DIRICHLET PROBLEMS

BY

GRIFFITH C. EVANS

CONTENTS

I. PRIMARY CONCEPTS

1. **Intrinsic energy.** One can encompass the central Dirichlet problem in a brief and simple exposition by basing the approach on the so-called subharmonic functions, according to the method of Perron [26], T. Radó and F. Riesz [32] and C. Carathéodory [4]. The essential feature which makes these methods possible is the mean-value theorem of harmonic functions, that the value of such a function at the center of a sphere is the mean of its values on the surface; and this notion has been even more directly exploited by Phillips and Wiener [28] and Kellogg [19]. In fact, it can hardly be avoided, whatever the approach may be.

There is some advantage, however, in postponing its use until it is necessary to differentiate between Laplace's equation and Newtonian potential on the one hand and related equations or integral expressions on the other. How long a postponement is possible is illustrated in the treatment given by Frostman, in his thesis [13], where the uniqueness of a minimum energy distribution serves to answer a number of important questions, as well for other laws of attraction as for the Newtonian. In fact, perhaps it may be argued that, from a physical point of view, energy in a field or on a body is a more natural concept than Laplace's equation, and that the problems which are solvable in terms of this concept, such as the distribution of electric mass on a conductor and the induction exerted by one charge on another, are the natural problems. This would of course be too narrow a view for mathematicians, but it happens that the bridge from these particular problems to the more general ones is short. Apparently also when one has crossed the bridge he is in a position to survey problems of a different degree of generality than those which have been the object of direct attack. Here we refer not merely to what is called the generalized Dirichlet problem, in which the harmonic function is uniquely determined by continuous boundary data but cannot itself remain continuous at the boundary, but also to problems in which the boundary data themselves are not continuous but where again the solution is unique within a certain class.

Of course, the energy principle and the Dirichlet integral go back many years, and recent papers like those of Frostman [13], de la Vallée Poussin [36], and the author [11] reaffirm them. There is a good deal of overlapping in them, leaving a number of rather difficult questions of priority, if such matters are regarded as of the highest importance. No attempt will be made to settle them here. It is worthy of remark, however, that the relation of potential to general distributions of mass was considered as early as 1911 by Plemelj [29]. And it is quite evident that no one can disregard the importance of the insight of Lebesgue, in recognizing spines and barriers [21, 22, 23], or of Kellogg in his reorientation of the theory of regular points and barriers [16, 17], or of Wiener in generalizing the notion of capacity [40, 41] and showing its relation to fundamental issues. Attention should be called to the work of Vasilesco, and especially of Bouligand, and Zaremba, contributors over a series of years to the solution of the Dirichlet problem [2, 38, 42]. The discussions of Hilbert and Lebesgue on the minimum of the Dirichlet integral, which have now become classics, are even more important for their relation to the calculus of variations than to the Dirichlet problem. One may speak similarly of the classic integral equation theory of Fredholm, of which the culmination is the memoir of J. Radon [33]. In this paper, however, we follow a somewhat different mode of development. For an exposition of the growth of the theory up to the last decade, the reader is referred to the report and book of O. D. Kellogg [17, 18].

With this remark, we proceed to exploit the common ground referred to above, as much with the purpose of learning things by the way as for the sake of the fundamental problem of determining a solution of Laplace's equation by continuously given boundary values.

We consider, then, the potential and the energy of mass distributions. The potential of a positive mass distribution $\mu(e)$, distributed arbitrarily on a bounded set E, may be written in the form

$$u(M) = \int (1/MP)d\mu(e_P), \tag{1}$$

in which the integral is defined in the customary manner as

$$\lim_{N\to\infty} \int h_N(M, P) d\mu(e_P),$$

where $\{h_N(M, P)\}$ constitutes an increasing sequence of bounded continuous functions, with

$$\lim_{N\to\infty} h_N(M, P) = 1/MP, \qquad M \neq P,$$
$$= + \infty, \qquad M = P.$$

The $\mu(e)$ may be described technically as a not negative completely additive function of sets (thus of finite total amount), with $\mu(e \cdot CE) = 0$, and the integration may be extended over the whole of space, or merely over E. The mass distribution $\mu(e)$ has a *nucleus* F_1, contained in the closed cover F of E consisting of those points P of F such that an arbitrarily small sphere of center P constitutes a set for which the mass is not zero; this nucleus F_1 is evidently closed. It is convenient to admit for $u(M)$ the value $+ \infty$, as for instance in the case where $\mu(e)$ consists in part of a point mass on some point Q, that is, $\mu(e) > 0$ for $e = Q$ and $u(M) = + \infty$ for $M = Q$.

We define the energy, or "intrinsic" energy, of the mass distribution $\mu(e)$ on E by the iterated integral

$$I = I(\mu) = \int_E d\mu(e_P) \int_E \frac{d\mu(e_Q)}{QP} = \int_E u(P) d\mu(e_P) \tag{2}$$

leaving out the customary factor 1/2 for convenience. This integral may also have the value $+ \infty$ if the mass is somewhere sufficiently concentrated. By means of distributions of mass of minimum energy, we are led to the conductor potential of the infinite domain complementary to F and to the Creen's function of that domain.

A completely additive set function $f(e)$, distributed on E, which is not necessarily positive, may be written as the difference of two of positive type: $f(e) = \mu(e) - \nu(e)$. Hence the potential and energy of distributions $f(e)$ may be written as the difference or in terms of sums and differences of integrals with respect to positive set functions, and will be said to converge when these converge separately. There are thus introduced the mutual energies

$$I(\mu, \nu) = I(\nu, \mu) = \int_E d\mu(e_P) \int_E \frac{d\nu(e_Q)}{QP} = \int_E u(Q) d\nu(e_Q) = \int_E v(P) d\mu(e_P),$$

where $u(Q)$, $v(P)$ are the potentials of $\mu(e)$ and $\nu(e)$ respectively, with

$$I(\mu \pm \nu) = I(\mu) + I(\nu) \pm 2I(\mu, \nu)$$

if the respective integrals are convergent.

By comparison with the corresponding Dirichlet integrals extended over all space, it is easily verified [10, I] that

$$I(\mu) = 4\pi \int_W (\text{grad } u)^2 d\tau,$$

$$I(\mu, \nu) = 4\pi \int_W (\text{grad } u) \cdot (\text{grad } v) d\tau,$$

and hence that

$$I(\mu) + I(\nu) \geqq 2I(\mu, \nu) \geqq 0.$$

It follows that if $I(\mu)$ and $I(\nu)$ converge, the same is true of $I(\mu+\nu)$ and $I(f) = I(\mu-\nu)$, and, further, that $I(f)$ is essentially positive and is equal to zero only in the case that $f(e) \equiv 0$.† Moreover if $I(f)$ and $I(g)$ are convergent, we have similarly

$$(3) \qquad \begin{aligned} I(f + g) &= I(f) + I(g) + 2I(f, g), \\ |\, 2I(f, g) \,| &\leqq I(f) + I(g). \end{aligned}$$

2. **Weak convergence and equicontinuity.** The discussion of the minima of Stieltjes integrals is based upon the notion of weak convergence. A sequence of completely additive set functions $\{f_n(e)\}$ converges weakly on E to $f(e)$ if for every continuous function $\phi(P)$ we have

$$(4) \qquad \lim_{n\to\infty} \int_E \phi(P) df_n(e) = \int_E \phi(P) df(e).$$

It is a fundamental theorem that, given on a closed bounded set F a collection of completely additive set functions whose total variations $\int |\, df_n(e)\,|$ are bounded uniformly, there exists an $f(e)$, also completely additive on F, and a subsequence $\{f_n^*(e)\}$ from the $f_n(e)$ which converges to $f(e)$ weakly. In particular, it may be remarked that if E is any subset of F, measurable with respect to $f(e)$ (that is, such that $\int_E |\, df(e)\,|$ converges), which is such that $\int_{E'} |\, df(e)\,| = 0$ where E' is the frontier of E, then

$$\lim_{n\to\infty} f_n^*(E) = f(E).$$

Consider now such a weakly convergent sequence of not negative set functions $\{\mu^n(e)\}$ and suppose that $u(P)$, not necessarily continuous, is the limit of an increasing sequence of continuous functions $\phi_N(P)$. It is a simple result, known as Fatou's theorem, that

$$\liminf_{n\to\infty} \int_F u(P) d\mu_n(e_P) \geqq \int_F u(P) d\mu(e),$$

where $\mu(e)$ is the limit distribution. In fact, given $\epsilon > 0$, an N exists such that

$$\int_F u(P) d\mu(e) < \int_F \phi_N(P) d\mu(e) + \epsilon$$

† Frostman proves the corresponding fact for more general laws of attraction by means of an integral identity which enables him to write $I(f)$ as a square [13].

if the left-hand member is convergent; but also

$$\liminf_{n\to\infty} \int_F u(P)d\mu_n(e) \geqq \lim_{n\to\infty} \int_F \phi_N(P)d\mu_n(e) = \int_F \phi_N(P)d\mu(e),$$

and the result is obtained by letting N become infinite. An obvious modification takes care of the case when $\int u du = +\infty$.

The same theorem applies to iterated integrals like (2), and it is an immediate consequence of the theorem that the potential of a positive mass distribution is *lower semicontinuous*, that is

$$\liminf_{N\to\infty} u(M) \geqq u(Q).$$

Another fundamental lemma of analysis is based on the notion of equicontinuity. A family of functions $\{\phi_n(P)\}$ is equicontinuous in a closed bounded region D if for P, P' in D and distance $PP' \leqq \delta$ a function $\omega(\delta)$ exists, independent of n and tending to zero with δ, such that $|\phi_n(P') - \nu_n(P)| \leqq \omega(\delta)$. If then there is a subsequence which converges at a single point of D, there is a subsequence $\{\phi_n^*(P)\}$ which converges uniformly in D to some function $\phi(P)$ which has the same modulus of continuity. The proof is based, as is evident, on the Cantor diagonal process.

3. **Elementary notions about harmonic functions.** For convenience in notation, we denote by $\Gamma(\rho, Q)$ the spherical domain of center Q and radius ρ and by $C(\rho, Q)$, its surface. A function which is harmonic in a domain T has at any point Q the value which is the mean of its values over any $\Gamma(\rho, Q)$ which is contained with its $C(\rho, Q)$ in T, or over the $C(\rho, Q)$ of such a $\Gamma(\rho, Q)$. Conversely, if a function $u(M)$ is summable spatially, and for every Q in an arbitrary closed domain D contained in T is equal to its spatial average over spheres $\Gamma(\rho, Q)$, $(0 < \rho < \delta)$, where δ is a constant (depending on D), then $u(M)$ is harmonic in T; likewise if it is equal to its average over spherical surfaces $C(\rho, Q)$, $0 < \rho < \delta$ for such values of ρ as the integral exists. It is a familiar consequence of these mean-value relations that a function which is harmonic in T cannot take on its upper or lower bounds at a point of T, unless it is identically constant. Equally familiar is the application of this fact to the Dirichlet problem: there cannot be two different functions harmonic in T (tending to zero at ∞ if T is an exterior domain) which take on continuously the same limiting values for the same manner of approach to the boundary—even if more than one limiting value may correspond to a given boundary point.

Let now $u(M)$ be harmonic and $|u(M)| \leqq K$ in T. Let F be a closed set in T and η less than the distance of F from the boundary of T. It is easily seen that if the distance MM' is $\leqq \delta \leqq \eta$, the difference $|u(M) - u(M')| \leqq \omega(\delta)$ where $\omega(\delta) = \delta \cdot 6K/\eta$. In fact,

$$\frac{4\pi\eta^3}{3}\left| u(M') - u(M) \right| = \left| \int_{\Gamma(\eta, M')} u(P)dP - \int_{\Gamma(\eta, M)} u(P)dP \right|$$
$$\leqq \int_{\Sigma} \left| u(P) \right| dP$$

where Σ is the region "between" the two spheres, that is,

$$\Sigma = \Gamma(\eta, M) + \Gamma(\eta, M') - \Gamma(\eta, M)\cdot\Gamma(\eta, M').$$

The region Σ has the volume $4\pi\eta^2(2d)$, d being the distance MM'. Hence finally,

$$\frac{4\pi\eta^3}{3}\left| u(M') - u(M) \right| \leqq K4\pi\eta^2(2\delta),$$

which yields the desired result.

But this result amounts to a statement that all harmonic functions in T with a common bound are equicontinuous. Hence, given an infinity of functions harmonic in T and bounded in their set, it follows from the theorem on equicontinuity that there exists a sub-sequence which converges to a function harmonic in T, and the convergence is uniform on any closed bounded set contained in T.

That the limit function $u(M)$ is harmonic in T is easily established. In fact, in any subregion of T whose boundary is distant by at least η from the boundary t of T, considering the convergent sub-sequence $u(M)$, we have for $\rho < \eta$,

$$u(M) = \lim_{n\to\infty} u_n(M) = \lim_{n\to\infty} \frac{3}{4\pi\rho^3}\int_{\Gamma(\rho, M)} u_n(P)dP = \frac{3}{4\pi\rho^3}\int_{\Gamma(\rho, M)} u(P)dP,$$

so that $u(M)$ is equal to its spherical volume average and is therefore harmonic.

When we consider the sequence solution of the Dirichlet problem we shall need to employ a sequence of domains approximating to the given one. We denote as usual the frontier of T by t, with corresponding notation in general for other domains; the set t will be taken as bounded, and the domain T may be either an interior or an exterior domain. A sequence of domains is known as a *nested sequence approximating to* T if (i) T_{n+1} contains $T_n + t_n$, (ii) each point of T lies in some T_n. A sequence of nested approximating domains S_n can first be constructed out of collections of cubes, since T is an open set, possibly not all of the same size, if S_n is to be connected—a domain being a connected open set. These approximating domains, by means of surfaces defined by the equations

$$K_n = \int_{s_n} \frac{dP}{MP^2}$$

may then be replaced by others with analytic boundaries, and by a proper choice of the K^u, these may be made free of singularities [18, pp. 319, 276].

