HYDRODYNAMICAL STABILITY

BY

J. L. SYNGE

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INTRODUCTION

The preparation for the Semicentennial of an address on the applications of mathematics has involved a difficult decision. Applied mathematics is so vast a subject that anything in the nature of a general review was quite out of the question. It seemed wiser to take some comparatively small field, already well formulated mathematically but offering problems still unsolved, and present it with some degree of completeness. Furthermore, it seemed best to take a subject belonging definitely to applied mathematics, that is, a subject in which any results we can obtain are of interest not only to the mathematician but also to the physicist and the engineer.

Hydrodynamical stability is such a subject. It is concerned with the initial stage of turbulence—its generation from steady flow—but not with turbulent motion, once established. It presents mathematical problems of no small difficulty: triumphs are few and disappointments many. Had a greater fraction of the mathematical energy of the last half-century been directed to these problems, no doubt our present knowledge of the behavior of fluids would be much greater. It is not too late to recommend these problems to the attention of mathematicians, especially those who derive satisfaction from the thought that their work binds them to a wider brotherhood of scientists.

It is not enough to present these problems in their final reduced form. It is essential that the background should be swiftly surveyed. There are three stages in any theory in applied mathematics: (i) creation of a mathematical model, or, equivalently, the formulation of axioms or laws; (ii) mathematical deduction of the behavior of the model; (iii) comparison of these deductions with observation. It is with the second stage that the mathematician is mainly concerned. But he must never lose sight of the other stages. He must always be ready to offer a modification of the fundamental laws if his deductions fail to fit observation. Hence, we must cast a critical glance over the fundamental equations of hydrodynamics in order to appreciate how far they involve fundamental mechanical laws (not to be lightly tampered with) and how far they involve convenient, but not inevitable, assumptions. The necessity for this survey and the desirability of having at hand the essential formulas in convenient notation will explain why so much space is devoted to these matters before coming to grips with the actual problems of stability.

The formulation of adequate axioms for the motion of a viscous fluid exercised the minds of Newton, D'Alembert, and Euler, but it was not until about a century ago that Navier and Stokes developed a successful mathematical model of a fluid. The reader may refer to an interesting account of the history of hydrodynamics by R. Giacomelli and E. Pistolesi [1, vol. 1, pp. 305–394]. There can never be a last word in regard to the axioms of any physical theory. All we can ask of them is that they lead to conclusions in agreement with observation. Sooner or later more refined observations will find the weak point in any set of physical axioms. Nature is far too complicated to be completely described in a few equations. But we may pertinently ask this question: Do the Navier-Stokes equations lead to deductions in obvious discord with observation? It might seem that this is an easy question to answer: actually it is difficult, and the fault lies with the mathematician rather than with the experimental physicist. Only in a few cases has the mathematician been able to make deductions from the Navier-Stokes equations. On account of the weakness of the methods available, the mathematician has tended to simplify the question of stability unduly, concentrating much attention on problems which do not admit a direct physical check. But on the whole we may say that the equations of Navier and Stokes have stood the test so far, their conspicuous triumph being in the work of G. I. Taylor [2] (see §8 below). On the other hand the work of R. von Mises [3, 4] and L. Hopf [5] (see §11 below) may make us doubtful as to the validity of these equations. Every investigation on hydrodynamical stability has a tang of excitement: the result obtained may confirm or undermine a theory now a century old.

The present address does not cover all the attacks that have been made on the problem of hydrodynamical stability: a more complete bibliography has been given by H. Bateman [6], and reference may also be made to a paper by F. Noether [31]. The aim has been rather to give a general conspectus of the subject with the maximum simplicity of presentation, and to direct attention to the discussion of disturbances more general than those usually treated. The rather weak but general and simple conditions for stability contained in §§6, 9, and 11 are believed to be new, and the essential connection between the methods of Part III and those of Part II does not seem to have been previously pointed out.

Part I. The physical problem and its mathematical formulation

1. **The observation of instability.** An historical account of experimental work on hydrodynamical stability has been given by L. Schiller [7]. Only some outstanding facts will be cited here. *Couette motion* is a steady motion with circular stream-lines of a fluid occupying the region between two rotating coaxial cylinders; *Poiseuille motion* is a steady motion with straight stream-lines through a fixed straight tube.* In each of these cases the steady motion is easily determined mathematically, and, under certain circumstances, there is good agreement between theory and observation. But under other circumstances the simple motion predicted theoretically is not observed at all or disappears on the slightest disturbance. This we explain by saying that such a motion, although possible, is *unstable.*

The stability of Couette motion has been investigated experimentally by G. I. Taylor [2] and J. W. Lewis [8]. It is found that, corresponding to any given speed of the outer cylinder, there is stability if the speed of the inner cylinder is small enough. When the speed of the inner cylinder is increased to a certain critical value, depending on the radii of the cylinders, the speed of the outer cylinder, the relative senses of rotation of the cylinders, and the kinematical viscosity of the fluid, the simple steady motion is replaced by an arrangement of annular vortices. On further increase of the speed of the inner cylinder, the motion becomes irregularly turbulent. Apparently the appearance of the vortices represents the incidence of instability: it is of some theoretical interest that the first step towards instability is the setting-up of a new steady motion. The critical speeds are shown graphically by Taylor and Lewis, the experimental results being in remarkable agreement with Taylor's mathematical work (see §8 below).

As regards Poiseuille motion, we shall refer here only to flow through a tube of circular section and make the briefest possible statement. Stability is found experimentally to depend on the value of the dimensionless *Reynolds number*†

* These definitions are more general than those sometimes employed.

