence of analogies, however superficial at first sight, between different theories, is strong presumptive evidence of an underlying, abstract identity that should be uncovered and followed out in all its implications.

In the spring of 1894, Moore lectured on groups at the University of Chicago. The lectures appear to have partly inspired Moore's paper on the group of isomorphisms of a given group (discovered simultaneously and independently by Hölder). They also partly inspired something of greater significance for the development of groups in America. In a footnote to his paper, Moore states: "In connection with the members of that course [Spring, 1894], Messrs. Brown, Dickson, Joffe, Slaught, and I worked up the linear fractional configuration [for certain cases]. I take this opportunity to thank them for their cooperation, and especially Mr. Dickson, who quite recently completed the tables as given above." Moore himself made the fruitful discovery in 1898 that every finite group of linear transformations on n variables has a Hermitian invariant.

Dickson's doctoral dissertation, *The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group*, was published in 1896, and in 1901 appeared his treatise, *Linear Groups with an Exposition of the Galois Field Theory*, still the standard work and source in its province. In this, and in his numerous papers on linear groups in a finite field, Dickson emphasized the advantages of working ab initio with the general abstract case which, by Moore's theorem on finite fields, could be taken as the Galois field of prime-power order. He was thus enabled to unify and greatly extend the work of his predecessors, including Jordan, on certain categories of linear groups.

From such favorable beginnings, comparatively modest in volume, the river of American contributions to the theory of groups took its rise. During the second decade of the Society's existence, the stream was in full spate; and although the crest of the flood passed with the end of the decade, abnormally high water was recorded from several stations all through the following decade. Had the deluge continued to rise at its initial rate, editors of mathematical journals would have been drowned in their burrows before 1938 by an annual flood of 2,400 papers on groups. Nature came to the rescue, as it usually does in a dire emergency; the slow, difficult growth of younger ideas diverted tributaries to the main river of the late 1890's and early 1900's, and disaster was averted. If past experience is any criterion, it is not healthful for any science to canalize over forty per cent of its energy in one narrow bed.

The digestion and codification of the vast mass of fairly general theories and minutely detailed theorems in permutation groups, linear groups, and abstract groups now scattered through the literature is a task for a corps of encyclopaedists. In glancing over some of this old work, any reviewer must be arrested by the numerous interesting results now in rapid process of being completely forgotten. It is quite possible that some of these will be rediscovered as by-products of more powerful methods of investigation; but in the meantime it would seem that the theory is ripe for a harvesting. Other fields of mathematics (determinants, arithmetic) have been gleaned so that it is possible for those interested to locate immediately what has been found and to imagine new things to be discovered. As groups appear to be entering a new phase along with the rest of abstract algebra, it might reward three or four competent men to undertake the drudgery of a comprehensive technical history of the subject. An exhaustive history—the only kind likely to be of permanent value—is probably beyond the combined capacities of any two men.

In the decade 1898–1907, the enumeration of substitution groups, and of particular kinds of such groups, was vigorously continued by Miller and his pupils, some of whom quickly became foci of activity, while Dickson led in the field of linear groups. There was also an intensive search for simple groups. This period was conspicuous for the minute examination of special categories of groups, with what end in view, if any, is not now clear. Frequently, an apparently adventitious feature common to a few substitution groups of low order or degree was seized upon as the defining characteristic of some species of groups and the species was duly trapped and mounted. From such taxonomic work there presently developed a tendency to greater abstractness, in the definition of certain well known groups, such as the alternating and symmetric, by systems of independent generators. Problems of a totally different and deeper nature, which might be described as the asymptotic enumerative theory of groups, were attacked in researches still incomplete, notably by W. A. Manning and H. F. Blichfeldt. In substitution groups, many investigations centered in transitivity and primitivity (or their opposites), a direct outgrowth of the earlier enumerative work of the preceding decade. Prime-power groups and their subgroups were intensively investigated. A notable result for its time in this direction, presently overshadowed by the more general theorem of Burnside, was Cole's proof of the simplicity of groups of order p^3q^β. Group characters were not a frequently used tool in this period, although a paper of Dickson's in 1903 reformulated the method for American algebraists.

Two significant straws in the general stream of this prolific decade attracted some notice at the time, but nothing comparable to what they might have done had the present outlook on algebra been foreseen thirty years ago. One was E. V. Huntington's paper of 1902 on an independent set of postulates for a group; the other, Moore's of a few months later on the same topic. These were followed by many investigations of a broadly similar kind, including Dickson's postulates of 1905 for groups and fields. The four years from 1902 to 1905 saw the abstract outlook in American algebra clearly visualized.

A small sample of the topics most frequently treated in groups during this second decade of the Society's existence will be exhibited presently, to give some idea of the extraordinary richness, and perhaps also of the equally extraordinary confusion, of the general output. Direction and coordination, except in the work of the more productive men, appear to have been almost wholly lacking. It remains for our successors to sift this unorganized mass of material and reduce it to some semblance of order. It also remains for them to judge whether the following forecast from Klein's address to the Chicago Congress in 1893 has been borne out in the forty-five years from then to the present. "Proceeding from this idea of groups, we learn more and more to coordinate different mathematical sciences. So, for example, geometry and the theory of numbers, which for long seemed to represent antagonistic tendencies, no longer form an antithesis, but have come in many ways to appear as different aspects of one and the same theory."

Has this proved true in any creative sense? If it has, where are the significant contributions to the geometry of numbers, to take the specific instance cited, that have come into arithmetic through the theory of groups? If there are few or none such at present, this may be due only to the preoccupation of specialists in groups with groups as an end in themselves. And if it is pointless to expect from specialists in a particular field applications in which they are not interested, it seems equally pointless to attempt to justify activity in that field by an appeal to hypothetical applications which do not exist. Autonomy here, as elsewhere, is the safest rule.

To sample the output of this decade 1898–1907, a selection of topics pointing in the direction of abstractness, dealing with the general structure of finite groups, may be noted first. Miller's innovations in commutator subgroups (1896) were further exploited. J. W. Young investigated the group of isomorphisms of (certain special) prime-power groups, also group holomorphisms, while Miller discussed the holomorph of a cyclic group; H. C. Moreno and Miller investigated non-abelian groups all of whose subgroups are abelian, and W. B. Fite introduced and analyzed the metabelian groups. Miller gave a notable extension of Sylow's theorem. Dickson investigated the group defined by the multiplication table of a group, and in his prolific work on linear groups, attained the maximum of generality by considering such groups in Galois fields and in an arbitrary field. The postulational treatments of groups by Huntington, Moore, and Dickson have been mentioned previously, and will be considered more in detail in a later section. The characteristic subgroups of an abelian group were investigated by Miller.

In another direction, Manning attacked groups, not from the angle of order or degree as was customary at the time, but from that of class, beginning work which was continued through the following decades to the present together with a serial research on the order of primitive groups. Blichfeldt introduced new methods into the determination of the order of linear homogeneous groups.

Several writers, including Fite, Miller, and L. I. Neikirk, investigated prime-power groups, but usually not the general case. O. E. Glenn considered the groups of order p^2q^r (p, q prime). Abstract definitions for the general symmetric and alternating groups were given by Moore, while Dickson did the same for the simple groups of orders 504 and 660, and exhibited a triply infinite system of simple groups. Dickson also determined all the subgroups of the simple group of order 25,920. W. Findlay discussed the Sylow subgroups of the symmetric group; primitive and imprimitive substitution groups were investigated by H. W. Kuhn, H. L. Reitz, and Manning (with respect to class), among others, while Blichfeldt considered imprimitive linear homogeneous groups. Several categories of linear groups, including the ternary orthogonal, the hyperorthogonal, and new species, were exhaustively investigated by Dickson.