This construction of analytic approximating domains is not difficult, and in order to save space we shall take it for granted.

4. **Distributions on conductors. The conductor distribution.** We follow the method of Frostman [13] and consider possible distributions $\mu(e)$ of positive mass on a bounded closed set F, such that the total mass $\mu(F)$ is unity. If there is no distribution for which I is finite, we say that F is of *zero capacity*; otherwise F is of positive capacity. In the latter case it is easy to see that I is bounded away from 0, but without regard to this fact let I_0 be the lower bound of I for all possible distributions $\mu(e)$; evidently $I_0 \geqq 0$. Let $\{\mu_n(e)\}$ be a sequence of these mass distributions such that the energies $I_n = I(\mu_n)$ tend to this lower bound I_0.

The sequence of mass functions $\mu_n(e)$, being uniformly bounded in total, contains a sub-sequence which converges weakly to a mass function $\mu_0(e)$, of total mass unity; and we may take our original sequence as reduced to this sub-sequence. Then, by Fatou's theorem

$$I_0 = \lim_{n\to\infty} I(\mu_n) \geqq I(\mu_0).$$

On the other hand, $\mu_0(e)$ is one of the set functions to be admitted as a possible mass distribution, so that

$$I(\mu_0) \geqq \text{l. b. } I(\mu) = I_0.$$

Hence $I_0 = I(\mu_0)$.

We turn now to the possible values of the potential on F_1, the nucleus of $\mu_0(e)$. Let $V(M)$ denote this potential.

Let $\delta\mu(e)$ be a mass distribution on F such that $I(\delta\mu)$ is finite. Then $I(\mu_0 + \delta\mu)$ is also finite, and from (3),

$$\Delta I = 2\int_F V(P)d\delta_\mu(e_P) + I(\delta\mu).$$

Since $I(\mu_0)$ is a minimum, we must have

$$\Delta I \geqq 0 \quad \text{if} \quad \int_F d\delta_\mu(e_Q) = 0 \quad \text{and} \quad \mu_0(e) + \delta\mu(e) \geqq 0.$$

It is clear that a $\delta\mu(e)$ of the form

$$\delta\mu(e) = \epsilon\left\{\phi(e) - \frac{\phi(F)}{\mu_0(F)}\mu_0(e)\right\}, \qquad \epsilon < 0,$$

is admissible for ϵ small enough, provided that $\phi(e)$ is an arbitrary positive distribution for which $I(\phi)$ exists. And since $I(\delta\mu)$ contains ϵ^2 as a factor, it follows that we must have

$$\int_F V(P)d\delta\mu(e_P) \geqq 0.$$

This integral may be rewritten in the form

$$\int_F V(P)d\left\{\phi(e_P) - \frac{\phi(F)}{\mu_0(F)}\mu_0(e_P)\right\} = \int_F \{V(P) - I_0\} d\phi(e_P)$$

since $I_0 = \int_F V(P)d\mu_0(e_P)$ and since

$$\int_F V(P)d\phi(e_P) = I(\mu_0, \phi)$$

converges. It follows that $V(P) \geqq I_0$ everywhere on F except possibly on a subset of capacity zero.

In particular, if F happens to have interior points, it has positive capacity (since a uniform distribution on a sphere has evidently finite energy); and if P is such an interior point of F, we shall have $V(P) \geqq I_0$. For it is deduced immediately that $V(P)$, being a potential of positive mass, is greater than or equal to its spherical average on a small sphere about P, and the latter is greater than or equal to I_0.

On the other hand, if for some point on F_1, the nucleus for $\mu_0(e)$, we have $V(P) > I_0 + \epsilon$, there will be a neighborhood including a sphere of center P where $V(P) \geqq I_0 + \epsilon$ which will contain a certain mass μ_1, and we shall have

$$I(\mu_0) \geqq (I_0 + \epsilon)\mu_1 + I_0(1 - \mu_1) > I_0$$

which is impossible. Hence $V(P) \leqq I_0$ on F_1.

That the distribution $\mu_0(e)$ is unique is easily shown. In fact, if there were another possible distribution $\nu_0(e)$ with the same minimum energy I_0, of total mass unity but with potential $U(P)$, we should have $I(\mu_0 - \nu_0) \geqq 0$. But

$$\begin{aligned} I(\mu_0 - \nu_0) &= I(\mu_0) + I(\nu_0) - 2I(\mu_0, \nu_0) \\ &= I_0 + I_0 - \int_{F_1} U(P)d\mu_0(e) - \int_{G_1} V(P)d\nu_0(e). \end{aligned}$$

On F_1, $U(P) \geqq I_0$ except on a set of capacity 0, and similarly for $V_0(P)$ on G_1, where G_1 is the nucleus of $\nu_0(e)$. Hence

$$\int_{F_1} U_0 d\mu_0 + \int_{G_1} V_0 d\nu_0 \geqq 2I_0$$

and $I(\mu_0 - \nu_0) \leqq 0$. From the two inequalities, $I(\mu_0 - \nu_0) = 0$ and therefore $\mu_0(e) - \nu_0(e) \equiv 0$, which would be a contradiction.

In resumé, we have, then,

$$\begin{aligned} &V(P) \geqq I_0 \textit{ everywhere on } F \textit{ except on a subset of zero capacity,} \\ (5)\qquad &V(P) \leqq I_0 \textit{ on } F_1, \textit{ the nucleus of } F, \\ &V(P) = I_0 \textit{ at interior points of } F_1. \end{aligned}$$

Remarks on zero capacity. To say that a set E measurable Borel is of capacity zero if the energy of any distribution of mass on it is infinite, is the same as Frostman's definition, that it is of capacity zero if every closed set contained in E is of capacity zero. In fact, if E will sustain no positive mass of finite energy, the same is true of any closed set contained in E. Suppose conversely, that no closed set F in E will sustain such a mass but that E will itself. We know that for any distribution $\mu(e)$ on E,

$$\mu(E) = \text{u.b. } \mu(E), \ F \text{ closed, contained in } E.$$

Hence there will be some of the mass on an F contained in E. But $I(F)=\infty$, and $I(E)\geqq I(F)$, so that $I(E)=\infty$. Hence if no F will sustain a positive mass with finite energy, E also will not.

A similar remark applies to de la Vallée Poussin's definition of zero capacity [35, p. 226]. A set E is of zero capacity if it will sustain no distribution of mass with potential bounded. But if there is a distribution with bounded potential the same one has bounded energy less or equal to the upper bound of the potential times the total mass. And if there is a distribution with bounded energy, there is such a one on some closed set F contained in E, and on F there is one of minimum energy. Hence there is a potential $V(P)$ whose upper bound on the nucleus of F is I_0. That this potential remains bounded by I_0 over all space will be seen from later considerations. Hence if E is of positive capacity by either definition, it is by the other.

These definitions of zero capacity make a convenient extension of the concept of capacity as first defined for a closed set by Wiener [40]. They are equivalent to the definition of *improper set* as given and used by Vasilesco [37], and based on the concept of Wiener.

We now continue the study of $V(P)$ by proving two simple propositions.

LEMMA 1. *Let Q be a point of the frontier of F_1, and $V(P)$ be continuous at Q for P in F_1. Then $V(P)$ is continuous at Q for arbitrary approach of P to Q.*

Consider first the portion $F_{1\rho}$ of F_1 within $\Gamma(\rho, Q)+C(\rho, Q)$ and denote the potential of the mass within this portion by $V(\rho, P)$. Let M be an arbitrary point not on F_1 and let Q be the nearest point of $F_{1\rho}$ to M. Then evidently for P in $F_{1\rho}$,

$$Q_1P \leqq MP + MQ_1 \leqq 2MP,$$

$$V(\rho, M) \leqq 2V(\rho, Q_1).$$

Next, we notice that $V(\rho, Q)$ approaches zero with ρ since $V(Q)$ is finite [35]. For we can choose the function $h_N(Q, P)$ of §1 so that $\int h_N(Q, P)d\mu_0$ is as little less than $V(\rho, Q)$ as we please, independently of ρ. But this integral tends to zero with ρ, since there can be no point mass at Q.

Finally, we notice that since $V(\rho, P)$ is continuous at Q for P on $F_1(V(P)-V(\rho, P)$ being evidently continuous at Q), we have, given $\epsilon>0$,

$$V(\rho, Q) < \epsilon, \qquad V(\rho, Q_1) < \epsilon$$

for ρ small enough, and Q_1 chosen close enough to Q.

But as M tends to Q, also Q_1 approaches Q. Hence eventually we shall have $V(\rho, M)<3\epsilon$. And since again $V(M)-V(\rho, M)$ is continuous at Q, we shall have

$$\limsup_{M\to Q} V(M) \leqq V(Q) + 3\epsilon, \qquad \limsup_{M\to Q} V(M) \leqq V(Q).$$

But

$$\liminf_{M\to Q} V(M) \geqq V(Q),$$

and $V(M)$ is continuous at Q.

The second proposition is a simple generalization of a condition of Poincaré and Zaremba [30, p. 228; 42, p. 204]. It is due to Raynor [34] and is a special case of a theorem about exceptional points of mass distributions [10, I]. Consider the bounded closed set F, and denote by T the infinite domain exterior to it, the boundary t of it being a portion of the frontier of F.

LEMMA 2. *Let $C(\rho, Q, E)$ be the measure of the portion of the spherical surface which consists of points of E, and require that*

$$\text{(P)} \qquad \limsup_{\rho\to 0} \frac{C(\rho, Q, F)}{C(\rho, Q, T)} > 0. \textit{ Then } V(Q) \geqq I_0.$$

Suppose the contrary, that $V(Q)=I_0-h$. A set of points on $C(\rho, Q)$ of positive superficial measure is evidently of positive capacity, since it will support with finite energy a distribution of mass of uniform density. There is then a spherical neighborhood $\Gamma(\rho_1, Q)$ such that, given $\epsilon>0$, except for sets of measure zero on any $C(\rho, Q)$,

$$V(P) \geqq I_0, \quad P \text{ in } F\cdot\Gamma(\rho_1, Q),$$
$$V(P) \geqq I_0 - h - \epsilon, \quad P \text{ in } T\cdot\Gamma(\rho_1, Q),$$

the latter inequality coming from the fact that $V(P)$ is lower semicontinuous. And since $V(Q)$ is greater than or equal to its spherical average on a sphere about Q,

$$C(\rho, Q)V(Q) \geqq I_0C(\rho, Q, F) + (I_0 - h - \epsilon)C(\rho, Q, T), \qquad \rho \leqq \rho_1.$$

But $V(Q)=I_0-h$, therefore

$$0 \geqq hC(\rho, Q, F) - \epsilon C(\rho, Q, T),$$
$$\frac{\epsilon}{h} \geqq \frac{C(\rho, Q, F)}{C(\rho, Q, T)}.$$

Since ϵ is arbitrary, it follows that the limit, as ρ tends to zero, of the right-hand member is zero, which contradicts the hypothesis.

These lemmas enable us to complete the description of $V(P)$ for sets F which are sufficiently regular. Suppose that at every point of F, and therefore at every point of F_1, the conditon (P) is satisfied. Then we shall have

$$\begin{aligned} V(P) &\equiv I_0, \quad P \text{ on } CT, \\ V(P) &< I_0, \quad P \text{ in } T, \\ F_1 &\equiv t. \end{aligned} \tag{6}$$

In fact, $V(Q) = I_0$ by Lemma 2, for every Q in F_1, and is continuous at every such Q by Lemma 1. Since $V(P)$ is harmonic elsewhere it is continuous throughout all space and less than or equal to I_0.

Moreover, every portion of t bears some of the mass μ_0, and every point of t belongs to F_1. Suppose, contrariwise, that a point Q of t were distant η from F_1. Then since $V(P)$ is harmonic at Q and not identically equal to I_0 in T it must be less than I_0 at Q. But the value of the energy could be lowered by transferring some of the mass from F_1 to the neighborhood of Q. It follows that $V(P) \equiv I_0$ for P on t, and therefore at points P of F, which are not also points of t; (since at such points $V(P)$ is harmonic) there is no mass $\mu_0(e)$ in the neighborhood of such points, and F_1 is identical with t.

The conductor potential. The quantity $1/I_0$ is called the *capacity* of the closed set F. The *conductor potential* is the function $v(P) = V(P)/I_0$ and has the value unity on the nucleus of F except possibly on a set of zero capacity. The distribution $\mu_0(e)/I_0$ is called the *conductor distribution*.

If E is not a closed set we follow Frostman in defining its capacity $K(E)$ as the upper bound of the capacities of closed sets contained in E.

5. **The Green's function.** The Kelvin transformation is a device for transforming by inversion. If $u(M)$ is harmonic in a domain T, and T' is the domain obtained as the collection of points inverse to T with respect to a certain sphere of center O and radius a, so that the respective distances from O satisfy the relation $rr' = a^2$, the function

$$w(M') = C\frac{r}{a}u(M) = C\frac{a}{r'}u(M), \qquad C \text{ const.},$$

is harmonic in T'. By means of this device we can obtain the Green's function $g(M, P)$ from the conductor potential $v(P')$, and vice versa.