† Care must be taken in examining results to see what definition of R is employed. Other definitions, differing by constant numerical factors, may be used.

$$R = U_m D/\nu, \tag{1.1}$$

where U_m is the mean velocity, D the diameter of the tube, and ν the kinematical viscosity. It is found [6, p. 336; 1, vol. 3, p. 178] that there is stability if $R<R_c$ and instability if $R>R_c$, where R_c is the *critical* Reynolds number: its value is

$$R_c = 2320. \tag{1.2}$$

On account of the mathematical difficulties involved in the theoretical treatment of the above problems, much attention has been devoted to the discussion of analogous plane problems (§11). Suitable experiments cannot be performed to test mathematical conclusions in these cases, but the problems are of mathematical interest and may be expected to throw light, by analogy, on the more complicated problems.

2. **Equations of motion of a viscous fluid.** Indicial notation will be used, in which Latin suffixes will have the range 1, 2, 3 and Greek suffixes the range 1, 2, with the usual summation convention for repeated suffixes.

For any continuous medium, let x_i be rectangular cartesian coördinates, t the time, ρ the density, u_i the components of velocity, X_i the components of body-force per unit mass, and E_{ij} the stress-tensor ($E_{ji}=E_{ij}$). According to the eulerian plan, x_i and t are the independent variables and the other quantities functions of them.

Application of Newton's laws of motion gives, as the *equations of motion of any continuous medium*,

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho X_i + \frac{\partial E_{ij}}{\partial x_j}; \tag{2.1}$$

with these is associated the *equation of continuity*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \tag{2.2}$$

(or equation of conservation of mass).

To complete the system of equations, a connection must be set up between the stress and the motion. To this end the *rate-of-deformation tensor* is defined as

$$e_{ij} = \tfrac{1}{2}(\partial u_j/\partial x_i + \partial u_i/\partial x_j), \tag{2.3}$$

and the *pressure* p is defined as the invariant

$$p = - E_{kk}/3. \tag{2.4}$$

In a *perfect* or *inviscid* fluid the stress across any plane is normal, from which it follows that in a perfect fluid

$$E_{ij} + p\delta_{ij} = 0 \tag{2.5}$$

where δ_{ij} is the Kronecker delta,

$$\delta_{ij} = 1 \text{ if } i = j; \qquad \delta_{ij} = 0 \text{ if } i \neq j. \tag{2.6}$$

As *stress-deformation hypothesis for a viscous fluid*, we assume that the quantities $E_{ij}+p\delta_{ij}$ are linear homogeneous functions of the quantities e_{ij}, or, formally,

$$E_{ij} + p\delta_{ij} = \mu_{ijkl}e_{kl}, \tag{2.7}$$

where μ_{ijkl} are the components of the *viscosity-tensor*, satisfying the symmetry conditions

$$\mu_{ijkl} = \mu_{jikl} = \mu_{ijlk}. \tag{2.8}$$

We shall assume that the fluid is *isotropic*: then μ_{ijkl} is an isotropic tensor, and hence [9, p. 70]

$$\mu_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}), \tag{2.9}$$

where λ, μ are invariants. Then (2.7) reads

$$E_{ij} + p\delta_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}, \tag{2.10}$$

where $\theta = e_{kk} = \partial u_k/\partial x_k$, the expansion of the fluid. Contracting (2.10) with $j=i$ and using (2.4), we find $3\lambda+2\mu=0$, and we have from (2.10) as *stress-deformation relation in an isotropic fluid* [10, p. 574]

$$E_{ij} = -p\delta_{ij} - 2\mu\theta\delta_{ij}/3 + 2\mu e_{ij}; \tag{2.11}$$

μ is the *coefficient of viscosity*. We shall regard μ as a constant, although actually it depends on temperature.

The daring simplicity of (2.7)—its linear character—should be noticed. In a similar situation in the theory of elasticity, it is admitted that the linear stress-strain relation (generalized Hooke's law) is physically valid only for small deformations.

Substituting from (2.11) in (2.1) and associating (2.2), we have as *equations of motion of an isotropic viscous fluid* [10, p. 577]

$$\begin{aligned} &\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = X_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\nu}{3}\frac{\partial \theta}{\partial x_i} + \nu\Delta u_i, \\ &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0, \end{aligned} \tag{2.12}$$

where ν is the *kinematical viscosity* ($\nu = \mu/\rho$) and Δ is the Laplacian operator ($\Delta = \partial^2/\partial x_j \partial x_j$).

We have in (2.12) four equations for five unknowns, ρ, u_i, p, the body-force X_i being supposed assigned. Another equation, such as a relation between ρ and p, is necessary to make the problem of the motion of a fluid mathematically definite. We shall assume the fluid *homogeneous* and *incompressible*, so that ρ is a constant. We have then for the four unknowns u_i, p the four *Navier-Stokes equations of motion of an isotropic homogeneous incompressible fluid* [10, p. 577]

$$(2.13) \qquad \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = X_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta u_i, \qquad \frac{\partial u_j}{\partial x_j} = 0.$$

These are the basic equations to be employed in the discussion of hydrodynamical stability. They have been derived in the above manner in order to show the various hypotheses underlying them. Each simplifying hypothesis represents a possible source of discrepancy between the model fluid and the physical fluid. Indeed, L. V. King [11] has suggested that compressibility may be of importance in the discussion of stability: in that case we should use (2.12) instead of (2.13). We shall, however, follow the usual course and use (2.13).

The *boundary conditions* to be associated with the equations of motion in any form are as follows. *Across a surface separating two fluids or a fluid and a solid, the three components of velocity* and the three components of stress across the surface of separation are continuous.* In symbols, u_i and $E_{ij}n_j$ are continuous, n_j being the direction cosines of the normal to the surface.

As we shall deal only with a single fluid bounded by rigid walls, in which the stress may be supposed adjusted to satisfy the condition of continuity, we shall be concerned only with continuity of velocity.

The equations (2.13) are given for rectangular cartesian coördinates. To pass systematically to any curvilinear coördinates, we may use the methods of tensor calculus [13, chap. 20]. Thus, if the coördinates are x^i and the line element

$$ds^2 = g_{ij}dx^i dx^j,$$

the Navier-Stokes equations (2.13) read

$$(2.14) \qquad \frac{\partial u^i}{\partial t} + u^j D_j u^i = X^i - \frac{1}{\rho} D^i p + \nu D^j D_j u^i, \; D_j u^j = 0,$$

where $u^i = dx^i/dt$, X^i are the contravariant components of body-force, D_i the operation of covariant differentiation, and $D^i = g^{ij}D_j$.