The foregoing small sample may serve to give some idea of the prevailing interests in the theory of groups during the second decade of the Society's existence. In the following decade, before the sharp recession set in, the work was of the same general character. As the third decade passed, the interest in linear groups steadily declined. So also, with the exception of a few persistent and indefatigable specialists, did that in substitution groups. In particular, Cole's program of exhaustively determining all the substitution groups of a given order or degree by the classical methods, seemed by the middle of the fourth decade to have been pushed to the limit of human endurance and even slightly beyond. Thus, in 1912, E. R. Bennett determined the primitive groups of degree twenty. A specimen of what can be done by more powerful current technique, originating partly in the work of P. Hall, is offered in the census of groups of orders 101–161 (omitting 128), by A. C. Lunn and A. K. Senior in 1934, and of orders 162–215 (omitting 192) in 1935.

A very small sample of the further production in groups will have to suffice to indicate the continued activity of men already in the field in 1907, the enlisting of new recruits, and the general nature of the problems considered. It does not follow, of course, that omissions are or were less meritorious than the few inclusions necessary for a random sample: questions of merit may be left to the tribunal of posterity—if posterity is to bother its head about such questions.

In 1908, Dickson represented the general symmetric group as a linear group. Fite in 1909 discussed the irreducible linear group in an arbitrary domain. A. Ranum in 1910 gave an unusual application of groups to the classes of congruent quadratic integers modulo a composite ideal. Manning (1911) found a limit for the degree of simply transitive primitive groups, and Miller, among many other things, wrote on cosets. At the suggestion of Cole, L. P. Siceloff in 1912 examined the orders 2,001–3,640 for simple groups. In 1913–1914, H. H. Mitchell, using in part geometrical methods, determined the finite quaternary linear groups and the primitive collineation groups in more than four variables containing homologies. Fite in 1914–1915 continued his work on prime-power groups. Miller, Blichfeldt, and Dickson in 1916 produced in collaboration a treatise, *Theory and Applications of Finite Groups*, each author covering the topics in which he had specialized. R. W. Mariott in 1916 investigated the group of isomorphisms of the prime-power groups of order p^4, and in 1919 Miller wrote on the number of subgroups of prime-power groups. Except for the unabated productivity of a few of the older specialists, the general output of group literature began to decline. There were, however, several minor relative maxima in the curve of production. What followed 1920 was for the most part much like the work of the decade preceding 1920. Thus, in 1923, H. A. Bender wrote on the Sylow subgroups of certain groups, M. M. Feldstein on the invariants of the linear homogeneous group modulo p^k, and Miller on fundamental theorems in substitution groups. In 1924 Bender wrote on special prime-power groups, and Cole made one of his last contributions to the theory with a determination of the simple groups up to order 6,232. In 1925, L. Weisner discussed the Sylow subgroups of the general symmetric and alternating groups. In 1927 Bender attacked the problem of determining the groups of order p^5 (p prime); in 1928 Brahana wrote on certain perfect groups, and M. J. Weiss on certain primitive groups. In 1929 (and 1933) Manning applied his methods to the degree and class of multiply transitive groups. In 1929 Manning discussed the primitive groups of class 14, and in 1933, C. F. Luther, those of class u. In 1934, Brahana investigated metabelian groups, and Lunn, in collaboration with Senior, gave a method for determining all solvable groups with examples of its application. Metabelian groups were discussed again in 1935 by C. Hopkins; and in 1936, P. Hall of England and G. Birkhoff investigated the order of the group of isomorphisms of a given group.

The last three specimens in this sample are from 1937 and illustrate the newer tendencies and, for once in the history of American contributions to finite groups, a mathematical response to a physical question: F. D. Murnaghan devised a practicable method for calculating the characters of the symmetric group, of use in the quantum theory. Finally, A. H. Clifford wrote on the representations induced in an invariant subgroup, and H. Weyl on the commutator algebra of finite group collineations.

As the fifty-year period closes, Miller is as active in groups as he was almost at its beginning; the other most prolific American leader in the general field of groups, Dickson, turned his major efforts in other directions several years ago. Of the earlier leaders, Cole died in 1926, Moore in 1932. If the foregoing outline, necessarily bare, has recalled to younger American workers in groups their indebtedness to the quaternion of leaders responsible for one of the most active periods in the history of algebra in this country, it will have served its purpose. And if some of the earlier work now seems a trifle old-fashioned, it appears probable that the same will be true of much that now is fresh and vivid when the Society celebrates its hundredth anniversary. However, this extremely mild prophecy may prove as unfortunate as a similar one to be quoted presently from H. Poincaré.

4. **Postulational methods.** The modern postulational technique is almost exactly the same age as the Society, although American mathematicians did not enter this particular field until about thirteen years after it was opened. It is customary to date the beginning of the formulation of postulate systems from the monographs of G. Peano, *Arithmetices Principia nova Methodo exposita*, and *I Principii di Geometria logicamente expositi*, published in 1889. The subtle logical analysis of the foundations of arithmetic by G. Frege is almost contemporaneous, but its later importance has been chiefly epistemological.

Peano's pioneering efforts initiated a great many other things in addition to the specific purpose for which they were undertaken. If modern abstract algebra, like the modern abstract form of other mathematical disciplines, grew from the precise formulation of sets of postulates for the various fields of classical algebra, as claimed by some familiar with the subject, then Peano's early work is indisputably its root. Lobatchewsky, apparently, was the first to establish the independence of any postulate, but this is hardly relevant here. Again, much of the present critical work on the foundations of mathematics, also a considerable part of symbolic logic, can be traced back to sure beginnings in Peano's first logical analyses in a symbolism especially designed for the purpose of precise logical analysis.

A few of Peano's contemporaries, particularly G. Loria, to be followed shortly by M. Pieri, A. Padoa, and M. Pasch, welcomed the innovation enthusiastically, in spite of the queer new symbols in which it was clothed, and prophesied a brilliant future for the newcomer. The costume of the new science no doubt struck conservative observers as slightly bizarre; but a mere eccentricity in dress seems a pretty feeble excuse for showering a timid stranger with empty bottles. The reaction of many on seeing something they had not seen before was that of the cockney navvy: "'Ere's a strainger, Bill. Let's 'eave 'arf a brick at 'im." Following Poincaré's bold lead, it became a mark of superior intelligence to refer scornfully to Peano's symbolic language as "Peanese," and to discount the serious efforts of those (including B. A. W. Russell) who, going far beyond Peano's primitive symbolism, sought to express mathematics in a language which would enable mathematicians to apprehend what it was they were always talking about.

"The symbolic language created by Peano," Poincaré wrote, "plays a very great part in these researches · · · . The essential element of this language is certain signs representing the several conjunctions—'if,' 'and,' 'or,' 'therefore.' It is possible that these signs may be convenient; but that they are destined to revolutionize all philosophy is another story." It was: the foundations of mathematics since at least the time of Kant had been considered a province of philosophy. Other departments of philosophy also are experiencing the impact of unambiguous symbolism at present.