We suppose the boundary t of T to lie in the finite space and take F as the complement of T. We take M in T as the center of inversion, and a so that $\Gamma(a, M)$ lies with its boundary in T, and let F' be the inversion image of F, $V(P')$ the conductor potential of F'. With $r = MP$, $r' = MP'$, the function

$$g(M, P) = \frac{r'}{a^2}(1 - v(P')) = \frac{1}{r}(1 - v(P')) \tag{7}$$

will be the Green's function for T with pole at M. In fact, $g(M, P)-1/r$ is harmonic in T and, if T is an unbounded domain, $g(M, P)$ vanishes as $P\to\infty$. Also $g(M, P)\to 0$ if P tends to a point of t such that at the corresponding point of t', $v(P')\to 1$.

In particular, if F satisfies the condition (P), the same will be true of F' and therefore $g(M, P)$ will take on continuously the value zero as P tends to a point of t.

It will be noticed that the Kelvin transformation and the Green's function afford the simplest method of treating domains whose boundaries t reach into the infinite domain. This will be apparent in the sequel, and it will be unnecessary to develop the details.

II. The generalized Dirichlet problem

1. **Regularity of a boundary point.** A point Q of t is said to be regular for T with respect to the conductor potential $v(M)$ if, for M in T, $\lim_{M\to Q} v(M)=1$; similarly, regularity with respect to the Green's function lies in the relation $\lim_{P\to Q} g(M, P)=0$. It will be shown that these notions are the same.

2. **Conductor potential and Green's function as sequence solution.** Given T, an exterior domain without restriction except that its boundary is a bounded set and its complementary set F is of positive capacity, let T_n be a nested sequence of exterior approximating domains whose complementary domains F_n satisfy the condition (P) at every point; in fact, the boundaries t_n may be taken without loss of generality as analytic. We take $\mu_n(e)$ as the corresponding minimal energy distributions of total mass unity, with potentials $V_n(M)$, and I_n as the corresponding energies.

Since T_{n+1} contains T_n, F_n contains F_{n+1} and F; hence the minimal distribution $\mu_0(e)$ on F is a possible distribution on F_{n+1}, and the minimal distribution μ_{n+1} also lies on F_n; hence

$$I_n \leqq I_{n+1} \leqq I(\mu_0), \qquad \lim I_n \leqq I(\mu_0). \tag{1}$$

A subsequence of the $\mu_n(e)$ converges weakly to some distribution $\mu(e)$ of total mass 1, which lies only on t, since $\mu_n(e)$ lies only on t_n. Since t is a portion of F, we have

$$I(\mu) \geqq I(\mu_0) = I_0.$$

But also, from Fatou's theorem, restricting n to this subsequence,

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} I(\mu_n) \geqq I(\mu),$$

and combining these inequalities with (1),

$$\lim_{n\to\infty} I_n = I(\mu) = I(\mu_0).$$

The minimizing distribution is, however, unique. Hence $\mu(e)\equiv\mu_0(e)$.

Moreover, since $1/MP$ remains continuous for a suitably large n if M does not lie on t,

$$\lim_{n\to\infty} V_n(M) = V(M), \quad M \text{ not on } t,$$

and by Fatou's theorem,

$$\liminf_{n\to\infty} V_n(M) \geqq V(M), \quad M \text{ on } t.$$

Since $V_n(M) \leqq I_n$ throughout all space, it follows that $V(M) \leqq I_0$, and if $V(Q) = I_0$, $\lim V_n(Q) = I$. By the same reasoning as used for smooth boundaries, we see now that any sphere $\Gamma(\rho, Q)$ with center at a point of t, which contains a portion of F of positive capacity, will also contain some of the mass $\mu_0(e)$.

Turning to the conductor potentials themselves, $v_n(M) = V_n(M)/I_n$, and the corresponding capacities $K_n = 1/I_n$, we note that the $v_n(M)$, K_n constitute monotonic decreasing sequences and from this we can deduce immediately the facts about conductor potentials.

The case where $K(F) = 0$ needs special mention. In this case there is no distribution of positive mass for which $K(F)$ is finite or its potential bounded; we shall therefore define the conductor potential as identically zero. With this convention we still have

$$\lim_{n\to\infty} v_n(M) = v(M), \quad M \text{ in } T, \quad \text{and} \quad \lim_{n\to\infty} K_n = K.$$

Suppose in fact, that $\lim_{n\to\infty} v_n(M)$ is not identically zero in T. Then the set function obtained by the weak convergence of $\mu_n(e)/I_n$ will not be identically zero, but since its potential remains bounded, it will be less than or equal to 1. This contradicts the assumption that $K(F) = 0$. Consequently, also $\lim_{n\to\infty} I_n = \infty$ and $\lim_{n\to\infty} K_n = 0$.

Throughout all space, $v(M) \leqq 1$. *For points* M *not on* t, $\lim_{n\to\infty} v_n(M) = v(M)$ *and for points on* t, $\lim_{n\to\infty} v_n(M) \geqq v(M)$. *The capacity* K *of* F *is equal to the capacity of* t *and satisfies the relation* $K = \lim_{n\to\infty} K_n$. *If* $v(Q) = 1$, Q *is a regular point of* t *with respect to the conductor potential; for every point of* T, $v(P) < 1$.

We now pass to the Green's functions for T_n, T by the Kelvin transformation, utilizing suitably chosen sets T_n', F_n', T', F'. The domain T may be either an interior or an exterior domain. Since the T_n form a nested sequence of approximating domains, the same is true of the T_n', and since the condition (P) holds at every point of t_n', we have, for T,

$$\lim_{n\to\infty} g_n(M, P) = g(M, P), \quad M \text{ in } T, \quad P \text{ in } T, \tag{2}$$

and if (with M fixed) Q' is a point of t' which is regular for $v(P')$, the corresponding point Q of t is regular for $g(M, P)$, and vice versa.

It is an elementary fact, established by means of Green's lemma, that for regions T with smooth boundaries, the Green's function is symmetrical in M and P. From (2) it follows that the same relation $g(M, P)=g(P, M)$ remains valid whatever the domain T.

If Q is a point of t which is regular with respect to $g(M, P)$, it is also regular with respect to $g(M_1, P)$, where M_1 is any point of T.

Let T_1 with boundary t_1 be the first of our nested sequence, chosen so as to contain both M and M_1 in its interior. Let B_1 be the upper bound of $g(M_1, P)$ and b the lower bound of $g(M, P)$, for P on t, so that, since the $g_n(M_1, P)$ form a monotonic increasing sequence, we have

$$g_n(M_1, P) \leqq g(M_1, P) \leqq (B_1/b)g(M, P), \quad P \text{ on } t_1.$$

But on t_n we have also

$$g_n(M_1, P) \leqq (B_1/b)g(M, P),$$

since the left-hand member is zero and the right-hand member positive. Since both members are continuous in the closed region bounded by t_n, t_1 and harmonic between these boundaries, the same inequality persists for that region. Hence, letting n become infinite, we have for all points P in $T-T_1$,

$$g(M_1, P) \leqq (B_1/b)g(M, P).$$

From this inequality the theorem follows immediately.

If T is an exterior domain and Q is a regular point of t for T with respect to the conductor potential, it is so also with respect to the Green's function, and conversely.

In order to prove this theorem it is necessary merely to repeat the reasoning just given, substituting the function $1-v(P)$ for $g(M, P)$ in the right-hand members. In order to prove the converse, the functions $1-v(P)$, $1-v_n(P)$ are used in the left-hand members.

It is useful, however, to have a direct statement of the relation between Green's function and conductor potential, without the use of inversion. To this effect, let Γ denote a sphere of radius R and boundary C on the surface of which is spread uniformly a mass of total amount R, thus of density $\kappa=(4\pi R)^{-1}$. This is the conductor distribution for C and its potential is given by a function which has the value unity for M in Γ:

$$v_\Gamma(M) = \kappa \int_C d\sigma/MP.$$

Consider now the function

$$u_n(M) = v_\Gamma(M) - \kappa \int_C g_n(M, P)d\sigma,$$

where for convenience we define $g_n(M, P)$ as zero in the complement of T_n.

We have

$$u_n(M) = \kappa \int_C [(1/MP) - g_n(M, P)] d\sigma_P,$$

and this function is harmonic for M in T_n. We take Γ large enough so that Γ contains t in its interior, and therefore also t_n, n large enough. But as M approaches a point Q_n of t_n, $g_n(M, P)$ is bounded and tends to zero; hence $\lim_{M\to Q_n} u_n(M) = v_\Gamma(Q) = 1$. Accordingly $u_n(M)$ is identical with $v_n(M)$, the conductor potential for t_n.

Let now n become infinite. The function $v_n(M)$ tends to the conductor potential $v(M)$ for every M in T, and since $g_n(M, P)$ changes monotonically with n as a function of P the integral relation remains valid, and, for M in T,

$$(3) \qquad v(M) = v_\Gamma(M) - \kappa \int_C g(M, P) d\sigma_P, \qquad \kappa = (4\pi R)^{-1}.$$

The device used in obtaining (3) enables us to state a slightly more general result:

If Q is a regular point of t for T, with respect to $v(M)$, and $\Gamma = \Gamma(\rho, Q)$, $C = C(\rho, Q)$, $\kappa = (4\pi\rho)^{-1}$, the function

$$(4) \qquad b_\rho(M) = 1 - \left\{ v_\Gamma(M) - \kappa \int_{C\cdot T} g(M, P) d\sigma_P \right\}$$

has the following properties for arbitrary ρ:

$$b_\rho(M) \text{ is harmonic and } \geqq 0 \text{ in } T, \text{ vanishing at } \infty;$$

$$(4') \qquad 1 > b_\rho(M) > 1 - \frac{\rho}{QM} > 0, \quad M \text{ in } T, \quad QM > \rho;$$

$$\lim_{M\to Q} b_\rho(M) = 0, \quad M \text{ in } T.$$

In fact, as in the proof of (3), we have

$$b_\rho(M) = 1 - \kappa \int_{C\cdot T} [(1/MP) - g(M, P)] d\sigma_P - \kappa \int_{C - C\cdot T} (1/MP) d\sigma_P,$$

which is harmonic in T; also for $QM > \rho$,

$$\rho/QM = v_\Gamma(M) > v_\Gamma(M) - \kappa \int_{C\cdot T} g(M, P) d\sigma > 0,$$

which establishes the second relation of (4′). Moreover, for $QM < \rho$, $v_\Gamma(M) = 1$ and (4) reduces to the equation

$$b_\rho(M) = \kappa \int_{C\cdot T} g(M, P) d\sigma.$$

As $M \to Q$, the function $g(M, P)$ is bounded, since P lies on $C(\rho, Q)$. $g(M, P)$ tends to zero. Moreover, since Q is regular with respect to $v(M)$ and therefore with respect to $g(M, P) = g(P, M)$, as a function of M for any P in T. Thus we have the third of the relations (4′).

3. **The Dirichlet problem for smooth boundaries.** Given a bounded closed set t and a function $f(P)$ continuous on t, the function may be extended by definition so as to be continuous throughout all space. This is a well known theorem of Lebesgue [20]. The function may be constructed as the limit of a step-by-step choice of values and, in particular, may be defined so as to vanish outside a sphere which contains t in its interior.

Consider a region T, exterior or interior, which is bounded by a finite number of closed surfaces sufficiently smooth so that the condition (P) is satisfied for the complementary regions F at every point of the boundary t. *Given $f(P)$ continuous on t, there is a unique function which is harmonic in T, vanishes at ∞ if T is unbounded, and takes on continuously the boundary values $f(P)$.*

The uniqueness has been established already, and the proof of the existence may be given briefly by constructing a sequence of harmonic functions. To this end, let $f(P)$ be continuously extended over space, and vanish outside a large sphere. We may write $f(P)$ as the uniform limit of a sequence of functions $F_k(P)$, which likewise vanish outside a sufficiently large sphere, independent of k, and have continuous first, second, and third derivatives over all space; for instance, we may choose $F_k(P)$ as the third iterated average over spheres with center P and radius $1/k$ [2′].

To each $F_k(P)$ we apply the operation expressed in the formula

$$u_k(M) = F_k(M) + \frac{1}{4\pi}\int_W g(M, P)\, \nabla^2 F_k(P)\, dP \tag{5}$$

in which $g(M, P) = 0$ for P in the complement F of T. For points M in T we have

$$\nabla^2 \int_W g(M, P)\, \nabla^2 F_k(P) = \nabla^2 \int_T (1/MP)\, \nabla^2 F_k(P)\, dP = -4\pi \nabla^2 F_k(M);$$

hence for such points $\nabla^2 u_k(M) = 0$, and $u_k(M)$ is harmonic in T. It evidently vanishes continuously at ∞ if T contains the infinite region, since $g(M, P) \to 0$ as $M \to \infty$ when P is at a finite distance from the origin. The integral part of the expression (5) tends to zero, however, as M approaches a point of t; for $g(M, P)$ is dominated by the summable function $(MP)^{-1}$, and vanishes continuously as M approaches a point Q of t for every P except $P = Q$. The function $u_k(M)$ is therefore the desired solution corresponding to $F_k(P)$. Since the $F_k(P)$ are bounded independently of k, the same is true of the $u_k(P)$.

Consider now the family of functions $u_k(M)$. The function $u_{k+p}(M)$

$-u_k(M)$, harmonic in T, is continuous in $T+t$ and takes on its upper and lower bounds only on t; hence with M in T, P on t,

$$| u_{k+p}(M) - u_k(M) | \leqq \text{u.b.} \, | F_{k+p}(P) - F_k(P) |.$$

Since the $F_k(P)$ converge uniformly, the same is true of the $u_k(M)$, and we have

$$\lim_{k\to\infty} u_k(M) = u(M),$$

where $u(M)$ is continuous on $T+t$, equals $f(M)$ on t, and vanishes at ∞.