The only curvilinear coördinates we shall require are the cylindrical coordinates r, ϕ, z, and for them the equations of motion are most easily obtained by the use of complex variables [14, p. 371]. Let u, v, w be the components of velocity in the directions of the parametric lines of r, ϕ, z

* Thus we exclude the possibility of a fluid slipping on a solid boundary; cf. Brillouin [12, vol. 1, pp. 42 ff.].

respectively and R, Φ, Z the components of body-force in these directions. Then with

$$x_1 = r\cos\phi, \qquad x_2 = r\sin\phi, \qquad x_3 = z,$$

we have

$$(2.15)\qquad \begin{aligned} u_1 + iu_2 &= e^{\phi i}(u+iv), & u_3 &= w, \\ X_1 + iX_2 &= e^{\phi i}(R+i\Phi), & X_3 &= Z, \\ \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} &= e^{\phi i}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \phi}\right), & \frac{\partial}{\partial x_3} &= \frac{\partial}{\partial z}, \end{aligned}$$

and we obtain at once from (2.13) the *Navier-Stokes equations in cylindrical coördinates*

$$(2.16)\qquad \begin{gathered} e^{-\phi i}D_t\{e^{\phi i}(u+iv)\} = R + i\Phi - \frac{1}{\rho}\left(\frac{\partial p}{\partial r} + \frac{i}{r}\frac{\partial p}{\partial \phi}\right) + \nu e^{-\phi i}\Delta\{e^{\phi i}(u+iv)\}, \\ D_t w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta w, \\ \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0; \end{gathered}$$

here $D_t = \partial/\partial t + u_j\partial/\partial x_j$, the operator of differentiation following the fluid, so that

$$(2.17)\qquad D_t = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial z},$$

and Δ is the Laplacian operator

$$(2.18)\qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\cdot$$

A motion will be said to possess *rotational symmetry* if the body-forces possess a potential Π independent of ϕ, and if u, v, w, p are independent of ϕ. The last of (2.16) then gives

$$(2.19)\qquad \frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0,$$

so that a function $\psi(r, z, t)$ exists such that

$$(2.20)\qquad u = -\frac{1}{r}\frac{\partial}{\partial z}(r\psi), \qquad w = \frac{1}{r}\frac{\partial}{\partial r}(r\psi).$$

We shall refer to ψ as a *stream-function*. For the body-forces we have

$$R = -\partial\Pi/\partial r, \qquad \Phi = 0, \qquad Z = -\partial\Pi/\partial z,$$

and $\Pi + p/\rho$ may be eliminated from the first two of (2.16)—actually three real equations—to yield the *equations of motion in the case of rotational symmetry*

$$(2.21)\qquad \begin{aligned} \left(D_t - \frac{u}{r} - \nu\Theta\right)\Theta\psi + \frac{2v}{r}\frac{\partial v}{\partial z} &= 0, \\ \left(D_t + \frac{u}{r} - \nu\Theta\right)v &= 0, \end{aligned}$$

where

$$(2.22)\qquad \begin{aligned} D_t &= \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}, \\ \Theta &= \Delta - \frac{1}{r^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}\,. \end{aligned}$$

Axial symmetry is a particular case of rotational symmetry, with the additional condition $v=0$, so that the velocity-vectors intersect the z-axis. In this case we have only one equation,

$$(2.23)\qquad (D_t - u/r - \nu\Theta)\Theta\psi = 0.$$

In *plane* or *two-dimensional motion*, in which a plane $x_3 = \text{const.}$ is taken as the plane of motion, the Navier-Stokes equations (2.13) take the form

$$(2.24)\qquad \begin{aligned} \frac{\partial u_\alpha}{\partial t} + u_\beta\frac{\partial u_\alpha}{\partial x_\beta} &= X_\alpha - \frac{1}{\rho}\frac{\partial p}{\partial x_\alpha} + \nu\Delta' u_\alpha, \\ \partial u_\beta/\partial x_\beta = 0, &\qquad \Delta' = \partial^2/\partial x_\beta\partial x_\beta, \end{aligned}$$

where u_α, p, X_α are functions of x_1, x_2, t. (We recall that Creek suffixes have the range 1, 2.) The equation of continuity implies the existence of a stream-function ψ such that

$$(2.25)\qquad u_1 = -\,\partial\psi/\partial x_2, \qquad u_2 = \partial\psi/\partial x_1;$$

the vorticity is

$$(2.26)\qquad \xi = \tfrac{1}{2}(\partial u_2/\partial x_1 - \partial u_1/\partial x_2) = \tfrac{1}{2}\Delta'\psi.$$

Assuming the body-forces conservative, we eliminate them and p from (2.24), obtaining as the single *Navier-Stokes equation for plane motion*,

$$(2.27)\qquad D_t\Delta'\psi = \nu\Delta'\Delta'\psi,$$

where

$$(2.28)\qquad D_t = \frac{\partial}{\partial t} + u_\beta\frac{\partial}{\partial x_\beta} = \frac{\partial}{\partial t} - \frac{\partial\psi}{\partial x_2}\frac{\partial}{\partial x_1} + \frac{\partial\psi}{\partial x_1}\frac{\partial}{\partial x_2},$$

or, equivalently,

$$D_t\xi = \nu\Delta'\xi. \tag{2.29}$$

3. **Some steady motions and their first-order equations of disturbance.** A *steady motion* is one in which velocity and pressure are functions of position only. Thus a steady motion $u_i = U_i(x_j)$, $p = P(x_j)$ must satisfy (2.13) in the form