Again, commenting upon Couturat's enthusiasm for Peano's pasigraphy as exemplified in the *Formulaire*, Poincaré observes: "I have the highest regard for Peano; he has done some very pretty things · · · . But after all he has not flown farther nor higher nor faster than the majority of wingless mathematicians, and would have done just as well on his legs." Exactly. It happened, however, that Peano and his successors were more interested in burrowing than in flying, and some of them have bored quite extensive tunnels into the foundations of mathematics. They have also, through their careful exploration of the postulates for groups, fields, and other constructs of rigidly technical mathematics, given to algebraists, at least, a wormseye view of the bases of their work, and a more thorough understanding of it than could have been obtained from a century of soaring in the vague empyrean of lofty speculation or pedestrian trudging along dusty highways.

In spite of significant applications of the new technique, principally by Peano's fellow countrymen, it had a grim struggle to rise in mathematical society; and when it did, shortly after the publication of Hilbert's *Grundlagen der Geometrie* in 1899, postulational analysis was stripped almost naked of logical symbolism in the works of professional mathematicians, to stand forth in a form which all could see and some admire. (In passing, it may be recalled that O. Veblen in 1904 did a much more thoroughgoing job in his system of axioms—not called postulates—for geometry.) Even at this comparatively advanced stage of respectability, the new method was not very cordially received by intuitionists. Poincaré praised Hilbert's geometrical effort with faint damns; and considerably later Klein looked back with nostalgic regret on the lush youth of group theory, when it proceeded on its carefree way untrammelled by any clear notion of what constitutes a group. Few working algebraists would claim that intuition plays no part or only a minor one in algebra. On the other hand, few who follow modern abstract algebra would admit that intuition is the whole play. The critical analysis of postulate systems in the past third of a century has not only rectified weaknesses in earlier intuitive work, but has also suggested profitable new fields for exploration, for example, the semigroups introduced by Dickson in 1905.

The earliest published contribution by an American to postulational methods is the paper by E. V. Huntington, *Simplified definitions of a group*, in April, 1902 (Bulletin of the American Mathematical Society, vol. 8, pp. 296–300). It is stated: "Up to the present time no attempt seems to have been made to prove the independence of the postulates employed to define a group, and as a matter of fact the definition usually given contains several redundancies." Huntington gave three postulates and proved their independence. In a subsequent paper (ibid., June, 1902, pp. 388–391), Huntington gave a second set, of four postulates. This second set was presented to the Society at its April meeting, 1902, where Moore also gave a set of six independent postulates, published in the Transactions of the American Mathematical Society, vol. 3 (1902), pp. 485–492. Moore analyzed his own postulates and those of Huntington in relation to them, adding to Huntington's first set the explicit statement of the closure postulate.

Moore (loc. cit., p. 488) mentions "earlier definitions of Pierpont and myself," and cites the report of J. Pierpont's lectures at the Buffalo Colloquium of September, 1896, on the Galois theory, subsequently published in the Annals of Mathematics, (2), vol. 2 (1900), especially p. 47, where a definition of finite groups by six postulates (compressed to three statements) is given. As no independence proof is offered, this earlier definition does not belong to modern postulational methods, as do the definitions of Huntington, Moore, and, later, Dickson. Thus Huntington's paper contains the earliest discussion of the *independence* of postulates for abstract groups. Huntington's papers of 1902 on complete sets of postulates for the theory of absolute continuous magnitude and for the theories of positive integral and rational numbers, contain the first definition of what Veblen later called "categorical." Those interested in the history of this topic may be referred to the papers of Dickson, Huntington, and Moore, in the Transactions of the American Mathematical Society, vol. 6 (1905). A recent remark by J. H. C. Whitehead (Science Progress, vol. 32 (1938), p. 495) that "The systematic study of abstract groups may be said to have been originated by L. E. Dickson," with a reference to the paper of 1905, would seem to require modification, if by "systematic study" is meant the explicit statement of a set of postulates with a proof of their independence. The interest here of this early work of Huntington, Moore, and Dickson is the impulse it imparted to abstract algebra in America.

The nature of the work on postulational analysis after the first papers already mentioned, can be seen from a series of samples, decade by decade. In practically all of the works cited, independence proofs were a major feature. In the decade 1902–1911, Huntington gave independent postulates for the integers and for the rational numbers, for real algebra, fields, ordinary complex algebra, two sets for abelian groups, and for the algebra of logic. A paper of his (Annals of Mathematics, 1906) on addition and multiplication in elementary algebra proposed a novel approach to the formal properties of numbers as operators. Dickson gave sets for groups, fields, and semigroups. Moore reconsidered his postulates for abstract groups.

Only that work in this field which is concerned more or less directly with algebra is noted. Thus no account is taken of the application of the postulational method to geometry, as in the work of Veblen and J. W. Young, nor of Moore's use of the method in his general analysis. Boolean algebra and its immediate neighbors (in so far as they concern algebra) are included, however, as from one historical aspect Boole's work in the algebra of logic is the original source. Boole, however, did not discuss independence, a characteristic feature of the modern work, nor did he give a set of postulates.

In the decade 1912–1921, two papers may be cited apart from the others for their interest beyond the special topics on which they were written. H. M. Sheffer's postulates for Boolean algebra, in which the now celebrated 'stroke' function was introduced, appeared in 1913. In 1921, E. L. Post published his *Introduction to a General Theory of Elementary Propositions*, one of the earliest papers on what are now called many-valued logics. This was done in complete independence of practically simultaneous work of a similar character in Europe. Another contrubution in this decade, B. A. Bernstein's determination in 1924 of all operations, with respect to which the elements of a Boolean algebra form a group, or more specially an abelian group, may be cited. The unique significance of the symmetric difference, appearing frequently today in topological algebra and elsewhere, is here first definitely pointed out.

A sufficient sample of postulational work in algebraic topics for the decade 1912–1921 shows several contributions by Bernstein to Boolean algebra, and an independence proof for Sheffer's postulates by J. S. Taylor, who wrote also on Boolean algebra. W. A. Hurwitz gave a concise set of postulates for abelian groups, proved independent by Bernstein. Taylor proved the independence of one of Bernstein's sets of postulates for abelian groups. Huntington continued as the most prolific contributor to postulational analysis. To this decade belong his analyses of cyclic order, serial order, and well ordered sets. N. Wiener investigated formal invariances in Boolean algebra, and defined fields in terms of the function $1-x/y$.

In this period, and through the next decade to the present, there appear to have been two dominant aims: to abstract a given theory in a minimum set of independent postulates; to state for a given theory the most flexible set of postulates, necessarily an appeal to taste or intuition. Beginning shortly before 1930, many of the investigations began to be tinged with metamathematics and/or syntax. This more recent direction in postulational methods originated in attempts to evaluate the logic or/and mathematics of A. N. Whitehead's and Russell's *Principia Mathematica.* It is still at its beginning, and appears to be pointing from the domain of algebra to another that does not directly concern algebraic technicians. As before, a small sample, following the chronological development, must suffice to indicate the general trend from 1922 to the present.

Bernstein continued his researches in Boolean algebras, investigating the relations of such to groups, serial relations, the theory of functions of one variable, the representation of finite operations and relations, duality, fields (in Boolean algebras), and in 1934 gave a set of four postulates. In 1932 both Bernstein and Huntington discussed *Principia Mathematica*, the former in relation to Boolean algebra, the latter postulationally, work continued in 1934, when also the relation of C. I. Lewis' strict implication to Boolean algebra was investigated. Huntington (1935) showed the interdeducibility of the Hilbert-Bernays system and that of *Principia Mathematica*. H. B. Curry about 1930 began publishing his combinatorial logic. A. Church in 1925 introduced the concept of irredundancy of a set of postulates, distinct from Moore's complete independence. In 1935, detecting a subtlety overlooked by previous writers, R. Garver gave a remarkable set of three postulates for groups. In the same year, M. H. Stone gave postulates for generalized Boolean algebra, having applied this algebra to topology in 1934. In 1936 he developed the representation theory of Boolean algebra, in line with modern algebra. In still another direction, J. H. M. Wedderburn in 1934 developed Boolean linear associative algebra. Also in a new direction was the generation of an n-valued logic by one binary operation by D. L. Webb in 1935; for $n=2$ this is Sheffer's result of 1913. Finally, in 1936, returning to the topic which started the postulational method in America in 1902, Garver applied his discovery to special types of groups.