Since the harmonic functions $u_k(M)$ are bounded in their set, they possess, by I, §3, a subsequence which converges on any given closed bounded region in T to a harmonic function. Hence $u(M)$ is harmonic in any closed bounded portion of T. It is therefore harmonic in T. It is thus the desired solution of the Dirichlet problem.

4. **The generalized problem.** Let T be any domain with its boundary t in the finite space, and $f(P)$ be given continuously on t. We may regard $f(P)$ extended as described in §3, continuous over the whole space, and vanishing outside a bounded region. Accordingly, it is natural to consider a sequence of functions $u_n(P)$ corresponding to a sequence of nested domains T_n approximating to T, and taking on as boundary values the extended $f(P)$. In the portion of space common to T and any given sphere $\Gamma(R, Q)$, arbitrarily large, these functions are bounded. Hence, by I, §3, a subsequence $u_n^*(P)$ converges within that region to a harmonic function $u(P)$.

Let H be the upper bound of $|f(P)|$, and consider for a moment the case where T is an exterior domain. On the spherical surface $C(R_0, O)$, R being chosen large enough so that t lies in $\Gamma(R_0, O)$, we have $|u_n(P)| \leqq H$. Since the $u_n(P)$ vanish at ∞, they must all satisfy the relation

$$-H\frac{R}{r} \leqq u_n(P) \leqq H\frac{R}{r}, \quad \text{for} \quad OP = r \geqq R,$$

for otherwise $u_n(P)-H(R/r)$ would have a positive maximum, or $u_n(P)+H(R/r)$ a negative minimum, at a point of T. We take then a sequence of values $R_0<R_1<R_2< \cdots$, tending to ∞. Corresponding to each $\Gamma(R_i, O)$ there is a sub-sequence of the $u_n^*(P)$ which converges to a harmonic function, and since within $T\cdot\Gamma(R, O)$ these functions coincide with $u(P)$, they provide a unique extension of $u(P)$ as a function harmonic throughout T. But also, we have

$$-H\frac{R}{r} \leqq u(P) \leqq H\frac{R}{r}$$

so that $u(P)$ vanishes at ∞.

It has been shown [40] that the function $u(P)$ is uniquely defined by the values given to $f(P)$ on t alone, independently of the choice of the continuous extension of $f(P)$ in W and of the choice of nested domains T_n.

It is called the *sequence solution* for T, belonging to boundary values $f(P)$. It is unnecessary to consider the uniqueness property at this point, however, since it results incidentally from the other theorems that must be given.

A point Q of t will be said to be *regular* for T if $u(P)$ takes on the boundary value $f(Q)$ as P approaches Q from T, whatever may be the function $f(P)$, continuous on t. It will be proved that if Q is regular for T with respect to the conductor potential $v(P)$ or the Green's function, it is regular in this more general sense. All points of t, however, are not necessarily regular. In particular, Lebesgue has shown that if t consists in part of a conical spine, projecting into an exterior domain T with vertex sufficiently sharp, the conductor potential does not tend to the value unity as P in T approaches the vertex of the spine along its axis. We have seen that the conductor potential is itself a sequence solution, corresponding to boundary values $f(P)=1$.

5. **Barriers.** We say that a function $b(M,Q)$ is a barrier [21; 17, p. 609] for the domain T at a point Q of t if there is a spherical neighborhood Ω with center Q in which the following conditions hold:

(i) The function $b(M, Q)$ is continuous for M in $T\cdot\Omega$ and approaches zero as M approaches Q.

(ii) The function $b(M, Q)$ is superharmonic† (or harmonic) in $T\cdot\Omega$.

(iii) Outside of any sphere $\Gamma(\rho, Q)$, it has in $T\cdot\Omega$ a positive lower bound. The definition as given here has been slightly modified from the usual one so as to make the existence of a barrier obviously a local property. But this definition is entirely equivalent to the customary one in which T is substituted for $\Omega\cdot T$, for the function which is everywhere the lesser of two superharmonic functions is superharmonic, and the properties of $b(Q, M)$ may be extended to the whole of T merely by defining it within Ω as the lesser of the two functions $b(Q, M)$ and b, where b is the lower bound of $b(Q, M)$ on the boundary of Ω, and as the radius of Ω times b/QM in the rest of T.

The following two theorems are essentially theorems of Vasilesco [37, p. 94] and Bouligand [1], respectively. Historically the first is a corollary of the second.

If T is an exterior domain and the point Q is regular for T with respect to the conductor potential, there exists a barrier $b(Q, M)$.

The proof of this proposition is immediate by means of equations (4), (4′). Let $\rho_1, \rho_2, \cdots$ be a monotonic sequence of positive values of ρ tending to zero, and $r_1, r_2, \cdots$ a second sequence of positive values such that

† A continuous function $U(P)$ is superharmonic in a domain T if in every closed region D contained in T we have $U(P)\geqq u(P)$, where $u(P)$ is harmonic in the interior of D, continuous in D, and less than or equal to $U(P)$ on the boundary of D.

$r_1+r_2+\cdots$ is convergent. Writing $b_k(M)$ for the function defined in (4) with $\rho=\rho_k$, we have the barrier

$$b(Q, M) = \sum_{1}^{\infty} r_k b_k(M). \tag{6}$$

In fact, the series in (6) is uniformly convergent, has a value which is greater than $r_k(1-\rho_k/QM)$ for M in T with $QM>\rho_k$, and is continuous and harmonic in T. Also, on account of the uniform convergence, $b(Q, M)\to 0$ as M in T tends to Q.

A function which behaves like the conductor potential at a point Q where it approaches unity may be called a *weak barrier*. In the definition of barrier the requirement (iii), is replaced by the less restrictive condition that the function be positive in T. The gist of the above theorem is that the existence of a weak barrer implies the existence of a barrier in the customary sense.

The following is a corollary of the above statement:

If T is an exterior or interior domain and Q is a regular point for T with respect to the Green's function, there exists a barrier $b(Q, M)$.

In fact, a barrier is transformed into a barrier by the Kelvin transformation, so that if T is an interior domain, the existence of a barrier at the corresponding point of the boundary of the exterior domain T' insures its existence at Q for T. If T is itself an exterior domain, the two kinds of regularity are, as we have seen, the same.

Consider now the function $G(P_0, M)=(1/r)-g(P_0, M)$, with $r=P_0M$, P_0 in T. This is a sequence solution for boundary values $1/r$. We have just proved that if $\lim_{M\to Q} G(P_0, M)=1/P_0Q$, there exists $b(Q, M)$. Consequently if Q is regular for T, there exists a barrier at Q.

We prove next the following theorem:

If there is a barrier $b(Q, M)$, the point Q is regular for T.

The proof is direct. Without loss of generality we may assume $f(Q)=0$. We make an extension of $f(P)$ as before and consider a sequence $u_n(P)$, corresponding to a sequence of nested domains approximating to T. For convenience we define $u_n(P)$ as $f(P)$ outside T_n. Let B be the upper bound of $|f(P)|$, and thus of all the $|u_n(P)|$, and let b be the lower bound of $b(Q, M)$ on the portion of spherical surface $T\cdot C(\rho, Q)$.

Given $\epsilon>0$, take ρ small enough so that $|f(P)|<\epsilon$ for P in $\Gamma(\rho, Q)$ and consider the function

$$h(M) = \epsilon + (B/b)b(Q, M).$$

For M in $T\cdot\Gamma(\rho, Q)$ we have, for all n,

$$-\, h(M) \leqq u_n(M) \leqq h(M).$$

In fact, $|u_n(M)| < \epsilon$ in $(T-T_n)\cdot\Gamma(\rho, Q)$ and in particular on $t_n\cdot\Gamma(\rho, Q)$, since for such points $u_n(M)=f(M)$. Also, on $(T_n+t_n)\cdot C(\rho, Q)$ we have $|u_n(M)| \leqq (B/b)b(Q, M)$. Since the above inequality holds on the boundaries $t_n\cdot\Gamma(\rho, Q)$ and $(T_n+t_n)\cdot C(\rho, Q)$ it holds throughout every domain bounded by them, and therefore throughout $T_n\cdot\Gamma(\rho, Q)$; for $u_n(M)$ and $-u_n(M)$ are in such domains harmonic where $b(Q, M)$ is harmonic or superharmonic. The inequality therefore holds for all M in $T\cdot\Gamma(\rho, Q)$.

We may now let $n\rightarrow\infty$, so that a subsequence of the $u_n(M)$ has the limit $u(M)$. Hence also,

$$|u(M)| \leqq h(M), \quad M \text{ in } T\cdot\Gamma(\rho, Q),$$

$$\limsup_{M\rightarrow Q} |u(M)| \leqq \epsilon,$$

and since ϵ is arbitrary,

$$\lim_{M\rightarrow Q} u(M) = 0,$$

which was to be proved.

It is thus proved that *a necessary and sufficient condition that a sequence solution should approach $f(Q)$, as M approaches Q from T, for an arbitrarily given continuous $f(P)$ on t, is that a barrier $b(M, Q)$ exists.*

As a final theorem in this section, we now prove the following statement:

The set of points of t at which a barrier for T fails to exist must be of capacity zero.

Suppose the contrary, that the set E of the theorem is of positive capacity. It will contain a closed subset F of positive capacity, by definition. There will then be a point Q of F such that the subset $F\cdot\{\Gamma(\rho, Q)+C(\rho, Q)\}$ will be of positive capacity for arbitrary ρ. Otherwise, by the Heine-Borel theorem, the set F could be covered by a finite number of spheres such that the portion of F in each sphere would be of zero capacity. But this is impossible since a portion of the conductor distribution on F must lie in one or more of these spheres. We choose ρ small enough so that a portion of T lies outside $\rho(\rho, Q)$, and denote by T' the domain exterior to the set $F\cdot\{\Gamma(\rho, Q)+C(\rho, Q)\}$, denoting this set itself by t'. It is evident that T' contains T.

The conductor potential of t' has, however, the value 1 except possibly on a subset of zero capacity, so that a barrier for T' exists except at these points of t. But since T' contains T, these are by definition barriers for T. Our hypothesis that there were no barriers for T at points of E is thus contradicted.

This theorem is essentially Kellogg's lemma, which he made a crucial point of the theory of the Dirichlet problem [16, p. 406; 18, p. 337; 8; 9].

6. **The uniqueness theorem and the solution of the problem.** The central theorem is the theorem of Kellogg [18, p. 335]:

If K is the upper bound of a function $U(M)$ in a bounded domain, in which $U(M)$ is bounded and harmonic, the set E of boundary points at which the superior limit of $U(M)$ is greater than or equal to $K-\epsilon$, for any $\epsilon>0$, has positive capacity.

The set E clearly is closed. Suppose that for some $\epsilon>0$ it has zero capacity. Then its exterior domain T' contains T, for if any bounded domain had a boundary lying entirely in E, E would be of positive capacity. Let $\{T_n'\}$ constitute a set of nested domains approximating to T' with boundaries everywhere satisfying condition (P), and denote by $v_n'(M)$ the corresponding conductor potentials, which are defined throughout all space and equal to unity on the complement of T_n'. In the domains constituting $T\cdot T_n'$, we have

$$U(M) \leqq K - \epsilon + \epsilon v_n'(M),$$

for this relation holds on the boundary of each such domain. Accordingly, letting n become infinite,

$$U(M) \leqq K - \epsilon + \epsilon v'(M), \quad M \text{ in } T,$$

where $v'(M)$ is the conductor potential of E.

But if E is of capacity zero, we have $v'(M)\equiv 0$ by §2, and $U(M)\leqq K-\epsilon$ in T. This is impossible since K is the upper bound of $U(M)$ in T. Hence the capacity of E is positive.

If T is an exterior domain and $U(M)$ vanishes at ∞, it is deduced as a corollary that the theorem still applies if K is a positive upper bound of $U(M)$, for we may consider $U(M)$ within $T\cdot\Gamma$ where Γ is a sphere chosen large enough so that $U(M)<K$ on its surface.

From this is deduced immediately the following statement:

If $U(M)$ is bounded and harmonic in T (vanishing at ∞ if T is an exterior domain) and has the limit 0 as M approaches a boundary point Q, except for points Q which constitute a set of capacity zero, then $U(M)\equiv 0$.

For if $U(M)\not\equiv 0$, either $U(M)$ or $-U(M)$ has a positive upper bound in T.

We have, in conclusion, the following summary of results:

The sequence solution $u(M)$, already defined, takes on continuously the given continuously assigned boundary values $f(P)$ at all points of t where there is a barrier for T, and that is at all points except possibly those of a set of capacity zero. There cannot be two different functions, harmonic in T, and bounded (vanishing at ∞ if T is an exterior domain) which take on the same continuously assigned boundary values except on a set of capacity zero. The *sequence solution is independent of the choice of the approximating sequence* $\{T_n\}$ *and of the manner of continuous extension of* $f(P)$.

III. The representation of harmonic functions

1. **Harmonic functions discontinuous on the boundary.** The function

$$\frac{\partial}{\partial\theta}\frac{1-r^2}{1+r^2-2r\cos\theta}\equiv\frac{2r(1-r^2)\sin\theta}{(1+r^2-2r\cos\theta)^2},$$

which is harmonic within the unit circle, nevertheless cannot be written as the difference of two functions which are positive and harmonic within the unit circle; it also has the property that it takes on continuously the boundary value zero along every radial direction. In the sphere, A. J. Maria has given an example of a function which is harmonic and vanishes as it approaches the boundary along any direction not tangent to it [24]. Thus arises the question of the classification of harmonic functions and the specification of appropriate boundary conditions which determine them uniquely.