$$U_j \frac{\partial U_i}{\partial x_j} = X_i - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\Delta U_i, \qquad \frac{\partial U_j}{\partial x_j} = 0, \tag{3.1}$$

together with the boundary conditions. To study small disturbances of the steady motion, we seek solutions of (2.13) of the form

$$u_i = U_i + \epsilon u_i' + \epsilon^2 u_i'' + \cdots, \qquad p = P + \epsilon p' + \epsilon^2 p'' + \cdots, \tag{3.2}$$

where ϵ is a constant parameter and u_i', u_i'', $\cdots$, p', p'', $\cdots$ are functions of position and time. We demand that (3.2) shall satisfy (2.13) and the boundary conditions for all values of ϵ in a range $0 < \epsilon < \epsilon_1$. Formal substitution of (3.2) in (2.13) gives a sequence of sets of equations, each set corresponding to a definite power of ϵ and having boundary conditions associated with it. The set corresponding to ϵ^0 is (3.1); the set corresponding to ϵ^1 is

$$\frac{\partial u_i'}{\partial t} + U_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\Delta u_i', \qquad \frac{\partial u_j'}{\partial x_j} = 0, \tag{3.3}$$

with the boundary conditions $u_i' = 0$ on any rigid wall, fixed or moving in a prescribed manner, which bounds the fluid. The equations (3.3) may be called the *first-order equations of disturbance.*

A complete treatment would require consideration not only of (3.3), but also of the equations corresponding to higher powers of ϵ, together with establishment of the convergence of (3.2) and justification of the term-by-term differentiations. It is customary, however, to confine attention to the first-order equations, which, it is reasonable to suppose, determine the behavior of disturbances initially very small. The discussion of these equations is certainly a necessary preliminary to a more complete discussion.

We shall now discuss some familiar steady motions and their first-order equations of disturbance. Body-forces are assumed absent throughout.

(a) **Couette motion.** Let fluid occupy the region between two coaxial circular cylinders of radii a_1, a_2, $(a_1 < a_2)$, which are rotating with constant angular velocities n_1, n_2 about their common axis. Using cylindrical coordinates r, ϕ, z in which the cylinders are $r = a_1$ and $r = a_2$, we see that (2.16) and the boundary conditions are satisfied by

$$u = 0, \qquad v = V, \qquad w = 0, \qquad p = P, \tag{3.4}$$

where

$$V = Ar + B/r, \qquad P = \rho \int (V^2/r) ar. \tag{3.5}$$

$$A = \frac{n_2 a_2^2 - n_1 a_1^2}{a_2^2 - a_1^2}, \qquad B = -\frac{a_2^2 a_1^2 (n_2 - n_1)}{a_2^2 - a_1^2}.$$

This is *Couette motion*. We note that the ψ of (2.20) is zero.

For a general disturbance of the Couette motion, we write

$$u = \epsilon u' + \cdots, \qquad v = V + \epsilon v' + \cdots, \tag{3.6}$$

$$w = \epsilon w' + \cdots, \qquad p = P + \epsilon p' + \cdots,$$

and obtain from (2.16) the first-order equations of disturbance. We shall not write them here, as we find it more convenient in §6 to appeal directly to (3.3) when the disturbance is general.

For a disturbance possessing *rotational symmetry* we may use (2.21), substituting

$$\psi = \epsilon\psi' + \cdots, \quad v = V + \epsilon v' + \cdots, \quad u = -\frac{\partial\psi}{\partial z}, \quad w = \frac{1}{r}\frac{\partial}{\partial r}(r\psi). \tag{3.7}$$

Hence we obtain the first-order equations of disturbance

$$\left(\frac{\partial}{\partial t} - \nu\Theta\right)\Theta\psi' = -2(A + B/r^2)\frac{\partial v'}{\partial z},$$

$$\left(\frac{\partial}{\partial t} - \nu\Theta\right) v' = 2A\frac{\partial \psi'}{\partial z}, \tag{3.8}$$

$$\Theta = \Delta - \frac{1}{r^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2},$$

with the boundary conditions

$$\frac{\partial\psi'}{\partial z} = \frac{\partial}{\partial r}(r\psi') = v' = 0 \quad \text{for} \quad r = a_1 \quad \text{and for} \quad r = a_2. \tag{3.9}$$

(b) **Poiseuille motion.** Let fluid fill a fixed cylindrical tube of arbitrary section, the axis of x_3 being parallel to the generators. Let C be the boundary of the normal section. We see that (2.13) and the boundary conditions are satisfied by

$$u_\alpha = 0, \qquad u_3 = U_3, \qquad p = P, \tag{3.10}$$

where

$$U_3 = \tfrac{1}{2} A\Phi, \qquad P = -\mu A x_3 + B, \tag{3.11}$$

A and B being any constants, and Φ a function of x_1, x_2 satisfying

$$\Delta'\Phi = -2, \qquad \Delta' = \partial^2/\partial x_\beta \partial x_\beta, \tag{3.12}$$

with the boundary condition $\Phi=0$ on C.* This is *Poiseuille motion.*

For a disturbance of the steady motion, we write

$$(3.13)\quad u_\alpha = \epsilon u_\alpha' + \cdots, \quad u_3 = U_3 + \epsilon u_3' + \cdots, \quad p = P + \epsilon p' + \cdots,$$

and obtain from (3.3) the first-order equations of disturbance

$$(3.14)\quad \begin{aligned} &\frac{\partial u_i'}{\partial t} + U_3 \frac{\partial u_i'}{\partial x_3} + \delta_{i3} u_\beta' \frac{\partial U_3}{\partial x_\beta} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \Delta u_i', \\ &\partial u_j'/\partial x_j = 0, \qquad \Delta = \partial^2/\partial x_j \partial x_j, \end{aligned}$$

with the boundary conditions $u_i'=0$ on C.