An inspection of the current literature shows no decline in the application of the postulational method. Much of the contemporary work has an ulterior motive, apparently, in topology or elsewhere; nevertheless, it is in the same broad tradition as that of the past thirty-six years. It may be said that American work in this general field has been sharper and clearer on the whole than that done elsewhere.

5. **Algebraic invariants.** In 1887 the second part of P. Gordan's *Vorlesungen über Invariantentheorie* was published. His finiteness theorem dates from 1868. In 1888 our Society was founded. This might have been an auspicious event for the future of algebraic invariants in the America where Sylvester had lavished much of his talent on the theory, had not Hilbert in the same year published his devastating proof of Gordan's theorem and his own basis theorem for an infinity of forms in any finite number of indeterminates.

It is sometimes said that Hilbert's attack slew the theory of algebraic invariants, and it is a fact that, with a few minor revivals, publication in the subject fell off rapidly. But why an encystment of existence theorems should be fatal to any body of mathematics is not clear: analysis is as tolerant of them as some share-croppers are of trichinae. Problems that were lively enough before Hilbert, slowly expired after he treated them.

In recent years the theory of algebraic invariants has experienced a sporadic resurrection. Part of this, in America, is due to a return to difficult problems left aside decades ago, as in the work of J. Williamson. Applications to atomic physics are also responsible for reviving interest, although the physicists do not seem very enthusiastic about mastering a new mathematical technique. The like is true for some aspects of permutation groups; and these quantal applications in both fields are sometimes urged as a teleological justification for the enormous labor of two generations of industrious tillers before the quantum theory even sprouted. This justification is difficult to comprehend, unless, of course, "time's arrow" points in the opposite direction from that in which the second law of thermodynamics invites it to point. However, if the new applications inspire algebraists to the invention of less inhuman algorithms than some of those in the classical literature, the debt of physics to algebra will have been discharged.

It is not feasible to give more than an almost random sample of the heterogeneous collection of results contributed by Americans to the theory of algebraic invariants during the fifty-year period. From the very nature of the subject, anything much to one side of the direction pointed out by Hilbert was bound to be detailed and special. An exception was the comprehensive theory of Dickson, *A Theory of Invariants*, published in 1909, in which the coefficients of the forms were taken in an arbitrary field, finite or infinite. This of course has contacts with modular invariants, not considered in this section.

Only two essentially new techniques (noted later) were applied to algebraic invariants during the fifty years. In the classical tradition, Bolza in 1888, at Klein's suggestion, investigated binary sextics with linear transformations into themselves, and Sylvester published his Oxford lectures on reciprocants in the journal he had founded, thus keeping touch with America and the older interests. Hilbert's ideas appear to have been first introduced into this country in 1892 by H. S. White, with a symbolic proof of Hilbert's method for deriving invariants and covariants of ternary forms.

In 1893, W. E. Story, who had been a pupil of Sylvester, simplified part of Hilbert's finiteness proof by extending the method used by Hilbert for two indeterminates to any finite number.

It is somewhat like witnessing a youthful indiscretion, in view of its author's subsequent achievements, to find W. F. Osgood expounding the symbolic notation of S. Aronhold and A. Clebsch (why not of A. Cayley?) in the same year for American readers, and calculating the system of two simultaneous ternary quadratics by the symbolic method. This, however, appears to have been only a temporary lapse, and Osgood never again fell from analytical grace.

The stream continued to flow, but it was only a trickle compared to what had swamped the American Journal during its first decade. McClintock in 1891 discussed lists of covariants and their computation. In 1894 F. Franklin, a leader in the earlier period, reviewed F. Meyer's exhaustive report on invariants. White in 1896 considered the cubic resolvent of a binary quartic. All through this period there were applications of the theory of invariants, mostly by the symbolic method, to geometry, particularly by A. B. Coble in the second half; but as the intent of these was not primarily algebraic, they are not included here.

To continue with the sample: in 1903, E. Kasner developed the cogredient and digredient theories of multiple binary forms; in 1909 D. D. Leib obtained the complete system of invariants for two triangles, and Dickson discussed combinants. E. D. Roe in 1911 presented a new invariantive function. Dickson's theory of modular invariants, developed during this period, is noted in another section. Glenn in 1914–1919, with a high degree of algebraic elegance, discussed finiteness theorems and gave a symbolic theory of finite expressions. H. B. Phillips in 1914 gave an example of what may be considered one of the innovations in technique, when he applied vector methods to the invariants and covariants of ternary collineations. I. H. Thomsen in 1916 wrote on the invariants of a ternary quantic, Dickson, in 1921, on those of a ternary cubic. C. C. MacDuffee in 1923 gave a theory of transformable systems and algebraic covariants of algebraic forms. J. Williamson in 1929, and later, obtained complete systems for certain simultaneous systems, for example, for two quaternary quadratics, also for two n-ary quadratics. Some of the items in the foregoing sample are on the borderline of other disciplines, or perhaps beyond it. Nevertheless, their algebraic content is high enough to justify inclusion. The same is true of the next.

Although its object was not the calculation of algebraic concomitants, the symbolic method devised by H. Maschke in 1904 for differential invariants has great interest as an algebraic algorithm. This was one of the essentially new techniques of the period.

Another new technique was the application of the tensor calculus to the derivation of algebraic invariants in 1928, and again in 1938, by C. M. Cramlet. The generalization of the deltas of L. Kronecker, which play a part in this work, appears to have occurred independently and almost simultaneously to Cramlet, Murnaghan, H. P. Robertson, and Veblen shortly after the advent of general relativity.

Whether the applications to physics are to induce any considerable revival in the theory of algebraic invariants in this country remains to be seen, and likely will be seen, one way or the other, within a decade. Physics also is subject to the fluctuations of fashion and necessity.

6. **Modular invariants.** One of the most interesting phenomena of the past thirty years of American algebra is the sudden rise and equally sudden decline of the theory of modular invariants. Activity in this field is at present in abeyance, apparently for lack of an extension (if there is one) to covariants of M. L. Sanderson's theorem on correspondences between modular and formal invariants, and of a proof of the conjecture that all congruent covariants admit symbolic representation. As the complete history of the topic up to 1922 is readily accessible in Dickson's *History of the Theory of Numbers* (vol. 3, 1923), and as only seven papers on the subject have been published by Americans since 1922, a few general remarks will suffice here.

Until E. Noether in 1926 applied her methods to a finiteness proof, modular invariants were practically an American preserve. As far as seems to be known, the first specimen of a modular invariant was the absolute invariant discovered in 1903 by Adolf Hurwitz (see his *Werke*, vol. 2, pp. 374–384) while investigating the number of roots of a congruence to a prime modulus. Unaware of Hurwitz' discovery at the time, Dickson in 1907 inaugurated the theory in what is now its classical form (Transactions of the American Mathematical Society, vol. 8, pp. 205–232). Of the 53 papers noted in the statistical data, 24, including the *Madison Colloquium Lectures* of 1914, were contributed by Dickson, 11 by Glenn, 5 by O. C. Hazlett, and 3 by W. L. G. Williams. The remaining contributors wrote their doctoral dissertations in the subject under Dickson.