In this Part III, we discuss the representation of positive harmonic functions in spatial domains. The treatment will be based on memoirs by F. W. Perkins, who considered the Dirichlet problem with continuously assigned values on boundary elements for boundaries where a point needs to be counted multiply [27], by A. J. Maria and R. S. Martin who discuss the representation of positive harmonic functions within domains for which the boundary points count simply and satisfy certain appropriate conditions [25], and by J. W. Green who combines the methods of both papers with the aid of a representation in a more abstract space with a consequent increase in the generality of results [15]. These papers afford a treatment in three dimensions of the problems considered by the author in two dimensions [7] who bases his work on the prime-end theory and conformal mapping of Carathéodory [3] and the treatment of Poisson's integral in terms of Lebesgue theory of integration by Fatou [12]. Fatou and Zaremba [42] may properly be regarded as having initiated the study of discontinuous problems. Wiener [39] and the present author [6] discuss them as functionals of their boundary values. The sweeping out method of Poincaré [30] leads to them naturally.

2. **Induction of unit mass.** Following Poincaré, but with more general domains, de la Vallée Poussin [35, p. 205] makes the sweeping out of unit mass a resolvent for the Dirichlet problem, thus obtaining a generalization of the mass distribution the density of which for smooth boundaries is the normal derivative of the Green's function. Following Frostman he again analyzes the question as a problem of minimum energy [36, p. 30; 11, p. 497].

We consider first a bounded region T. We fix unit negative mass at a point M of T. In order to avoid the complication of an unbounded set we take F as the complement of T within a large closed sphere $\Gamma+C$, with boundary C. On F let there be distributed a mass $m(e)$ of total value $+1$ in such a way as to make the energy a minimum, disregarding the energy of the point charge itself, which is, of course, infinite. In other words, we minimize the quantity

$$(1) \qquad I_M = \int_F \int_F \frac{dm(e_Q)dm(e_P)}{QP} - 2\int_F \frac{dm(e_P)}{MP}.$$

With the same considerations as those advanced for the conductor potential it becomes clear that there exists a unique distribution on F which makes I_M a minimum; the corresponding potential, which is given by the formula

$$(2) \qquad V(M, P) = -\frac{1}{MP} + \int_F \frac{dm(e_Q, M)}{QP},$$

in which $m(e)$ is denoted by $m(e, M)$, has a constant value C_M on the nucleus F_1 of F except at a possible subset of capacity zero, where it is less than C_M. On the rest of F, except for a possible set of capacity 0 it is greater than or equal to C_M.

Suppose now, as a particular case, that the points of F on t satisfy the condition (P). Then all the points of F satisfy the condition, and $V(M, P)$ as a function of P is continuous on F_1 with respect to F_1. In the same way as for the conductor potential, it is deduced that $V(M, P)$ is continuous in the whole space W and therefore takes on its maximum value C_M on F_1. This value must be zero, for otherwise, since $V(M, P)$ vanishes at ∞, the total algebraic mass within Γ could not be zero. The potential $V(M, P)$ is therefore identically zero outside t, and no portion of t is without mass. We note that we have the relation

$$(2') \qquad g(M, P) = -V(M, P),$$

for the Green's function and $-V(M, P)$ are harmonic in T except for the term $1/MP$, and both vanish continuously on t. We have $I_M < 0$.

As for the mass function $m(e, M)$ itself, we find that it is a harmonic function of M within T, assuming still that t satisfies the condition (P).

In fact, if $-\nu(e)$ is any distribution of negative mass, of total amount -1, on a closed set F' interior to T, and $m(e)$ the induced distribution of positive mass of total value $+1$ on F, we find again that for the corresponding minimizing energy the potential

$$V(P) = -\int_{F'} \frac{1}{PM}\, d\nu(e_M) + \int_\Gamma \frac{dm(e_Q)}{QP}$$

has the constant value $C=0$ on t and remains 0 outside t. Since this condition is satisfied by writing

$$(3) \qquad m(e) = \int_{F'} m(e, M) d\nu(e_M),$$

and the minimizing distribution is unique, we know that (3) is valid.

Take now F' as $C(\rho, M')$ where M' is an arbitrary point of T and ρ is less than the distance of M' from t, and write

$$\nu(e) = (1/4\pi\rho^2) \int_{C(\rho, M')\cdot e} 1 dM.$$

The potential of this distribution at a point P outside $C(\rho, M')$ is $1/M'P$, so that the induced distribution on t is $m(e, M')$, as if the mass $-\nu(e)$ were all concentrated at M'. Hence from (3),

$$m(e, M') = \int_{C(\rho, M')} \frac{m(e, M)}{4\pi\rho^2} dM.$$

But the last equation is a statement that $m(e, M')$ is its own spherical surface average, and $m(e, M)$ is accordingly harmonic in T.

A final result we can obtain by specializing still further the nature of t. Later, by means of the sequence method the result can be made to have more general significance. Let us assume that the boundary of t satisfies a condition which we shall call (P_0).

(P_0) At each point of t a sphere can be drawn which passes through that point and contains in its interior no point of T or on its boundary no other point of t.

The following is a special instance of de la Vallée Poussin's theorem [35, p. 203].

Let T be a bounded domain which satisfies the condition (P_0); *let Q be a point of t and t_ρ the portion of t which is contained in* $\Gamma(\rho, Q)$. *Then, for arbitrary ρ, $m(t_\rho, M)\to 1$ as $M\to Q$ from T.*

In order to prove this statement, let P' be the center and r the radius of the sphere of condition (P_0) at Q. Since the potential $V(M, P)$ is identically zero for P not in T, we have

$$\int_{t_\rho} \frac{dm(e_P, M)}{PP'} + \int_{t-t_\rho} \frac{dm(e_P, M)}{PP'} = \frac{1}{MP'}.$$

But since for P on t_ρ, $PP'\geqq r$ and on $t-t_\rho$, $PP'\geqq r+\delta$, where δ depends on ρ and is positive, we have

$$\frac{m(t_\rho, M)}{r} + \frac{m(t - t_\rho, M)}{r + \delta} \geqq \frac{1}{MP},$$

and letting M approach Q as a limit

$$\frac{1}{r} \lim_{M\to Q} m(t_\rho, M) + \frac{1}{r+\delta} \lim_{M\to Q} [1 - m(t_\rho, M)] \geqq \frac{1}{r} \cdot$$

But this is impossible unless $\lim_{M\to Q} m(t_\rho, M) = 1$.

This theorem provides an immediate representation of the solution of the Dirichlet problem, for the function

$$u(M) = \int_t f(P) dm(e_P, M), \tag{4}$$

with $f(P)$ given as continuous on t, is harmonic in T and takes on continuously the values $f(Q)$ as M approaches Q on t. In fact,

$$u(M) = f(Q)m(t, M) + \int_t [f(P) - f(Q)] dm(e_P, M),$$

and if ρ is chosen small enough so that $|f(P) - f(Q)| < \epsilon$ for P in t_ρ,

$$\left| \int_{t_\rho} [f(P) - f(Q)] dm(e_P, M) \right| < \epsilon$$

while the rest of the integral, over $t - t_\rho$, approaches zero as M approaches Q. Hence

$$\lim_{M\to Q} | u(M) - f(Q) | < \epsilon.$$

By considering a sequence of nested approximating domains $\{T_n\}$ to a general bounded domain T, the domains T_n satisfying the condition (P_0) the following results may be obtained, exactly as in the analysis of II, §2'.

The induced distribution of unit mass $m(e, M)$ for a bounded domain T lies entirely on t and is unique. Any sphere $\Gamma(\rho, Q)$ with center at a point of t which contains a portion of F of positive capacity will contain a portion of the mass. The potential $V(M, P)$ of unit negative mass at M and the induced mass is zero on t except on a possible set of capacity 0 where it is negative; it is negative except on t.

Consider any cube w enclosing points of t such that none of the mass $m(e, M)$ lies on any of its faces. By the weak convergence there is a sub-sequence $\{n^*\}$ of the sequence $\{n\}$ such that the corresponding $m_n(w, M)$ converge to $m(w, M)$. Since the harmonic functions $m_n(w, M)$ are bounded, it follows that $m(w, M)$ is harmonic. This fact being true for "almost all" cubes, it follows that $m(E, M)$ is harmonic for any open set, and consequently for any set measurable Borel. But $m(E, M) = m(E \cdot t, M)$ since all the mass lies on t.

Making use of (4) for the boundaries t_n, it follows from the weak convergence, since t_n may be replaced by W, that (4) still applies where $u(M)$ denotes the sequence solution for T. This is de la Vallée Poussin's formula [35, p. 205]. The uniqueness of the function $m(e, M)$ for general boundaries was established however by general considerations on the sweeping out process [10, II].

The set function $m(e, M)$ is harmonic and equation (4) represents the sequence solution of the generalized Dirichlet problem. A necessary and sufficient condition that Q be a regular point of t for T is that

$$\lim_{M \to Q} m(e, M) = 1, \tag{5}$$

where $e = t \cdot \Gamma(\rho, Q)$, for all ρ. The Green's function is still given by (2′).

So far, T has been considered as a bounded domain. If T is an exterior domain, its boundary t may be enclosed within a sphere $\Gamma(R, O)$ and R allowed to become infinite so that $T \cdot \Gamma(R, O)$ has T as a limit. The induction problem is solved for the region $T \cdot \Gamma(R, O)$ and the weak convergence, as R tends to infinity, gives the solution for T, the total mass on t being the limiting value of the portion of the mass unity which lies on t for R finite. De la Vallée Poussin [35, p. 208] shows that this amount $m(t, M)$ is less than unity. But otherwise the situation is unchanged and the proofs of equations (4) and (5), with trivial modifications, remain valid.

3. **Boundaries with multiple points. The generalized space.** We shall restrict ourselves for convenience to a bounded domain T. A boundary element γ for T is defined by Perkins to be a sequence of partial domains of T: $T_1, T_2, \cdots$ with boundaries $t_1, t_2, \cdots$ such that the following hold:

(i) T_n contains T_{n+1} and the diameter of T_n tends to 0 with $1/n$.

(ii) Each t_n contains at least one point of t.

(iii) $t_{n+1} - t \cdot t_{n+1}$ is at a positive distance from $t_n - t \cdot t_n$.

The set $t_n - t \cdot t_n$ is the portion of t_n which lies in T and with its limit points constitutes what is called the *auxiliary* boundary of T. The requirement (iii) provides that the auxiliary boundaries have no points in common.

Two boundary elements are said to be identical if each T_n of one contains all the partial domains of the other of index sufficiently great; if the condition holds one way, it holds the other. A partial domain is said to contain γ if it contains some T_n of the sequence. There is one and only one point P of t which is common to all the t_n of a boundary element γ, so that γ may be regarded as lying on P. But evidently, as in the simple case of a domain which is bounded partially by both sides of the same piece of surface, more than one γ may lie on the same P. In fact, even in a plane domain, a boundary point may be composed of a non-denumerable infinity of boundary elements [3, p. 363], and if the point is not accessible from the domain will not correspond to any boundary elements.

In particular, "pseudo-spherical" nested domains may be used for the definition of boundary elements, a pseudo-spherical domain being one of the above type whose auxiliary boundary is a portion of a sphere, the center being at a given point P—a statement which is evident from the definition of identical boundary elements. A sequence of such domains defining γ may be denoted by $\{\mathfrak{S}(\gamma, \rho_i)\}$ where γ lies on the center P.

The method of Green is to set up a new space $\mathfrak{T}$ composed of boundary elements and points of T and thus to obtain briefly and extend the results already obtained. The metric is established by defining distance in the following way.

Let π_1, π_2 be two elements of $\mathfrak{T}$ (that is, points or boundary elements of T). Let ρ_1 be the lower bound of ρ such that $\mathfrak{S}(\pi_1, \rho)$ contains π_2 and ρ_2 the lower bound of ρ such that $\mathfrak{S}(\pi_2, \rho)$ contains π_1. The distance (π_1, π_2) is defined by the formula

$$(\pi_1, \pi_2) = \tfrac{1}{2}(\rho_1 + \rho_2).$$

The distance between two elements of $\mathfrak{T}$ cannot be more than $2d$, where d is the diameter of T. In particular, in the above definition, $\rho_1 \leqq 2\rho_2$ and $\rho_2 \leqq 2\rho_1$ and (π_1, π_2) vanishes only when π_1, π_2 are the same point or boundary element in the sense defined by Perkins. Moreover if π_1, π_2, π_3 are three elements of $\mathfrak{T}$, the triangular inequality $(\pi_1, \pi_3) \leqq (\pi_1, \pi_2) + (\pi_2, \pi_3)$ can be verified. If π_1 is not a boundary element but a point of T, we denote by $\mathfrak{S}(\pi_1, \rho)$ the "pseudo-spherical" or spherical domain whose auxiliary boundary consists only of points of the sphere. If $\mathfrak{S}(\pi_1, \rho')$ contains π_2 and $\mathfrak{S}(\pi_2, \rho'')$ contains π_3, then $\mathfrak{S}(\pi_1, \rho' + \rho'')$ contains $\mathfrak{S}(\pi_2, \rho'')$ and therefore π_3. From this the inequality is easily derived.

The space $\mathfrak{T}$ is complete, that is, if the sequence $\{\pi_n\}$ is convergent there is an element π such that $\lim_{n\to\infty} (\pi, \pi_n) = 0$. If some or all of these π_n are not points of T, we may replace them by sufficiently nearby points of T, and if these latter have a limit element π, the former will have the same limit element on account of the triangular inequality. Hence we may suppose that the π_n are points P_n of T and these have a limit point P in T or on t. It is only if P is on t that there is something to prove.