For the more familiar Poiseuille motion, in which C is a circle, it is convenient to use cylindrical coördinates r, ϕ, z, the equation of the cylinder being $r=a$. Then (2.16) and the boundary conditions are satisfied by the steady motion

$$(3.15)\quad u = v = 0, \qquad w = W, \qquad p = P,$$

where

$$(3.16)\quad W = W_0(1 - r^2/a^2), \qquad P = -4\mu W_0 z/a^2 + \text{const.},$$

W_0 being a constant, the velocity at the center of the tube. We note that the stream-function ψ of (2.20) is $\psi=\Psi$ where

$$(3.17)\quad \Psi = aW_0(\tfrac{1}{2}r/a - \tfrac{1}{4}r^3/a^3).$$

On account of the singularity in the coördinate system for $r=0$, cylindrical coördinates do not seem particularly useful for the discussion of general disturbances. For disturbances with *rotational symmetry* in a tube of circular section, we substitute in (2.21)

$$(3.18)\quad \begin{aligned} &\psi = \Psi + \epsilon\psi' + \cdots, \qquad v = \epsilon v' + \cdots, \\ &u = -\partial\psi/\partial z, \qquad w = (1/r)\partial(r\psi)/\partial r, \end{aligned}$$

and obtain the first-order equations of disturbance

$$(3.19)\quad \begin{aligned} &\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z} - \nu\Theta\right)\Theta\psi' = 0, \\ &\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z} - \nu\Theta\right)v' = 0, \\ &\Theta = \Delta - \frac{1}{r^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}, \end{aligned}$$

* The problem of finding Φ is identical with the problem of finding the irrotational motion of a perfect fluid inside a rotating cylinder and with the torsion problem for an elastic cylinder.

with the boundary conditions

$$\frac{\partial\psi'}{\partial z} = \frac{\partial}{\partial r}(r\psi') = v' = 0 \quad \text{for} \quad r = a, \tag{3.20}$$
$$\frac{\partial\psi'}{\partial z} = \psi' = v' = 0 \qquad \text{for} \quad r = 0.$$

For disturbances with *axial symmetry*, we put $v'=0$ and obtain the single first-order equation of disturbance

$$\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z} - \nu\Theta\right)\Theta\psi' = 0, \tag{3.21}$$

Θ being as in (3.19) and the boundary conditions being as in (3.20).

(c) **Plane Couette motion and plane Poiseuille motion.** Let fluid fill the region between the parallel planes $x_2 = \pm h$.

Let these planes have constant velocities U_0, $-U_0$, respectively, in the direction of the x_1-axis. The equation of motion (2.27) and the boundary conditions are satisfied by the steady motion $\psi = \Psi$ where

$$\Psi = -\tfrac{1}{2}U_0 x_2^2/h. \tag{3.22}$$

This is *plane Couette motion*, or simple shearing motion. We shall refer to it as *P.C.M.*

Now let the planes $x_2 = \pm h$ be fixed. The equation of motion (2.27) and the boundary conditions are satisfied by the steady motion $\psi = \Psi$ where

$$\Psi = U_0 h(\tfrac{1}{3}x_2^3/h^3 - x_2/h), \tag{3.23}$$

where U_0 is a constant, the velocity at the center of the channel. This is *plane Poiseuille motion*, or pressure-flow. We shall refer to it as *P.P.M.*

Confining our attention to disturbances in the plane of the motion ($x_3 = \text{const.}$), we write, for the disturbance of any plane steady motion $\psi = \Psi$,

$$\psi = \Psi + \epsilon\psi' + \cdots, \tag{3.24}$$

and obtain from (2.27) the first-order equation of disturbance

$$\frac{\partial}{\partial t}\Delta'\psi' - \frac{\partial\Psi}{\partial x_2}\frac{\partial}{\partial x_1}\Delta'\psi' + \frac{\partial\Psi}{\partial x_1}\frac{\partial}{\partial x_2}\Delta'\psi' \tag{3.25}$$
$$- \frac{\partial\psi'}{\partial x_2}\frac{\partial}{\partial x_1}\Delta'\Psi + \frac{\partial\psi'}{\partial x_1}\frac{\partial}{\partial x_2}\Delta'\Psi = \nu\Delta'\Delta'\psi'.$$

We substitute from (3.22) and (3.23) in (3.25), at the same time introducing the dimensionless variables

$$x = x_1/h, \qquad y = x_2/h, \qquad \tau = \nu t/h^2, \tag{3.26}$$

and the Reynolds number, defined as

$$R = U_0 h/\nu. \tag{3.27}$$

Thus we obtain the first-order equations of disturbance

$$\left(\frac{\partial}{\partial \tau} + Ry\frac{\partial}{\partial x}\right)\Delta\psi' = \Delta\Delta\psi', \qquad P.C.M., \tag{3.28}$$

$$\left(\frac{\partial}{\partial \tau} + R(1 - y^2)\frac{\partial}{\partial x}\right)\Delta\psi' + 2R\frac{\partial\psi'}{\partial x} = \Delta\Delta\psi', \qquad P.P.M., \tag{3.29}$$

where

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2, \tag{3.30}$$

and the boundary conditions for both cases are

$$\partial\psi'/\partial x = \partial\psi'/\partial y = 0 \quad \text{for} \quad y = \pm 1. \tag{3.31}$$

4. **Formulation of the problem of stability.** In the case of a dynamical system with a finite number of degrees of freedom, a definition of stability of equilibrium or of steady motion is comparatively easy to give, because the motion is determined by the initial conditions—initial position and velocity. But in the case of a continuous medium, the determination of the motion requires a knowledge not only of the conditions at $t=0$, but also the boundary conditions for $t>0$. Some of these boundary conditions, namely, the velocity on fixed or moving walls, are assigned in the hydrodynamical problem. The rest of the boundary consists, in the physical problem, of those parts of the apparatus which the physicist does his best to render unimportant, namely, the ends of the cylinders in Couette motion and the ends of the tube in Poiseuille motion. In the mathematical problem we can do what the experimentalist cannot do—remove the ends to infinity. But that does not absolve us from considering boundary conditions at infinity; some such conditions must be furnished if the problem of fluid motion is to be definite.