It is of some interest to note that this field attracted two of the most active American women mathematicians of the period, Sanderson (1889–1914) and Hazlett. Miss Sanderson's single contribution (1913) to modular invariants has been rated by competent judges as one of the classics of the subject.

In conclusion, it may be said that the early expectations of some writers on the subject, that it would provide a powerful method of research in arithmetic, have yet to be realized.

7. **Theory of equations.** In an older day, algebra was almost synonymous with the theory of algebraic equations. Interest in the theory is still alive but apparently in a decline. During the period of the Society's existence, only about six per cent of all American contributions to algebra have been to the theory of equations, including expository articles on the Galois theory. Several textbooks on the elementary theory have been written by Americans, as the subject is still popular in colleges and universities. These need not be cited here.

Bolza's paper introducing the Galois theory to American students has already been noted; Pierpont in 1899 (Annals of Mathematics) presented a more detailed discussion. The theory became a standard part of the senior or graduate course in algebra, and by 1904 practically every Ph.D. graduated in mathematics from a reputable American university had at least a nodding acquaintance with the theory. The usual classical model of the theory was finally made available in accessible form by Dickson in his *Modern Algebraic Theories* (1926); a presentation of the newer approach was given ten years later by A. A. Albert in his *Modern Higher Algebra.* Remarkably little research in the Galois theory has been done by Americans, and there is even a dearth of significant applications to equations of special types. Possibly the theory was too abstract to afford a convenient mine for doctoral dissertations.

At the turn of the century, Moore was interested for a short time in the higher parts of the theory of equations, as a result of his work in groups and tactical problems. In 1899 he studied the resolvent of degree fifteen of the general equation of the seventh degree, and showed that the general equation of the eighth degree has a resolvent of degree fifteen.

As samples of contributions by other Americans to the theory of equations as a whole throughout the fifty-year period, the following may serve. In 1901 J. C. Glashan wrote on metacyclic quintics and B. G. Morrison on the removal of terms from an equation. Dickson in 1906 made a novel departure in the theory of equations in a finite field. O. Dunkel in 1909 discussed the imaginary roots of equations, and, in 1918, H. B. Mitchell investigated the imaginary roots of a polynomial. C. F. Gummer in 1922 wrote on the relative distribution of the real roots of systems of polynomials. Solutions of equations by infinite series were given in 1907 by P. A. Lambert, and in the same year Dickson investigated the Galois group of a reciprocal quartic. D. R. Curtiss in 1912 extended Descartes' rule of signs. J. L. Walsh in 1922–1923 discussed the location of roots, and gave inequalities for the roots of algebraic equations, either refining known methods or inventing new ones. In 1922 A. J. Kempner discussed the separation of the complex roots of algebraic equations. Glashan in 1923 investigated (what he called) isodyadic quintic equations, and, in 1924, isodyadic septimics. Kempner returned in 1935 to his topic of 1922. Possibly classifiable under this head are the researches of H. Blumberg (1916), O. Ore (1935), H. L. Dorwart (1935), and others, on criteria for the reduciblity of polynomials. The brevity of the foregoing list is no measure of the difficulty of some of the problems attacked and successfully solved.

Only one American writer of the period appears to have chosen algebraic equations as a major interest. Beginning about 1926, and continuing to his premature death in 1935 at the age of 34, Garver published a long and varied series of notes and papers on various topics in this field. These included investigations on the transformations of one principal equation into another, the binomial quartic as a normal form, Tschirnhaus transformations, cyclic cubics, the transformation of cubic equations, the Brioschi normal quintic, and many other investigations of a similar nature.

8. **Miscellaneous investigations.** This extensive division can be discussed only summarily, with mention alone of those topics out of the heterogeneous miscellany which attracted the largest numbers of investigators. The topics most frequently discussed appear to have been matrices, quadratic, Hermitian, and bilinear forms from the standpoints of reduction and equivalence, and determinants. The publication in 1907 of M. Bôcher's *Introduction to Higher Algebra* familiarized American undergraduates with the elements of these theories, and also with linear dependence and elementary divisors. The example of Bôcher's rigorous treatment did much to raise the standard of American instruction in algebra. Dickson's *Modern Algebraic Theories* of 1926 included a simplified and unified discussion of the forms mentioned, and was responsible for a considerable amount of research in this field.

The work in determinants has been as specialized as the nature of the subject would lead one to expect, with occasional interludes of novelty or generality. Moore in 1899 made "A Fundamental Remark concerning Determinantal Notations." When Moore called anything "fundamental," it was likely to be at least not trivial. The import of his "remark" was fully appreciated only when the tensor notations of general relativity became popular. Somewhat off the beaten track, polydimensional determinants occupied several writers. Thus in 1899, E. R. Hedrick gave a detailed treatment of three-dimensional determinants; L. P. Hall in 1917 discussed the general case; and L. H. Rice made the subject his own, contributing more to this field than any other American writer. In 1924, he obtained a generalization of Frobenius' theorem. Cramlet in 1927 reduced the theory of n-dimensional determinants to an exercise in tensor algebra. This does not imply, of course, that nothing remains to be done in the subject; but it appears to have been transformed.

Of papers in the classical tradition on ordinary determinants, the following may be cited as a sample. H. S. White in 1895 wrote on Kronecker's theorem, and W. H. Metzler in 1917 and 1924 on compound determinants; Dickson in 1903 generalized symmetric and skew symmetric determinants. L. M. Blumenthal in 1936 introduced new methods into the classical theory. A somewhat unusual problem claimed the attention of Dickson and some of his pupils, that of expressing certain forms as determinants and finding for what forms there is a rational solution. Thus Dickson in 1921 considered the case of cubic forms, also that of reducible cubic forms, and found all general homogeneous polynomials expressible as determinants with linear elements. H. S. Everett in 1922 considered the polynomial case. As regards the general classical theory as a whole, it appears that the most frequently useful properties of determinants are now best treated by elementary tensor methods. Much of the older theory seems to have become indeed classical, that is, dead.

A topic not yet exhausted is that of forms which repeat under multiplication (or which admit a rational composition). Dickson, Hazlett, and C. G. Latimer considered this (either in the general case, or in special instances) in 1920–1921, 1928, and 1929, respectively. The prime and composite polynomials defined and investigated in 1922 by J. F. Ritt, more in the spirit of iteration, are unrelated to these. The significance of the composition problem was pointed out by F. M. G. Eisenstein nearly a century ago.

A difficult type of tactical problem, originating in T. P. Kirkman's famous fifteen schoolgirls problem, has attracted a small group of American writers on account of its applications and its intrinsic interest. Moore was one of the earliest contributors; an application occurs in his paper (cited in the section on equations) on the general equations of degrees seven and eight. Cole in 1912 determined all the triad systems on thirteen letters, and in 1921 wrote on "Kirkman parades"—"crocodiles" would have been a more correct technical description than "parades," and one which Kirkman would have recognized. L. D. Cummings and White in 1914 discussed the groupless triad systems on fifteen letters; White, Cole, and Cummings determined the triads on fifteen letters in 1918, and in 1934, White those on thirty-one letters.