We form a subsequence of the $\{\pi_n\}$. Given $\rho_1 > 0$ we take n_1 so that $(\pi_{n_1}, \pi_{n_1+p}) < \rho_1/2$ for all p. Then $\mathfrak{S}(\pi_{n_1}, \rho_1)$ contains π_{n_1+p} and has for part of its boundary a portion of t, since $P_{n_1}P_{n_1+p} < \rho_1/2$ and $P_{n_1}P \leqq \rho_1/2$. Hence $\mathfrak{S}(\pi_{n_1}, \rho_1)$ may be taken as a domain T_1. We take now $\rho_2 < \rho_1/4$ and choose $n_2 > n_1$ so that $(\pi_{n_2}, \pi_{n_2+p}) < \rho_2/2$. The domain $\mathfrak{S}(\pi_{n_2}, \rho_2)$ may then be taken as T_2, for it is contained in T_1 and the two auxiliary boundaries, being parts of the respective spherical surfaces, have no points in common. Proceeding in this way we obtain a sequence of domains T_i which define a boundary element γ. The corresponding element π of $\mathfrak{T}$ satisfies the relation $(\pi, \pi_{n_i}) \leqq \rho_i$. Hence the π_{n_i} converge to π. By the triangular inequality, the π_n therefore converge to π.

The space $\mathfrak{T}$ is thus complete and bounded. It is evidently non-linear, and in general it is not compact—for instance, if there is a point of t which is not accessible from T, an infinite sequence of points of T which have this point only as limit point will not converge in $\mathfrak{T}$. Accordingly, for the sake of obtaining systematic results, $\mathfrak{T}$ will in the main be made compact as a restriction on the character of t. The following condition (A) is a necessary and sufficient condition for compactness [15, §2.2]:

(A) Let P be any point of t, and consider the set $T \cdot \Gamma(\rho, P)$. There exists $\delta > 0$ (depending on P and ρ) such that of the finitely or denumerably infinitely many domains whose sum is $T \cdot \Gamma(\rho, P)$ all but a finite number are at distances greater than or equal to δ from P.

The simplification of the compact spaces lies in the fact that simple covering theorems may be employed. Thus it may be proved in the usual way that a function which is continuous on a closed set contained in $\mathfrak{T}$ is uniformly continuous.

We assume then condition (A). The Riemann-Stieltjes integral

$$\int_{\mathfrak{F}} \phi(\pi)\, df(e_\pi)$$

extended over a closed set $\mathfrak{F}$ of $\mathfrak{T}$ may be defined in the customary manner, $\phi(\pi)$ being a continuous function of elements, and $f(e)$ being defined and completely additive on a closed family of subsets of $\mathfrak{F}$, that is, one containing sums, products and differences, finite or denumerably infinite, of the members of the family, and containing finite or denumerably infinite collections of non-overlapping sets of the family, of diameter arbitrarily and uniformly small, whose sum is $\mathfrak{F}$.

Types of sets of this last category may for instance be constructed by imposing on the euclidean space containing T a lattice whose cells have diameter less than δ, so that T is divided by the planes of the lattice and its own boundary into at most denumerably many domains. If these domains are numbered in a countable order, all the boundary elements of the first domain may be assigned to the first domain to form the first set, all the boundary elements of the second domain, not previously assigned, may be assigned to the second set, and so on. The diameter in $\mathfrak{T}$ of any of these sets of elements of $\mathfrak{T}$ is not greater than 2δ.

The usual properties of the Stieltjes integral follow immediately from the definition. Moreover, the property of weak convergence of a sequence of set functions may be developed with reference to this integral. Let $\{f_n(e)\}$ represent a sequence of completely additive positive functions of sets of elements e of $\mathfrak{T}$, bounded in their set. There exists a function $f(e)$ of the same character and a sub-sequence $\{f_n^*(e)\}$ such that

$$\lim_{n\to\infty}\int_{\mathfrak{F}}\phi(\pi)df_n^*(e_\pi) = \int_{\mathfrak{F}}\phi(\pi)df(e_\pi)$$

for every function $\phi(\pi)$ continuous on $\mathfrak{F}$; in particular $\mathfrak{F}$ may be identical with $\mathfrak{T}$.

A partial extension of weak convergence may be made to domains $\mathfrak{T}$ which do not satisfy the condition of compactness. Assume that there exists a sequence of sets $\Omega_1, \Omega_2, \cdots$ of elements of $\mathfrak{T}$, where each set contains the next, and a monotone sequence $\{\epsilon_i\}$ of positive numbers approaching 0, such that (a) $\mathfrak{T}-\Omega_i$ is a complete compact space; (b) $f_n(\Omega_i) < \epsilon_i$ for $n \geqq n_0 = n_0(i)$, where the $f_n(e)$ are bounded in the set and $\geqq 0$. Then there exists a function $f(e)$ such that $f(e)=f(e\cdot[\mathfrak{T}-\prod\Omega_i])$ and such that, for a sub-sequence,

$$\int_{\mathfrak{T}}\phi(\pi)df(e) = \lim_{n\to\infty}\int_{\mathfrak{T}}\phi(\pi)df_n^*(e) \tag{6}$$

where $\phi(\pi)$ is continuous on $\mathfrak{T}$.

4. **Application to Dirichlet problem.** Given $\mathfrak{T}$ compact and a function $f(\gamma)=f(\pi)$ continuous as a function of elements π on the boundary τ of $\mathfrak{T}$. There cannot be more than one function bounded and harmonic in $\mathfrak{T}$ which takes on continuously (in terms of the $\mathfrak{T}$ metric) the boundary values $f(\pi)$ at elements π of τ, even with the exception of elements which lie on a set of points of t of zero capacity. For Kellogg's uniqueness theorem deals with methods of approach to the boundary.

On the other hand, if a barrier for T exists at a point Q of t, the function $f(\pi)$ being continuously extended throughout T, it is but a repetition of the previous proof to show that the sequence solution $u(M)=u(\pi)$ takes on continuously the value $f(\pi')$ as $\pi\to\pi'$ from T, where π' is any of the boundary elements at Q. A barrier exists however, as we have already seen, except at points of t which form a set of zero capacity. Hence there is a unique solution of this extended Dirichlet problem. This, except for the application of Kellogg's lemma and the use of the space $\mathfrak{T}$, is essentially Perkin's theorem.

The uniquely determined solution is represented by means of the unit mass distribution induced on the elements of τ. In fact, for the domains of a nested sequence T, whose boundary points are not multiple—that is, each one counts as a single boundary element,—the representation already found may be used and transferred to the abstract space $\mathfrak{T}$:

$$u(M) = \int_{t_n} u(P)dm_n(e_P, M) = \int_{\mathfrak{T}} u(\pi)dm_n(e_\pi, \mathfrak{M}) = u(\mathfrak{M}).$$

But now, in terms of the weak convergence in the space $\mathfrak{T}$, letting $n \to \infty$,

$$u(M) = \int_{\mathfrak{T}} f(\pi) dm(e_\pi, \mathfrak{M}). \tag{7}$$

If the boundary elements are all regular the procedure is direct. If some of them are irregular, all the boundary elements corresponding to the exceptional set of zero capacity may be omitted from $\mathfrak{T}$, and the extension of the weak convergence method is still applicable. Certain domains, where boundary elements fail to exist on sets of restricted character, due, say, to the presence of inaccessible points, may also be brought under this method and yield again equation (7) [15, §8.2].

5. **Extension to representation of positive harmonic functions.** For functions which do not remain bounded, a representation in terms of an additive set function is still possible provided the function is the difference of two positive harmonic functions in T. If the boundary t of the region consists of a finite number of surfaces, each with uniformly bounded curvatures, the function within T is represented by a potential of a double layer on t. To the representation corresponds also a uniquely solvable extension of the Dirichlet problem: to each function $u(M)$ corresponds a unique completely additive function of point sets $F(e)$ on t which is the limit of the indefinite surface integral of the function over portions of surfaces in T neighboring t, and conversely. Boundary values of $u(M)$, which are given by the Lebesgue derivative of $F(e)$, exist almost everywhere on t for directions of approach that keep away from tangency to the boundary, but $u(M)$ is not determined by them [14]. The representation of the function has the form

$$u(M) = \int_t f(M, P) dF(e). \tag{8}$$

In the problem for plane domains, the method of conformal transformation is available, so that when the above representation is expressed in a form which is invariant of conformal transformation, that is, in terms of the integral of the normal derivative of the Green's function, it is possible to extend the representation without much difficulty to domains of a considerable degree of generality, but of finite connectivity.

Maria and Martin extended the representation (8) to a very general class of boundaries in an arbitrary number of dimensions, without reference to the connectivity of the domain, or, of course, to conformal transformation. The function $f(M, P)$, which in (8) is the ratio of normal derivatives of the Green's functions with variable pole M and fixed pole O, becomes the Daniell derivative of one set function with respect to another, corresponding to the fact that in two dimensions it is the ratio $dh(M, P)/dh(O, P)$ where $h(M, P)$ is the function conjugate to

$g(M, P)$. The boundary, however, has no irregular points or multiple points.

A slight extension of the general character of the domain and the elimination of the restriction on irregular or multiple boundary points is made by Green [15, §§9–11]. The conditions may be stated as follows:

(A) The corresponding $\mathfrak{T}$ is compact.

(B) Every boundary element of T satisfies a condition of Picard: If $u(P)$, $v(P)$ are positive harmonic functions in T which are bounded except in $\mathfrak{S}(\gamma, \rho)$ for all $\rho > 0$ and vanish at all regular boundary elements of T except γ, then $u(P)/v(P) = \text{const.}$

(C) There exists a sequence $\{T_i\}$ of nested domains approximating to T, with boundaries $\{t_i\}$, having the following properties:

(Ca) T_i admits the Maria-Martin representation of positive harmonic functions. (This would be the case, for instance, if each T_i had bounded curvatures.)

(Cb) Let $u_i(P)$ be a positive harmonic function in T_i, zero on t_i except at the point B_i. Let $\{u_i(P)\}$ converge in T to the harmonic function $u(P)$ and let $\{B_i\}$ converge to the boundary element γ. Then the $u_i(P)$ are uniformly bounded outside any $\mathfrak{S}(\gamma, \rho)$.

Under these conditions the method of weak convergence can be applied to the Maria-Martin representation of $u(M)$, if $u(M)$ is positive and harmonic in T, and a formula obtained of the form

$$u(M) = \int_t f(\gamma, M) d\mu(e_\gamma), \tag{9}$$

in which the $\mu(e)$ is a positive distribution of mass on boundary elements of t, and the function $f(\gamma, M)$ is continuous in the elements γ and harmonic in M.

The principal restriction on the domain seems to be the Picard condition (B), and this is worthy of a study on its own account. For example, Bouligand [2, p. 32] shows that when T consists of the domain between two spheres, tangent internally, the Picard condition is not satisfied at the point of tangency. The difficulty in this case is not trivial; the singularity subtends an angle of π around the singular point and is movable.

IV. Related problems

1. A problem of mixed boundary values for the sphere. The mixed boundary value problem in the case of plane domains, in which values of the function are given on part of the boundary and values of the normal derivative on the rest, may be reduced to the Dirichlet problem for the first part of the boundary considered as that of an infinite domain. The reduction is effected by the use of the Green's function for this new domain, in the case of circular boundaries, and the result extended to more general ones by conformal transformation [11, p. 487]. In space, the corresponding boundary value problem depends on the initial domain considered, varying from one domain to another. If the domain is the upper half space, $z \geqq 0$, and the boundary the plane $z=0$, the problem admits the same statement as in the two-dimensional case. In the case of the sphere, which we treat here, the function $u(M)$ is given on part of the surface of the sphere and the values of $u/2+rdu/dr$ on the rest. But in order to take account of the generality of the representation merely the integrals of these data will be given, rather than the data themselves, as in the discussion referred to above.

On the spherical surface C of radius a, bounding a domain Γ, let F be a closed set of positive capacity, as a spatial set. The set $C-F$ is the sum of a denumerable infinity $E=\sum I_k$ of domains I_k. Let C_r denote the concentric spherical surfaces $C(r, O)$, σ a two-dimensional region on C and σ_r the radial projection of σ on C_r. We shall deal with functions $u(M)$ which are harmonic within the sphere, of a class described by the following two properties:

(i) Any $u(M)$ is in Γ the difference of two functions, harmonic and nonnegative in Γ.

(ii) If σ is any region contained with its boundary in an I_k, the quantity $\int_{\sigma_r}|du/dr|\,d\sigma_r$ remains bounded as r tends to a.

It may be verified, by an integration over a volume contained between σ_{r_0} and σ_r, that the condition that $\int_{\sigma_r}|du/dr|\,d\sigma_r$ remain bounded implies a similar condition on $\int_{\sigma_r}|u|\,d\sigma_r$ and therefore that the condition (ii) is equivalent to requiring the boundedness of $\int_{\sigma_r}|\lambda u+r(du/dr)|\,d\sigma_r$ for any particular λ.

On the domains $\sum I_k=C-F$, we are given a completely additive set function $h(e)$. We shall show that there is a function $U(M)$ of the class defined by (i), (ii) such that

$$\lim_{r\to a}\int_{\sigma_r}\left\{\frac{U}{2}+r\,\frac{dU}{dr}\right\}d\sigma_r = h(\sigma) \tag{1}$$

for regions σ on $C-F$ whose boundaries s contribute no mass themselves to $h(e)$, that is, regions σ that $\int_s|dh(e\cdot s)|=0$. This function will vanish at every point of F which is at a positive distance from $C-F$. The Dirichlet data on F lead however naturally to a function for which the limit of the expression in the left-hand member of (1) is zero. Hence by subtracting the function $U(M)$ above, the problem is reduced to the representation problem of the preceding chapter for the domain complementary to F These statements will now be made precise.