It has been an almost universal custom to circumvent this difficulty of boundary conditions at infinity by limiting the discussion to disturbances which are *spatially periodic in the direction in which the fluid extends to infinity*, thus giving to the problem the required definite character. Strangely enough, this periodicity, introduced as a mathematical convenience, has apparently a physical reality, as shown in the experiments of Taylor [2] and Lewis [8].

The following is offered as a definition of hydrodynamical stability. Let R be a region occupied by a fluid in steady motion, specified by U_i, P, satisfying (3.1). Consider now the equations (3.3) to be satisfied by u_i', p'. With these equations we associate (i) initial conditions I for $t=0$, (ii) boundary conditions B for $t \geqq 0$. In B we include the condition of spatial periodicity in the infinite direction, and write the condition $B(\lambda)$, where λ is a parameter specifying the wave-length. (In the case of three-dimensional disturbances of plane Couette and Poiseuille motions, we would have two such parameters.) The conditions I are to be consistent with B for $t=0$ and with (3.3).

If *all* solutions u_i' of (3.3), satisfying I and $B(\lambda)$, are bounded for $t>0$, we say that the steady motion is "stable I, $B(\lambda)$"; but if there exists any solution u_i' of (3.3), satisfying I and $B(\lambda)$ and unbounded for $t>0$, we say that the steady motion is "unstable" (absolutely). If we so far weaken the conditions I that they demand nothing more than consistency with $B(\lambda)$ and (3.3) for $t=0$, and the continuity of the initial values of u_i', p', $\partial u_i'/\partial x_j$, then stability may be discussed with reference to $B(\lambda)$ alone. If all such solutions u_i' of (3.3), satisfying $B(\lambda)$, are bounded for $t>0$, we say that the motion is "stable $B(\lambda)$," and if this holds for arbitrary λ, we say that the motion is "stable" (absolutely).

It is not claimed that the word "stable" is used below with quite as precise a meaning as that given above. Our knowledge is so scanty that there has been a natural tendency to concentrate on what appear to be the most important aspects of the stability problem. Thus for simplicity of expression, we shall in Part II refer to a motion as stable if there is no characteristic value with positive real part, not seeking to generalize or critically examine the expansion theorem of Haupt [15].

In Part III we shall be concerned with "stability in the mean," as there explained.

Part II. The method of the exponential time-factor

5. **Introductory remarks.** The method of the exponential time-factor (also called the method of small oscillations) has long been a classical method for the determination of the periods of vibration of continuous systems. It was applied to the problem of hydrodynamical stability (for inviscid liquids) as long ago as 1880 by Rayleigh [16]. In this method we seek solutions of the first-order equations of disturbance (3.3) of the form

$$(5.1) \qquad u_i' = e^{\sigma t}F_i(x_1, x_2, x_3), \qquad p' = e^{\sigma t}G(x_1, x_2, x_3),$$

the constant σ and the functions F_i, G being in general complex. Real velocity and pressure are found by adding to these expressions their complex conjugates. On introducing the spatial periodicity mentioned in §4, the differential system is homogeneous and inconsistent unless σ takes certain characteristic values. If the real parts of *all* the characteristic values of σ are zero or negative, then there is stability for disturbances for which initially $u_i'=F_i$, $p'=G$, but if there is *any* characteristic value of σ with a positive real part, there is instability. Since F_i are characteristic functions, the initial conditions in question are somewhat special and in order to establish stability for arbitrary initial disturbances, it is necessary to discuss the expansion of arbitrary functions in terms of the characteristic functions and the validity of applying to the expansions the required differential operations. This question will not be discussed. We shall for simplicity refer to a steady motion as "stable" if all the characteristic values of σ have zero or negative real parts. As far as instability is concerned, no such question is involved, and hence the method of the exponential time-factor is ideally suited to the establishment of instability. Unfortunately, as we shall see, the methods available are such as to establish conditions under which the real parts of all characteristic values σ are zero or negative much more readily than conditions under which the real part of at least one σ is positive.

In Part III we shall discuss a different approach to the question of stability by the method of decreasing positive-definite integrals, this latter type of stability being *stability in the mean.* The mathematical connection between the two methods is closer than appears to have been realized. Certain arguments, actually developed by the methods of Part III, may also be presented by the methods of Part II. We shall, for simplicity, treat such arguments in Part II and show in Part III how the two lines of argument coalesce.

The method of Reynolds [17] will not be used. His mean values present certain difficulties, and, while the theory of turbulence demands their use, it would seem that first-order stability can be discussed as adequately and more clearly without them.

6. **Couette motion with general disturbance.** Taking the axis of x_3 along the axis of the cylinders, the Couette motion (3.4) may be written $u_i = U_i$ where

$$U_\alpha = -\epsilon_{\alpha\beta}x_\beta(A + B/r^2), \qquad U_3 = 0, \qquad r^2 = x_\gamma x_\gamma, \tag{6.1}$$

where $\epsilon_{\alpha\beta}$ is the permutation symbol ($\epsilon_{11}=\epsilon_{22}=0$, $\epsilon_{12}=-\epsilon_{21}=1$) and A, B are constants as in (3.5). Let us seek solutions of (3.3) of the form

$$u_i' = f_i(x_1, x_2)e^{\sigma t+i\lambda x_3}, \qquad p' = g(x_1, x_2)e^{\sigma t+i\lambda x_3}, \tag{6.2}$$

the functions f_i, g and the constant σ being in general complex and the constant λ real, taken positive without loss of generality. Substitution from (6.2) in (3.3) and elimination of f_3 and g lead to the two partial differential equations for f_α,

$$\begin{aligned} \sigma\left(\frac{\partial\theta}{\partial x_\alpha} - \lambda^2 f_\alpha\right) &= \nu(\Delta' - \lambda^2)\left(\frac{\partial\theta}{\partial x_\alpha} - \lambda^2 f_\alpha\right) \\ &\quad + \lambda^2\left(U_\beta\frac{\partial f_\alpha}{\partial x_\beta} + f_\beta\frac{\partial U_\alpha}{\partial x_\beta}\right) - \frac{\partial}{\partial x_\alpha}\left(U_\beta\frac{\partial\theta}{\partial x_\beta}\right), \end{aligned} \tag{6.3}$$

where

$$\theta = \partial f_\beta/\partial x_\beta = -i\lambda f_3, \qquad \Delta' = \partial^2/\partial x_\rho\partial x_\rho. \tag{6.4}$$

The boundary conditions are

(6.5) $$f_\alpha = \theta = 0 \quad \text{for} \quad r = a_1 \quad \text{and for} \quad r = a_2.$$

The consistency of (6.3) and (6.5) demands the satisfaction of a characteristic equation by σ, λ, ν.