The theory of matrices has been, and is, a popular subject with many writers, no doubt on account of its numerous applications in other fields of algebra and elsewhere. MacDuffee compiled a useful tract in 1933 (in the Ergebnisse series), and Wedderburn in 1934 published his Princeton lectures on matrices in the Colloquium series of the Society. Other expositions are also readily accessible, so that the theory is now part of the standard equipment of every student of algebra, and has been for several years.

The evolution of this theory from its almost accidental beginning in Cayley's work, of about eighty years ago, into one of the most useful tools in algebra, suggests that there is but little rhyme and less reason in the historical development of algebra. It also suggests that other apparent trivia may be the germs of equally significant and useful theories. Now that the theory of matrices is full-blown before us, it is easy for any empirical historian to see the inevitability of the growth from seed to flower. But the familiar historical explanation through linear transformations, etc., explains precisely nothing, unless the explainer is able to validate himself and his explanation by making a prediction from the present phenomena of algebra for observers half a century hence to confirm or refute. The like applies to a good deal of the rest of algebra during the past fifty years. The solution of the implied problem may be left to social historians of mathematics.

A small sample of the more recent contributions by Americans to the theory will indicate the nature of present interests in matrices in this country. In 1914 Wedderburn considered the rank of a symmetric matrix, as did Dickson in 1917. Wedderburn in 1915 discussed matrices whose elements are functions of one variable, in 1925 the absolute value of the product of two matrices, and also matrices in a given field. H. B. Phillips investigated functions of matrices in 1919. MacDuffee in 1925 wrote on the nullity of a matrix with respect to a field. In 1928 E. T. Browne wrote on the characteristic equation of a matrix, and W. E. Roth gave a new discussion of matric equations. T. A. Pierce considered matrices with cyclic characteristic equations in 1930. In 1935 J. Williamson gave a method for the simultaneous reduction of matrices to canonical form, and N. H. McCoy gave a rational canonical form for a function of a matrix; M. M. Flood discussed matric polynomials; M. H. Ingraham and K. W. Wegner considered equivalent pairs of Hermitian matrices; Albert considered the principal matrices of a Riemann matrix, as one of his numerous contributions to a subfield of the theory of matrices in which he leads in this country. He had discussed Riemann matrices as early as 1929. In 1936 Roth investigated the characteristic values of certain functions of matrices; Williamson considered matrices with respect to their idempotent and nilpotent elements. In 1937 Brown, Flood, and Parker wrote respectively on conjugate sets of matrices, column normal matric polynomials, and the characteristic roots of matrices. From this sample it is evident that present interest in matrices is both lively and varied. Applications, as to matric algebras, or to physics, have not been included here, as their chief interest is in the applications rather than in the general theory of matrices.

American interest in the algebra of bilinear and Hermitian forms seems to have developed from the occurrence of such forms in linear algebra, and the theory of groups. Possibly the earliest American contribution to bilinear forms is Taber's discussion in 1885 of the automorphic of a bilinear form, a topic reconsidered by Wedderburn in 1921. In 1906 Dickson gave a simplified treatment of quadratic, Hermitian, and bilinear forms. J. I. Hutchinson considered Hermitian forms of zero determinant in 1907; Dickson in the same year discussed quadratic forms in a general field. M. I. Logsdon in 1922 considered the equivalence of pairs of Hermitian forms, and Dickson in 1927 treated the singular case of pairs of forms. In 1926 Dickson produced a new theory of the rational equivalence of linear transformations of bilinear forms, stressing the desirable rationality. In his book (already cited) of the same year, he gave an ingenious simultaneous discussion of parts of the theory of quadratic, Hermitian, and bilinear forms. O. E. Browne in 1931 considered equivalent triples of bilinear forms. Recent work of R. Oldenburger is concerned with similar equivalence problems from a general point of view.

Last may be mentioned the consideration of similar problems in Galois fields, which possibly are more arithmetical than algebraic in character. Some of this is abstracted by Dickson in his *History of the Theory of Numbers*. As specimens may be cited Dickson's (1908) reduction of quadratic forms in a Galois field of order a power of two; A. D. Campbell's

(1928, and later) development of geometry in a Galois field; and the work of L. Carlitz (1933–) on carrying over to Galois fields certain classical problems of algebra, such as the expression of a polynomial (when possible) as a sum of squares.

It has not been feasible to give more than a glimpse of the rich variety of fields successfully explored in this miscellaneous region of algebra during the past fifty years. Inventiveness is as frequent here as in other fields. For example, and to conclude with two specimens of things left aside because they do not fit any of the categories, there is O. Ore's paper of 1933 on noncommutative polynomials, and Wedderburn's of 1917 on continued fractions in noncommutative systems.

9. **Linear algebra; abstract algebra.** Great progress in linear algebras and in their general theory was made in Europe and America during the fifty-year period, and a considerable part of this progress was due to American algebraists. In the evolution of linear algebra in this country during the past thirty years is clearly seen the progression from the particular to the general. Another tendency is also evident in the latest period: parts of linear and abstract algebra now closely mimic parts of arithmetic, and concepts which first evolved in the theory of algebraic numbers are taking on a sort of protective coloration in the theory of algebras. This partial arithmetization of algebra belongs to the theory of numbers rather than to algebra, and is not considered here beyond the immediately following remarks, which do not apply to the arithmetics of specific algebras, such as that of generalized quaternions, investigated by Dickson and his pupils. These are not considered here, as their interest is arithmetical.

It has been reported that the reason for some of the more profound of these arithmetoid researches in algebra was an attempt to provide finite proofs for certain results in classical arithmetic obtained hitherto only analytically. If such is the fact, the attempt has not yet succeeded. The delayed success of this attempt is parallel to another, namely, the invention of the theory of algebraic numbers to aid in the solution of certain problems in classical arithmetic. If none of these successive generalizations of rational arithmetic has as yet attained its initial object, it is conceivable that no lack of competence in the devotees of these generalizations is responsible for the failure. A more imaginative generation may discover that, so far as arithmetic is concerned, close imitation of arithmetical concepts leads back to arithmetically barren reformulations of fundamental and difficult problems in arithmetic. What arithmetic seems to want at present is neither a parody of itself nor the sincere flattery of a close imitation but the solution of at least one of its outstanding classical problems. However, the advance in linear algebra during the past fifty years was but little concerned with the wants of arithmetic. The main progress was in another direction.

In America at the beginning of the fifty-year period, linear algebra was still in the tradition of Peirce; in Europe, Poincaré's connection between hypercomplex number systems and continuous groups, discovered in 1884, was already being exploited, notably by G. Scheffers. After the first phase waned (about 1905), what may be called the modern American tradition in linear algebra began to appear. This has had little to do with continuous groups, and has been essentially algebraic in character. The rapid and varied development of this tradition is readily traced to the work of three men, Dickson, Wedderburn, and Albert—in the chronological order of their entry into the field.

The relation of the contributions by American algebraists to "algebras" since about 1927 to the general progress in the subject, during the same period, can be seen by consulting M. Deuring's *Algebren* (1935). Dickson's *Algebras and their Arithmetics* (1923; German edition, 1927) will partly serve the same purpose for 1914–1923, and, for the period up to 1914, the same author's *Linear Algebras* (Cambridge Tract No. 16) may be consulted. A history of the subject has yet to be written.

As in previous sections, only a sample of the profusion of results, special and general, can be noted. Even this much presents serious difficulties of just exposition, as one author has frequently generalized the work of another, or of himself, soon after it was published, and what at the time seemed like several long steps forward turned out to be only half a step. In spite of this "dot and carry one" feature of the general advance, progress was nevertheless fairly coherent, due no doubt to the conscious direction of the three men mentioned. At the least, the progress appears to have been markedly less haphazard than was the case in the theory of groups.