Let $u(M)$ be harmonic in Γ, belonging to the class (ii). *If for every circular region σ' in the domain I on $C-F$ we have*

$$\lim_{r \to a} \int_{\sigma_r'} \left(\frac{u}{2} + r \frac{du}{dr} \right) d\sigma_r = 0, \tag{2}$$

then $u(M)$ may be defined so as to be harmonic at the points of I, and admits a unique harmonic extension across I into the entire exterior of C by means of the formula

$$u(M') = \frac{r}{a} u(M), \tag{2'}$$

where M' is the inverse of M with respect to C. It vanishes at ∞.

We note first that if $v(M)$ is harmonic in Γ and $\int_{\sigma_r} |v(M)| d\sigma_r$ is bounded as r tends to a, and the quantity $\int_{\sigma_r'} v(M) d\sigma_r$ approaches 0 as r tends to a, then $v(M)$ admits a unique extension by the formula

$$v(M') = -(r/a)\, v(M)$$

being 0 on I.

In fact, we may form the average of $v(M)$ over regions of constant following angular radius τ:

$$v_\tau(M) = \frac{1}{2\pi(1 - \cos\tau) r^2} \int_{\sigma_r'} v(P) d\sigma \tag{3}$$

where $d\sigma = r^2 d\omega$. This function is seen to be harmonic, for τ fixed, as a function of M; it is bounded, by hypothesis, in any cone in Γ with vertex at the center of the sphere and base a closed region Σ in I, distant from the boundary of I by more than τ. Moreover along all radii to points of Σ, $\lim v_\tau(M) = 0$. An iterated average $v_{\tau,\tau'}(M)$, with τ' suitably small, retains these properties and is continuous on C_r within a region Σ_r' inside Σ_r, uniformly in r. It therefore has the boundary values 0 in Σ' continuously and admits a unique harmonic extension, taking at the inverse point M' the value $-(r/a)v_{\tau\tau'}(M)$ and remaining harmonic at points of Σ'.

We can thus let first τ' and then τ approach 0 so that the same properties are valid for $v(M)$ itself. But any point in I may be included in Σ', so $v(M)$ is harmonic at all the points of I.

Returning to the function $u(M)$ we see by direct calculation that the function $u/2 + r du/dr$ is harmonic and can thus be chosen as the $v(M)$ of the preceding paragraphs, and extended by the formula

$$v(M') = -\frac{r}{a}\left(\frac{u(M)}{2} + r \frac{du}{dr} \right),$$

vanishing identically on I. We have, however, the relation

$$\frac{\partial}{\partial r}(r^{1/2} u) = r^{-1/2}\left(\frac{u}{2} + r \frac{\partial u}{\partial r} \right) = r^{-1/2} v,$$

and therefore by integration, $u(M)$ approaches finite continuous limits as $r \to a$. Moreover, if we write $u(M')=(r/a)u(M)$ and calculate the value at M' of $u/2+r'du/dr'$, we find that it is precisely that given for $v(M')$; also $u(M)$ is made continuous at points in I. Since $u/2+rdu/dr$ takes on continuously the value 0 as r passes through the value a, it follows that du/dr has the same exterior limit as interior limit, and, by the law of the mean, that this common value is the value of du/dr when $r=a$. That the lateral derivatives are bounded is deduced at once from the formula $r^{1/2}u=\int_0^r r^{1/2}v dr$. The function $u(M)$ will therefore be harmonic on I [18, p. 267]. Since $u(M')=(a/r')u(M)$, it vanishes at ∞.

Conversely, if $u(M)$ is harmonically extensible across I by the formula $u(M')=(r/a)u(M)$, the quantity $u/2+rdu/dr$ vanishes identically on I.

Consider now the Green's function for the set F, taken as a set in space $g(M, P)=1/MP+G(M,P)$, where $G(M,P)$ is harmonic in M if M is not on F. If P is on $C-F$, the Green's function, as a function of M, admits the transformation $g(M', P)=(r/a)g(M, P)$, for when M' approaches a point of F, M approaches the same point (at perhaps an opposite boundary element) and $g(M', P)$ takes on the same limiting values as $g(M, P)$. Moreover, $g(M', P)$ becomes infinite like $1/M'P$ when $M' \to P$ on $S-F$, since by similar triangles $1/M'P=(r/a)(1/MP)$.

Let $h_k(e)$ be any completely additive function of sets on the domain I_k for each k, such that $\sum_k \int_{I_k} |dh_k(e)|$ is bounded, and form the function

$$(4)\qquad \begin{aligned} U(M) &= \sum_k \frac{1}{2\pi a}\int_{I_k} g(M, P)dh_k(e_P) \\ &= \frac{1}{2\pi a}\int_{C-F} g(M, P)dh(e), \end{aligned}$$

where $h(e)=\sum_k h_k(e\cdot I_k)$.

If σ, contained with its boundary s in an I_k, is such that the contribution of $h(e)$ to s itself is zero, we have

$$(5)\qquad \lim_{r\to a}\int_{\sigma_r}\left(\frac{U}{2}+r\frac{dU}{dr}\right)d\sigma_r = h(\sigma).$$

The function $U(M)$ satisfies the conditions (i), (ii).

That $U(M)$ satisfies the condition (i) is clear, since $g(M, P)\geqq 0$. That it satisfies (ii) comes without difficulty from the inequality

$$\int_{\sigma_r}\left|\frac{\partial U}{\partial r}\right| d\sigma_r \leqq \frac{1}{2\pi a}\int_{C-F}|dh(e_P)| \cdot\int_{\sigma_r}\left\{\left|\frac{\partial}{\partial r_M}\frac{1}{MP}\right|+\left|\frac{\partial}{\partial r_M}G(M, P)\right|\right\}d\sigma_r,$$

the function $G(M, P)$ being harmonic as a function of M, with bounded derivatives so long as M remains at a positive distance from F.

Let I be any one of the domains I_k, and select σ as in the statement of the theorem. We seek to evaluate the integral

$$J_r = \int_{\sigma_r} \left\{ \frac{U(M)}{2} + r\,\frac{\partial U(M)}{\partial r} \right\} d\sigma_r.$$

We have

$$\begin{aligned} 2\pi a J_r &= \int_{C-F} dh(e_P) \int_{\sigma_r} \left\{ \frac{g(M, P)}{2} + r\,\frac{\partial g(M, P)}{\partial r} \right\} d\sigma_r \\ &= -\left(\frac{r}{r'}\right)^{3/2} \int_{C-F} dh(e_P) \int_{\sigma_{r'}} \left\{ \frac{g(M', P)}{2} + r'\,\frac{\partial g(M', P)}{\partial r'} \right\} d\sigma_{r'} \end{aligned}$$

$$\begin{aligned} 4\pi a J_r &= \int_{C-F} dh(e_P) \int_{\sigma_r} \frac{1}{2} \left\{ g(M, P) - \frac{r}{a}\, g(M', P) \right\} d\sigma_r \\ &\quad + \int_{C-F} dh(e_P) \left\{ \int_{\sigma_r} r\,\frac{\partial g(M, P)}{\partial r}\, d\sigma_r - \int_{\sigma_{r'}} \left(\frac{r}{r'}\right)^{3/2} r'\,\frac{\partial g(M', P)}{\partial r'} d\sigma_{r'} \right\}, \end{aligned}$$

of which the first integral is zero. Likewise, in the second integral, since M and M' are bounded away from F in distance, if we write $g = 1/MP + G$, the only terms which do not cancel as r approaches a are those which correspond to the derivatives of $1/MP$. Hence

$$\begin{aligned} \lim_{r\to a} 4\pi a J_r = \lim_{r\to a} &\int_{C-F} dh(e_P) \\ &\cdot \left\{ r \int_{\sigma_r} \frac{\partial}{\partial r}\, \frac{1}{MP}\, d\sigma_r - r'\left(\frac{r}{r'}\right)^{3/2} \int_{\sigma_{r'}} \frac{\partial}{\partial r'}\left(\frac{1}{M'P}\right) d\sigma_{r'} \right\}. \end{aligned}$$

But

$$\begin{aligned} \int_{\sigma_r} \frac{\partial}{\partial r_M}\left(\frac{1}{MP}\right) d\sigma_r(M) &= -\int_{\sigma_r} \frac{1}{PM^2} \cos(PM, r)\, d\sigma_r = \Theta(P), \\ -\int_{\sigma_{r'}} \frac{\partial}{\partial r'_M}\left(\frac{1}{M'P}\right) d\sigma_{r'}(M) &= \int_{\sigma_{r'}} \frac{1}{PM'^2} \cos(PM', r')\, d\sigma_{r'} = \Theta'(P), \end{aligned}$$

where $\Theta(P)$, $\Theta'(P)$ are respectively the solid angles subtended by σ_r and $\sigma_{r'}$ from P.

It is evident then, that if σ is such that there is no portion of $h(e)$ on s, that is, $\int_s |dh(e\cdot s)| = 0$, since the sum of $\Theta(P)$ and $\Theta'(P)$ approaches 4π when r approaches a, if P is interior to σ, and 0 if P is exterior to σ, we have

$$\lim_{r\to a} 4\pi a J_r = 4\pi a h(\sigma),$$

as was to be proved.

We shall extend this lemma slightly. If σ is bounded by a simple closed curve s, with continuously turning tangent, the formula (5) must be replaced by the following

$$\lim_{r\to a} J_r = h(\sigma) + \tfrac{1}{2}\bar{h}(s), \tag{5'}$$

where σ denotes the set of points interior to s, and $\bar{h}(s)$ stands for $h(\bar{\sigma})-h(\sigma)$, where $\bar{\sigma}$ is the closed cover of σ. If, in addition, s is allowed to have a finite number of vertices P_i, at which σ subtends plane angles θ_i, and $h(P_i)$ denotes $h(e\cdot P_i)$, we have

$$\lim_{r\to a} J_r = h(\sigma) + \tfrac{1}{2}\left[\bar{h}(s) - \sum h(P_i)\right] + \sum (\theta_i/2\pi)h(P_i). \tag{5''}$$

Accordingly, when stated completely, $\lim_{r\to a} J_r$ is given as an additive function of curves s in the domain I rather than as a function of point sets. In accordance with (5″) the function of curves may be spoken of as having regular discontinuities, and in this case it is determined uniquely if it is known merely for curves s for which $\bar{h}(s)=0$.

Given then a function $u(M)$ of class (i), (ii) such that

$$\lim_{r\to a}\int_{\sigma_r}(u/2 + r\,du/dr)d\sigma_r$$

is given as $h(\sigma)$, in the sense of (5″), where $h(e)$ is a completely additive function of sets on $C-F$, and σ is contained with its boundary in any domain of $C-F$, the function

$$w(M) = u(M) - U(M)$$

will still be of class (i). By harmonic extension across the domains I_k of $C-F$, where $[w/2+r\,dw/dr]_{r=a}\equiv 0$, $w(M)$ becomes harmonic and of class (i) in the infinite domain complementary to F. Since $w(M') = (r/a)w(M)$, the function w vanishes at ∞. It is therefore representable by Dirichlet data on F, according to the methods of III, §5, in terms of a completely additive set function on F.

2. **The Pleateau problem.** The connection of the Plateau problem with harmonic functions and the Dirichlet problem has been well known since Weierstrass. In the simplest statement of the problem—which does not correspond with the simplest solution—it is to find a surface $z\ f(x, y)$, which satisfies the partial differential equation obtained by setting the mean curvature equal to zero and which, on the boundary of a given region in the x, y-plane takes on given values of z. The vanishing of the mean curvature results from the Euler equation associated with minimizing the area of the surface $z=f(x, y)$ which caps the given boundary curve. In a more general statement of the Plateau problem, it is to find the expression of a surface of zero mean curvature (a minimal surface)

which caps a given space curve, in terms of two curvilinear coordinates, that is, to map it on a plane, but not by direct projection. The most important question is the relation of the minimal surface to the surface of least area.

The problem has been solved and the fundamental questions have been answered by Douglas and Radó, throughout the same recent period of years. For these treatments we refer to a book by Radó [31]. The relation to harmonic functions and Dirichlet's principle has been most clearly brought out in a recent paper by Courant [5]. Accordingly we shall present briefly an introduction to the concepts developed there.

The area of a surface of sufficient smoothness may be written as the integral in terms of curvilinear coordinates u, v:

$$A = \int (EG - F^2)^{1/2} du\, dv,$$

where E, F, G are the customary expressions $E = x_u^2 + y_u^2 + z_u^2$, $F = x_u x_v + y_u y_v + z_u z_v$, $G = x_v^2 + y_v^2 + z_v^2$. But

$$(EG - F^2)^{1/2} \leqq (EG)^{1/2},$$

and, by a familiar property of the geometric and arithmetic means,

$$(EG)^{1/2} \leqq \tfrac{1}{2}(E + G).$$

Accordingly, Courant considers the minimum of $(1/2)\int(E+G)du\, dv$ and finds that under suitable conditions this minimum exists and yields the by-product $F=0$, $E=G$, with x, y, z harmonic functions of u, v. In considering the minimal sequences equicontinuity takes the place of the weak convergence of mass functions.

We may regard the surface as represented on the plane by means of a vector $\rho(u, v)$, where $\rho(u, v) = \{x(u, v), y(u, v), z(u, v)\}$, and discuss the integral

$$D(\rho) = \frac{1}{2}\int (E + G) du\, dv = \frac{1}{2}\int (\rho_u \cdot \rho_u + \rho_v \cdot \rho_v) du\, dv.$$

In particular, if the corresponding portion of the u, v-plane is a simply covered simply-connected domain, that domain may be taken as the unit circle, for the differential expressions

$$(x_u^2 + x_v^2)(du\, dv),\ (y_u^2 + y_v^2)(du\, dv),\ (z_u^2 + z_v^2)(du\, dv)$$

are seen by direct calculation to be invariant of conformal transformation of the u, v parameters.