We shall now establish conditions under which the Couette motion is stable for general disturbances, in the sense that all characteristic values of σ have negative real parts. Let us multiply (6.3) by $\bar{f}_\alpha dS$ (the bar denoting a complex conjugate and dS an element of area) and integrate over the part of the plane $x_3 = \text{const.}$ occupied by the fluid. On integration by parts, this gives, by virtue of (6.5),

(6.6) $$\begin{aligned}\sigma(I_2^2 + \lambda^2 I_1^2) = &- \nu(I_3^2 + \lambda^2 I_2^2 + \lambda^2 J_2^2 + \lambda^4 I_1^2)\\ &- \lambda^2 \int (U_\beta \partial f_\alpha/\partial x_\beta + f_\beta \partial U_\alpha/\partial x_\beta)\bar{f}_\alpha dS - \int U_\beta(\partial\theta/\partial x_\beta)\bar{\theta} dS,\end{aligned}$$

where we have denoted certain positive-definite integrals as follows:

(6.7) $$\begin{aligned}I_1^2 &= \int f_\alpha \bar{f}_\alpha dS, & I_2^2 &= \int \theta\bar{\theta} dS,\\ J_2^2 &= \int \frac{\partial f_\alpha}{\partial x_\beta}\frac{\partial \bar{f}_\alpha}{\partial x_\beta} dS, & I_3^2 &= \int \frac{\partial \theta}{\partial x_\alpha}\frac{\partial \bar{\theta}}{\partial x_\alpha} dS.\end{aligned}$$

Let us write

(6.8) $$\sigma = \sigma_1 + i\sigma_2,$$

and add to (6.6) its complex conjugate. Since $\partial U_\beta/\partial x_\beta = 0$, we have

(6.9) $$\begin{aligned}\int U_\beta(\bar{f}_\alpha \partial f_\alpha/\partial x_\beta + f_\alpha \partial \bar{f}_\alpha/\partial x_\beta) dS &= -\int (\partial U_\beta/\partial x_\beta) f_\alpha \bar{f}_\alpha dS = 0,\\ \int U_\beta(\bar{\theta}\partial\theta/\partial x_\beta + \theta\partial\bar{\theta}/\partial x_\beta) dS &= -\int (\partial U_\beta/\partial x_\beta)\theta\bar{\theta} dS = 0,\end{aligned}$$

and so we obtain

(6.10) $$\begin{aligned}\sigma_1(I_2^2 + \lambda^2 I_1^2)/\nu = &- (I_3^2 + \lambda^2 I_2^2 + \lambda^2 J_2^2 + \lambda^4 I_1^2)\\ &- \tfrac{1}{2}(\lambda^2/\nu)\int (\partial U_\alpha/\partial x_\beta)(\bar{f}_\alpha f_\beta + f_\alpha \bar{f}_\beta) dS.\end{aligned}$$

The last term may also be written

(6.11) $$- \tfrac{1}{2}(\lambda^2/\nu)\int U_{\alpha\beta}(\bar{f}_\alpha f_\beta + f_\alpha \bar{f}_\beta) dS, \qquad U_{\alpha\beta} = \tfrac{1}{2}(\partial U_\beta/\partial x_\alpha + \partial U_\alpha/\partial x_\beta),$$

$U_{\alpha\beta}$ being in fact the rate-of-deformation tensor for the steady motion.

We may note, in passing, the application of (6.10) to another problem. Consider a cylinder of any section (simply or multiply connected) rotating with constant angular velocity about an axis parallel to the generators. We know that a rigid-body rotation of fluid contained in the cylinder is a possible steady motion: this motion is given by (6.1) with $B=0$. For the disturbance of this steady motion, (6.10) holds. But $U_{\alpha\beta}=0$ in a rigid-body motion: hence $\sigma_1<0$ and the motion is stable for all values of ν and λ.

Returning to the case of Couette motion, we shall give certain inequalities satisfied by complex functions f_α, arbitrary save for the boundary conditions (6.5), θ being defined by (6.4). For any real constant χ,

$$\int (U_\alpha\theta + \chi f_\alpha)(U_\alpha\bar{\theta} + \chi\bar{f}_\alpha)dS \geqq 0, \tag{6.12}$$

and hence

$$\left| \int U_\alpha(f_\alpha\bar{\theta} + \bar{f}_\alpha\theta)dS \right| \leqq 2I_1\left(\int U_\alpha U_\alpha\theta\bar{\theta}dS\right)^{1/2} < 2\tilde{U}I_1I_2, \tag{6.13}$$

where $\tilde{U}$ is the maximum velocity in the steady motion: this maximum occurs on one of the cylinders and hence $\tilde{U}$ is equal to the greater of $|n_1a_1|$, $|n_2a_2|$. Also, for any real constant χ,

$$\int (U_\alpha f_\beta + \chi\partial f_\alpha/\partial x_\beta)(U_\alpha\bar{f}_\beta + \chi\partial\bar{f}_\alpha/\partial x_\beta)dS \geqq 0, \tag{6.14}$$

and hence

$$\begin{aligned} &\left| \int U_\alpha(f_\beta\partial\bar{f}_\alpha/\partial x_\beta + \bar{f}_\beta\partial f_\alpha/\partial x_\beta)dS \right| \\ &\qquad \leqq 2J_2\left(\int U_\alpha U_\alpha f_\beta\bar{f}_\beta dS\right)^{1/2} < 2\tilde{U}I_1J_2. \end{aligned} \tag{6.15}$$