As samples of the work up to 1905, the following may be offered. In 1899–1901, G. P. Starkweather discussed linear associative algebras of order six. H. Taber in 1904 extended Peirce's results concerning algebras of deficiencies, 0, 1 to those in which the domain of the coordinates is arbitrary. In the same year, H. E. Hawkes enumerated hypercomplex number systems in seven units, partly in defense of Peirce's methods against foreign critics (notably Cayley, Scheffers, and E. Study), who had been misled by Peirce's metaphysical propensities and occasional slips into the belief that his methods were inadequate for his specific purpose. In this paper and others (especially in the Transactions of the American Mathematical Society, vol. 3 (1902)), Hawkes removed the obscurities, and showed that "by using Peirce's principles as a foundation, we can deduce a method more powerful than those hitherto given for enumerating all number systems of the types Scheffers has considered." Hawkes thus placed Peirce's work (published first only in lithographed form, 1870, republished in the American Journal of Mathematics, vol. 4, (1881)) on a sound basis. He also evaluated Peirce's success in attaining his declared objective of exhaustively enumerating all types of number systems in a small number of units and of developing and applying calculi for some or all of these systems.

In a sense, this work of Hawkes balanced Peirce's account, and closed one chapter of the book opened by W. R. Hamilton and A. DeMorgan. Of the Hamilton-Peirce problem, the outstanding unsolved part is the somewhat vague project of devising "useful" applications; possibly the quantum theory, which has shown itself capable of swallowing even the eight-square imaginaries of Cayley without acute discomfort, may do what Hamilton and Peirce desired. While it is a question of applications, passing reference may be made to the ideas of J. W. Gibbs on multiple algebra. Up to the present, Gibbs' approach does not seem to have attracted professional algebraists. Whether this neglect is the fault of the algebraists or of Gibbs' ideas may be left to the judgment of posterity; the fact is that algebra has developed in other directions.

Still on the border of the newer tradition, W. M. Strong in 1901 wrote on nonquaternion number systems; S. Epsteen in 1903 on semireducible hypercomplex number systems; J. B. Shaw in 1903 on the theory of linear associative algebras, on nilpotent algebras, and on the application of matrices to algebras; Epsteen in 1904 on the definition of reducible hypercomplex number systems, and Shaw in the same year on algebras defined by finite groups. Dickson enters in 1905 with a more general discussion of hypercomplex number systems.

The year 1905 is also memorable in the subject for Wedderburn's proof that a Galois field is the only algebra with a finite number of elements that is a linear associative division algebra in the domain of real numbers, the analogue for finite algebras of the theorem of Frobenius and C. S. Peirce. The following year, Dickson made two contributions to a province of algebra in which he was to lead for over a quarter of a century, and in which he was to inspire many researches by his students, that of division algebras. In the first paper he found (among other things) new commutative algebras in which division is uniquely possible, and gave for division algebras a method of constructing an algebra of mk units from an algebra of m units when m exceeds two. He also determined the inequivalent, non-field, commutative algebras in six units with coordinates in the same abstract field. In the second paper of 1906, Dickson exhibited commutative linear algebras in $2n$ units, with coordinates in a general field, n of the units defining a field subalgebra. In a later investigation (1910; published, 1912) in the same general direction, he found linear algebras containing a modulus in which neither the associative nor the commutative law of multiplication is postulated, such that every element of the algebra satisfies an equation analogous to the usual characteristic equation. Among many new results of this paper is the possibility of division in Cayley's eight-square algebra.

The year 1907 seasoned linear algebra in America with a pinch of irony.

In that year J. B. Shaw's monumental *Synopsis of Linear Associative Algebra* all but immortalized the subject like a perfectly preserved green beetle in a beautiful tear of fossilized golden amber. Had this exhaustive synopsis been the last word on linear associative algebra, it would have made a noble epitaph. Instead of submitting to premature mummification and honorific burial, however, the subject insisted on getting itself reborn, or its soul transmigrated immediately, in Wedderburn's paper *On hypercomplex numbers*, published in vol. 6 (1907) of the Proceedings of the London Mathematical Society. The object and point of view of this paper may be recalled in its author's words:

"My object throughout has been to develop a treatment analogous to that which has been so successful in the theory of finite groups. An instrument towards this lay to hand in the calculus developed by Frobenius, and used by him with great effect in the theory of groups. This calculus is, with slight additions, equally applicable to the theory of hypercomplex number-systems, or, as they will be called below, algebras."

The calculus referred to is that of complexes. Parts of Wedderburn's paper had been read before the Mathematical Seminar at the University of Chicago early in 1905, "and owe much to Professor Moore's helpful criticism." At Dickson's suggestion, "algebra" was adopted as equivalent to Peirce's "linear associative algebra." Algebra, after the assimilation of this paper, was a very different thing from what it was before. Much of it took on the graces of civilized generality and unity. The striking gain in simplicity of method, with its consequent greater ease in deduction, marked by this contribution can be seen by comparing some of the proofs of classical theorems as given in the paper with proofs by the older methods.

Dickson's earlier researches in division algebras have already been mentioned. The intensive investigation of such algebras, and the discovery of wide new classes of them, was one of the outstanding contributions to American algebra of the past thirty years. Although it is out of its chronological order here, special note may be made of Dickson's paper of 1926, *New Division Algebras*, in which the subject took a new direction. Stating that "the chief outstanding problem of linear algebras is the determination of all division algebras," Dickson proceeded to greatly extend existing knowledge by showing how to construct from any solvable group of order n at least one type of division algebra of order n^2. He recalled that until 1905, when he discovered a division algebra in n^2 basal units, fields and real quaternions were the only division algebras known; the results of 1926 generalized those of 1905. Apart from such particulars—sufficiently general in themselves—the main advance in this work was its pragmatic demonstration of the basic importance of the theory of finite groups in the theory of division algebras. In 1930 the work was continued in a paper on *Construction of Division Algebras*, in which methods were devised for greatly reducing the necessary computations.

To return to the chronological sampling: not all of the work of the period 1907–1938 was in the newer tradition; the older continued to give interesting results. As these seem to lie rather to one side of the advance toward generality, they will not be considered further here. In the main line, there were generalizations from special fields to an abstract field for the coordinates of hypercomplex numbers. On the borderline between algebra and arithmetic, the theory of certain finite algebras, constructed by H. S. Vandiver in 1912, abstracted and generalized the instances suggesting it in the residue classes of arithmetic. The work of E. Kircher in 1917, concerned in part with decompositions in rings, was in a similar direction. Generalizing results of Dickson, Wedderburn showed in 1914 that Dickson's algebra of the same year, connected with an abelian equation, can be made primitive. Shaw in 1915 investigated parastrophic algebras (one of the six possible types with multiplication tables derived from the table of a given algebra by permutation of the indices in the triple suffix of the multiplication constants), and showed that, when applicable, such algebras provided a short method of determining when a given algebra is semisimple. In passing, it seems fortunate that the names of the six types, such as antipreparastrophic and antipostparastrophic, reminiscent of the early Greek of the theory of elasticity, have not proved indispensable.