Consider a simple closed space curve γ and suppose that there exists a surface S which it bounds, with the following properties:

The vector $\rho(u, v)$ maps S, bounded by γ, on the unit circle B of the u, v-plane with boundary C, so that $\rho(u, v)$ is continuous for (u, v) in $B+C$, has "piecewise" continuous first derivatives in B, with C mapped continuously and monotonically on γ. Further it is supposed that the Dirichlet integral $D(\rho)$ is finite.

Courant's first problem is to show the existence of a vector $\rho(u, v)$ of this kind for which $D(\rho)$ attains its lowest bound. In particular, since $D(\rho)$ is invariant of a conformal map of the circle into itself, it may be assumed that three prescribed points of C go into three prescribed points of γ.

If there are an infinite number of possible surfaces S, let d be the lower bound of $D(\rho)$, and $\rho_1, \rho_2, \cdots$ a sequence of vectors ρ_n such that $D(\rho_n)$ tends to d. Courant shows that the three point condition just mentioned implies that the sequence of functions ρ_n is equicontinuous on C, and that therefore there exists a suitable subsequence, which we denote again by ρ_n, that converges uniformly on C.

We shall have to omit the development, itself an important chapter in connection with the Dirichlet problem, which deals with the minimum value of the Dirichlet integral over a given domain T for functions which take on assigned boundary values. This excursion into the calculus of variations would carry us too far afield.

3. **Minimum Dirichlet integral.** With the advantage of knowing already under what conditions a solution of the Dirichlet problem exists, it is not difficult, however, to establish the minimum property of the corresponding Dirichlet integral. To this effect we give briefly the proof of the following theorem, the statement and proof being extensible to any number of dimensions:

Given a bounded domain T in the u, v-plane, and a continuous function $f(P)$ on its boundary, which can be continuously extended over T in such a way that within T it has piecewise continuous derivatives and $\int_T(f_u^2+f_v^2)du\,dv$ exists, let $X(u, v)$ be the corresponding sequence solution of the Dirichlet problem for the given boundary values $f(P)$. Then the quantity $D(X)=\int_T(X_u^2+X_v^2)du\,dv$ exists, and we have

$$D(X) \leqq D(f). \tag{D}$$

As a first case, we assume that the boundary t is composed of a finite number of closed analytic curves and that $f(P)$ is continuous with continuous first and second derivatives in a region T' which contains $T+t$. Then, even under less restrictive conditions, it has been shown by Kellogg [18′] that the corresponding solution $X(u, v)$ of the Dirichlet problem in T for $f(P)$ on t has bounded first derivatives in $T+t$.

Let $W=f-X$. We have for the Dirichlet integrals the relation

$$D(f) = D(X) + 2D(W, X) + D(W),$$

where

$$D(W, X) = \int_T (W_u X_u + W_v X_v) du\, dv$$

$$= \int_t W \frac{dX}{dn} ds - \int_T W \nabla^2 X\, du\, dv = 0.$$

Since $D(W) \geqq 0$, we have $D(f) \geqq D(X)$.

Return now to the theorem. We may suppose that $f(P)$ is extended continuously outside T. We replace $f(P)$ by its twice iterated average $f_h(P)$ over squares of center P and side $2h$. Within any subdomain none of whose points are as close to t as $2h2^{1/2}$, the function $f_h(P)$ has continuous first and second partial derivatives. Let T_h represent a domain, contained with its boundary in such a subdomain, whose boundary consists of a finite number of closed analytic pieces, and is such that $\lim_{h\to 0} T_h = T$. In fact, by taking h small enough, T_h may be any given T_n of the sequence process. Let X_h be the solution of the Dirichlet problem in T_h for $f_h(P)$ on t_h.

With an obvious extension of the notation, we have from the result established in the first case,

$$D_{T_h}(X_h) \leqq \mathrm{D}_{T_h}(f_h).$$

But also, interchange of the order of differentiation and averaging is permissible, that is,

$$(f_u)_h = \left(\frac{\partial f}{\partial u}\right)_h = \frac{\partial f_h}{\partial u} = (f_h)_u,$$

which is one of the advantages of the systematic use of the process [2′], and moreover it is an immediate application of Hölder's inequality, as used for instance in the theory of area [25′, p. 687], that

$$\int_{Th} (f_u)_h^2\, du\, dv \leqq \int_T (f_u)^2 du\, dv.$$

Hence

$$D_{T_h}(f_h) \leqq D_T(f); \qquad D_{T_h}(X_h) \leqq D_T(f).$$

If now we let $h \to 0$, the function X_h converges to the sequence solution $X(u, v)$ of the Dirichlet problem in T for $f(P)$ on t; in fact, $\lim_{h\to 0} f_h(P) = f(P)$ uniformly on $T + t$. But by Fatou's theorem, $D_T(X)$ exists and

$$\lim_{h\to 0} D_{T_h}(X_h) \geqq D_T(X).$$

This inequality, combined with the previous one, yields the inequality (D), which was to be proved.

4. **Minimal sequence of harmonic functions.** Returning to the analysis of Courant we denote by ρ_n^* the vectors whose components are harmonic in u, v but take on the same boundary values as those of $\rho_n(u, v)$. They are likewise admissible vectors. But as we have just seen,

$$D(\rho_n^*) \leqq D(\rho_n).$$

Hence the vectors $\rho_n^*(u, v)$ also constitute a minimal sequence. And since the functions x_n^*, y_n^*, z_n^* converge uniformly on C they converge uniformly to harmonic functions x, y, z in B.

By the Fatou theorem, since the integrands are positive,

$$d = \lim_{n\to\infty} D(\rho_n^*) \geqq D(\rho).$$

But also ρ is an admissible vector, since it is monotonic on C; hence $D(\rho) \geqq d$. It follows that $D(\rho) = d$, and a solution of the minimum problem is obtained.

It remains to find the complete description of the surface generated by $\rho(u, v)$, and for this the reader is referred to the article cited. The surface is shown to be a minimal surface by making use of variations $\delta\rho(u, v)$ of a special kind, following Radó; that is, $E = G$, $F = 0$. The mapping on the boundary C is shown to be strictly monotonic; in other words, an interval on C cannot correspond to a single point on γ. From the fact that $E = G$, $F = 0$, it is evident that $D(\rho)$ is the area of the minimal surface; but the question as to whether $D(\rho)$ is the minimum area is not answered at once. This question requires consideration of approximation theorems in terms, say, of the mapping of polyhedra inscribed in the curve γ. It should be mentioned that surfaces of a prescribed degree of connectivity are also treated simply, through the same minimum problem. For example, the surface caps $\rho_n(u, v)$ may be one-sided surfaces, so that the points of them may be considered as counting doubly.

The Plateau problem, as is now well known, may be handled for curves such that there is no finite area corresponding to them. The solution, of course, depends upon approximation theorems.

A natural extension of the Dirichlet and Plateau problems lies in the consideration of the corresponding boundary value problems and variational problems connected with partial differential equations of elliptic type. It is with regret that the author realizes that limitations on the length of this article preclude the exposition of very recent and most significant work in this direction.

Bibliography

1. Bouligand. *Sur le problème de Dirichlet.* Annales de la Société polonaise de Mathématique, vol. 4 (1925), pp. 59–112.

2. Bouligand. *Fonctions Harmoniques. Principes de Picard et de Dirichlet.* Mémorial des Sciences Mathématiques, no. 11, 1926.

2′. BRAY. *Proof of a formula for an area.* Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 264–270.

3. CARATHÉODORY. *Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis.* Mathematische Annalen, vol. 73 (1913), pp. 305–370.

4. CARATHÉODORY. *On Dirichlet's problem.* American Journal of Mathematics, vol. 59 (1937), pp. 709–730.

5. COURANT. *Plateau's problem and Dirichlet's principle.* Annals of Mathematics, (2), vol. 38 (1937), pp. 679–724.

6. EVANS. *Fundamental points of potential theory.* Rice Institute Pamphlet, vol. 7 (1920), pp. 252–329.

7. EVANS. *The Logarithmic Potential,* American Mathematical Society Colloquium Publications, vol. 6. New York, 1927.

8. EVANS. *Application of Poincaré's sweeping-out process.* Proceedings of the National Academy of Sciences, vol. 19 (1933), pp. 457–461.

9. EVANS. *Short proof of Kellogg's lemma.* Bulletin of the American Mathematical Society, vol. 40 (1934), p. 665. Also given by VASILESCO. *Sur la continuite du potentiel a travers les masses, et la démonstration d'un lemme de Kellogg.* Comptes Rendus Hebdomaires de l'Académie des Sciences de Paris, vol. 200 (1935), pp. 1173–1174.

10. EVANS. *Potentials of positive mass.* I. Transactions of the American Mathematical Society, vol. 37 (1935), pp. 226–253. II. ibid., vol. 38 (1935), pp. 201–236.

11. EVANS. *Modern methods of analysis in potential theory.* Bulletin of the American Mathematical Society, vol. 43, (1937), pp. 481–502.

12. FATOU. *Séries trigonométriques et séries de Taylor.* Acta Mathematica, vol. 30 (1906), pp. 335–400.

13. FROSTMAN. *Potential d'Équilibre et Capacité des Ensembles.* Lund, 1935.

14. GARRETT. *Necessary and sufficient conditions for potentials of single and double layers.* American Journal of Mathematics, vol. 58 (1936), pp. 95–129.

15. GREEN. *Harmonic functions in domains with multiple boundary points.* American Journal of Mathematics, in cou-se of publication.

16. KELLOGG. *On the general solution of the classical Dirichlet problem.* Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 397–406.

17. KELLOGG. *Recent progress with the Dirichlet problem.* Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 601–625.

18. KELLOGG. *Foundations of Potential Theory.* Berlin, 1929.

18′. KELLOGG. *On the derivatives of harmonic functions on the boundary.* Transactions of the American Mathematical Society, vol. 33 (1931), pp. 486–510.

19. KELLOGG. *Converses of Gauss's theorem on the arithmetic mean.* Transactions of the American Mathematical Society, vol. 36 (1934), pp. 227–242.

20. LEBESGUE. *Sur le problème de Dirichlet.* Rendiconti del Circolo Matematico di Palermo, vol. 24 (1907), pp. 371–402.

21. LEBESGUE. *Sur le problème de Dirichlet.* Comptes Rendus Hebdomaires de l'Académie des Sciences de Paris, vol. 154 (1912), p. 335.

22. LEBESGUE. *Sur des cas d'impossibilité du problème de Dirichlet.* Comptes Rendus des Séances de la Société Mathématique de France, (1913), p. 17.

23. LEBESGUE. *Conditions de regularité, conditions d'irrégularité, conditions d'impossibilité dans le problème de Dirichlet.* Comptes Rendus Hebdomaires de l'Académie des Sciences de Paris, vol. 178 (1924), p. 349.

24. MARIA. *Examples of harmonic functions.* Bulletin of the American Mathematical Society, vol. 38 (1932), pp. 839–843.

25. MARIA and MARTIN. *Representation of positive harmonic functions.* Duke Mathematical Journal, vol. 2 (1936), pp. 517–529.

25′. MORREY. *A class of representation of manifolds,* Part I. American Journal of Mathematics, vol. 55 (1933), pp 683–707.

26. PERRON. *Eine neue Behandlung der ersten Randwertaufgabe für* $\Delta u=0$. Mathematische Zeitschrift, vol. 18 (1923), pp. 42–54.

27. PERKINS. *The Dirichlet problem for domains with multiple boundary points.* Transactions of the American Mathematical Society, vol. 58 (1925), pp. 106–144.

28. PHILLIPS and WIENER. *Nets and the Dirichlet problem.* Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 2 (1923), pp. 105–124.

29. PLEMELJ. *Potentialtheoretische Untersuchungen.* Leipzig, 1911.

30. POINCARÉ. *Sur les équations aux derivées partielles de la physique mathématique.* American Journal of Mathematics, vol. 12 (1890), pp. 211–294.

31. RADÓ. *On the Problem of Plateau.* Berlin, 1933.

32. RADÓ and F. RIESZ. *Über die erste Randwertaufgabe fur* $\Delta u=0$. Mathematische Zeitschrift, vol. 22 (1925), pp. 41–44.

33. RADON. *Über die Randwertaufgaben beim logarithmischen Potential.* Wiener Sitzungsberichte, vol. 128 (1919), pp. 1123–1167.

34. RAYNOR. *Dirichlet's problem.* Annals of Mathematics, vol. 23 (1922), pp. 183–198.

35. DE LA VALLÉE POUSSIN. *Extension de la méthode du balayage de Poincaré et problème de Dirichlet.* Annales de l'Institut H. Poincaré, vol. 2 (1932), pp. 169–232.

36. DE LA VALLÉE POUSSIN. *Potential et problème géneralisé de Dirichlet.* Mathematical Gazette, vol. 22 (1938), pp. 17–36.

37. VASILESCO. *Sur les singularités des fonctions harmoniques.* Journal de Mathématiques Pures et Appliquées, vol. 9 (1930), pp. 81–111.

38. VASILESCO. *Le problème généralisé de Dirichlet.* Académie royale de Belgique, Classe des Sciences, Mémoires, vol. 16 (1937), pp. 1–55.

39. WIENER. *Discontinuous boundary conditions and the Dirichlet problem.* Transactions of the American Mathematical Society, vol. 25 (1923), pp. 307–314.

40. WIENER. *Certain notions in potential theory.* Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 3 (1924), pp. 24–51.

41. WIENER. *The Dirichlet problem.* Ibid., vol. 3 (1924), pp. 127–146.

42. ZAREMBA. *Sur le principe du minimum.* Bulletin Internationale de l'Académie Polonaise des Sciences de Cracow, II, (1909), pp. 197–264.

UNIVERSITY OF CALIFORNIA,
BERKELEY, CALIF.