Hazlett's first contributions to the subject appeared in 1914–1916, the latter being the classification and invariantive characterization of nilpotent algebras. The following year and again in 1930 she discussed associative division algebras. Most of her further contributions were arithmetical in character; in particular, she modified Dickson's definitions for the arithmetic of an algebra. Wedderburn in 1924 opened a new field with his investigation of algebras without a finite basis. Cayley's eight-square algebra and generalized quaternions were discussed by Dickson in 1918, partly for their arithmetical interest. Carrying out a detail of the program in division algebras, Garver in 1927 produced a division algebra of order sixteen.

Albert in 1929 investigated normal division algebras in $4p^2$ units, p any odd prime, and determined all normal division algebras in sixteen units. In 1930 he showed how to construct all noncommutative rational division algebras of order eight, and found all normal division algebras in thirty-six units of particular types. Continuing his researches in this same general direction, he obtained normal division algebras of degree four over an algebraic field. Albert and H. Hasse in 1932 collaborated in a determination of all normal division algebras over an algebraic number field. In 1933 Albert investigated noncyclic algebras; in 1934 and 1936, normal division algebras, and also certain simple algebras. It is impossible here to give further indications of Albert's interests in linear algebra constituting, as stated previously, the third of the major surges in the subject as it has progressed in America since 1900.

Although the arithmetical aspects have been deliberately left aside, a word must be added on some work in this general field which is of algebraic interest apart from its arithmetical objective. In addition to his studies of 1929–1932 on the matrices of an algebra, MacDuffee investigated ideals (1931) in linear algebras, and, with G. Shover in 1933, defined ideal multiplication in a linear algebra. Shover in 1933 discussed the class number; Latimer in 1934 proved its finiteness in a semisimple algebra.

The most prolific contributors of twenty years ago are still active; thus, in 1935 Dickson gave a detailed investigation of algebras with associativity not assumed, and Wedderburn in 1937 extended certain of his previous methods so as to apply to algebras in modular fields. Among topics in which there is a new or revived interest, that of Lie algebras may be mentioned. G. Birkhoff and N. Jacobson are among the current workers in this topic. An investigation in a different direction from any so far indicated is cited here out of its chronological order to suggest that its possibilities have not, perhaps, been sufficiently explored: J. W. Young's *New Formulation of General Algebra*, of 1927.

As the fifty-year period closes, linear algebra is in a much livelier condition that it was half a century ago. The causes for this revival of interest have been sufficiently indicated in the foregoing samples. To judge by the number of young recruits entering the field, interest in linear algebra will persist for some time to come, although what is now called modern algebra is attracting many.

Modern abstract algebra, especially in America, is too close to the present to have acquired a long list of contributors. Nevertheless, the comparatively few in this vigorous young school who do cultivate the subject are sufficiently prolific to compensate for lack of numbers. There is, in fact, some indication that the next decade or two may experience a deluge from this quarter rivaling or even surpassing that of thirty years ago when the group-dam burst. There is no dearth of problems for competently trained men: practically the whole of algebra offers itself for remodelling in the streamlined patterns of the newer abstract algebra. The latest models of the Galois theory, for instance, also of the composition theorems in groups, are easier on the eyes than the old.

This latest phase of algebra is distinctly European in origin, and practically all German, if the ideas of Galois be excluded as too remote historically. Its roots are in Dedekind's work of 1900 on dual groups,* Steinitz' paper of 1910, and Emmy Noether's abstract school, trained by her either personally or through her writings from about 1922 to her death in 1935 at the age of 53.

A striking feature of the newer developments is the frequent use of

* And in work on modules as early as 1877: *Ueber die Anzahl der Ideal-classen in den verschiedenen Ordnungen eines endlichen Körpers.*

concepts, such as homorphism, residue classes, and ideals, which came into mathematics through arithmetic. These concepts are now abstracted and understood. The younger generation of American algebraists has been strongly attracted to this latest development. As an item of historical interest, Moore's early work (1893) on Galois fields was the first in this country to foreshadow the present trend.

The two most prolific of the many workers, all of whom are young, in this new field may be mentioned, so the reporter for the hundredth anniversary of the Society may be able to look back and see just how close to the mark our present aim comes. Nobody writing fifty years ago on linear algebra could have foreseen that within two decades the whole subject would be radically recast. Still less could anyone have foreseen who was to do the recasting. So, in any sample census at the present time, the probability of entirely missing the mark seems high. The very man who so far has contributed nothing or only one short abstract of his work to the Bulletin, and whose name is not even mentioned, may be the very one who will revise the tradition once more.

With a long record of achievement in algebraic numbers (among other things) behind him, O. Ore is at present engaged in elaborating his theory of structures in his work on the foundation of abstract algebra. "In the discussion of the structure of algebraic domains one is not primarily interested in the *elements* of these domains, but in the relations of certain *distinguished subdomains* like invariant subgroups in groups, ideals in rings and characteristic moduli in modular systems. For all of these systems there are defined the *two operations* of *union* and *cross-cut* satisfying the ordinary axioms. This leads naturally to the introduction of new systems, which we shall call *structures*, having these two operations." (Annals of Mathematics, vol. 36 (1935), p. 406.) Such is the bare indication of the program, already successfully carried out in numerous details. Some of the results obtained, as stated by Ore, are restatements of previous results due to Dedekind, and to G. Birkhoff in his paper *On the combination of subalgebras* (Proceedings of the Cambridge Philosophical Society, vol. 29 (1934)).

G. Birkhoff develops an abstract theory of what he calls lattices, which partially overlaps the earlier work of T. Skolem and Fritz Klein. (A Klein "Verband" is a finite lattice; there are further contacts with Klein's systems.) Many applications of lattice theory have been made. The theories of Birkhoff and Ore also overlap, and among other points not yet settled is that of which terminology is to survive, structure or lattice. Whichever does, it may be anticipated that there will be no dearth of attractive abstract theorems unifying numerous particulars of current theories, with many new algebraic facts and applications, all succinctly expressed in the ultimate language of the theory. Should this language prove to be the final one of algebra, algebra will be as dead as Cheops. But the experience of the past fifty years is against any such disastrous outcome of the present intensely interesting development of algebra par excellence, abstract algebra.

One root of the present development can be traced back to Boole, another to the G.C.D. and L.C.M. in rational arithmetic and in the theory of ideals in algebraic number rings, and still another to Euler and Gauss in rational arithmetic. In this connection it seems a pity that Dedekind did not take algebraists and arithmeticians into his confidence as to what he was really up to, and what false leads he abandoned—if he did—in his theory of dual groups, as he did in his account of how he created his theory of algebraic numbers and ideals. And if the algebra of structures and lattices is ever exhausted, there will remain the wider field of n-valued "connections" suggested by the current generalizations of parts of Boolean algebra. However, sufficient unto the day is the abstractness thereof—to transpose Moore's aphorism on mathematical rigor to the abstract temper of our times.

According to the mortality tables, many of the younger generation of American algebraists will be living in 1988, when the Society celebrates its first hundredth anniversary. If these survivors (provided human stupidity permits anything with brains in its head to survive) can look back at the algebra of 1938 and find some of it as quaint as some of that which we now recall from 1888, all will have been well from 1938 to 1988.

To conclude, a suggestion may be offered the officers of the Society who are to be responsible for the hundredth anniversary. Choose young men, preferably under thirty, to review the progress of American mathematics from 1938 to 1988. For if there is anything that will give a man, young or old, a decent humility and a sane humor regarding his own efforts, it is an acquaintance with the work of his predecessors and contemporaries. The earlier this acquaintance is gained, the better for all concerned, including mathematics.

California Institute of Technology,
Pasadena, Calif.