Now on integration by parts the final integral in (6.10) is

$$\begin{aligned} \int (\partial U_\alpha/\partial x_\beta)(\bar{f}_\alpha f_\beta + f_\alpha\bar{f}_\beta)dS = &-\int U_\alpha(\bar{f}_\alpha\theta + f_\alpha\bar{\theta})dS \\ &-\int U_\alpha(f_\beta\partial\bar{f}_\alpha/\partial x_\beta + \bar{f}_\beta\partial f_\alpha/\partial x_\beta)dS, \end{aligned} \tag{6.16}$$

and hence by virtue of the inequalities (6.13), (6.15) we deduce from (6.10) the inequality

$$\begin{aligned} \sigma_1(I_2^2 + \lambda^2I_1^2)/\nu < &(\lambda^2\tilde{U}/\nu)I_1(I_2 + J_2) \\ &- (I_3^2 + \lambda^2I_2^2 + \lambda^2J_2^2 + \lambda^4I_1^2). \end{aligned} \tag{6.17}$$

Obviously $\sigma_1<0$ if $I_2^2+J_2^2+\lambda^2I_1^2-(\tilde{U}/\nu)I_1(I_2+J_2)$ is a positive-definite form in the real variables I_1, I_2, J_2; a sufficient condition for this is

$$\tilde{U}/(\nu\lambda) < 2^{1/2}. \tag{6.18}$$

Hence we see that *Couette motion is stable for a general disturbance of wave-length* $l(=2\pi/\lambda)$ *if*

$$(6.19)\qquad \tilde{U}l/\nu < 2\pi(2)^{1/2},$$

where $\tilde{U}$ *is the greater of the two linear velocities of the cylinders.* Thus any given Couette motion is stable for disturbances of sufficiently short wave-length.

Returning to (6.10) and writing the last term in the form (6.11), we may develop another attack. Choosing temporary axes we can make $U_{12}=0$ at a specified point in the fluid: then at that point (since $U_{\beta\beta}=0$)

$$(6.20)\qquad \begin{aligned} \left| U_{\alpha\beta}(\bar{f}_\alpha f_\beta + f_\alpha \bar{f}_\beta)\right| &= 2\left| U_{11}f_1\bar{f}_1 + U_{22}f_2\bar{f}_2\right| \\ &\leqq 2\left| U_{11}\right| f_\alpha \bar{f}_\alpha = 2\left| D\right|^{1/2} f_\alpha \bar{f}_\alpha, \end{aligned}$$

where $D=\det. U_{\alpha\beta}$. But the final expression in (6.20) is invariant, and so exceeds or is equal to the left-hand side for the original general axes. It is easily seen from (6.1) that $D=-B^2/r^4$ and so

$$(6.21)\qquad \left| \int U_{\alpha\beta}(\bar{f}_\alpha f_\beta + f_\alpha \bar{f}_\beta)dS \right| \leqq 2\left| B\right| \int r^{-2} f_\alpha \bar{f}_\alpha dS < 2\left| B\right| a_1^{-2} I_1^2 .$$

Thus by (6.10)

$$(6.22)\qquad \sigma_1(I_2^2 + \lambda^2 I_1^2)/\nu < (\left| B\right| /\nu)(\lambda/a_1)^2 I_1^2 - (I_3^2 + \lambda^2 I_2^2 + \lambda^2 J_2^2 + \lambda^4 I_1^2),$$

and hence $\sigma_1 < 0$ if

$$(6.23)\qquad \left| B\right| /(a_1^2 \nu \lambda^2) < 1.$$

Thus *Couette motion is stable for a general disturbance of wave-length* l *if*

$$(6.24)\qquad \frac{\left| n_2 - n_1\right| l^2}{\nu}\cdot\frac{a_2^2}{a_2^2 - a_1^2} < 4\pi^2.$$

The conditions (6.19) and (6.24) for stability are weak, but they have appeared worth noting on account of the ease with which they are obtained. Returning to (6.10) we may describe a method which has been used to give a fairly wide range of stability when applied to simpler problems than the general disturbance of Couette motion.

Let us consider the functions f_α arbitrary save for the boundary conditions (6.5), and a normalizing condition $I_1=1$. It is evident that the right-hand side of (6.10) has no finite lower bound, because we have only to choose for f_α rapidly oscillating functions to make the term $-I_3^2$ dominant. But it is evident from (6.22) that the expression in question has a finite upper bound. To find it, we apply the calculus of variations and see that the functions f_α providing this maximum satisfy the partial differential equations

$$(6.25)\qquad (\Delta' - \lambda^2)(\partial\theta/\partial x_\alpha - \lambda^2 f_\alpha) + (\lambda^2/\nu)U_{\alpha\beta}f_\beta + Kf_\alpha = 0,$$

where K is a constant; the maximum value in question is K. The boundary conditions associated with (6.25) are (6.5). We have here a new characteristic-value problem, namely, to find the characteristic values of K such that (6.25) may be consistent with (6.5). If all these characteristic values are negative, then the maximum of the right-hand side of (6.10) is negative, and hence $\sigma_1 < 0$.

Now it follows from the reasoning leading to (6.19) that if the motion of the cylinders and λ are assigned, the right-hand side of (6.10) is negative when ν is large enough. Thus, for large ν, all the characteristics K of (6.25) are negative. If we decrease ν, instability cannot appear until the greatest of the characteristics K passes through the value zero. Hence we may make this statement: *Couette motion is stable for general disturbances of wavelength $2\pi/\lambda$ provided that $\nu > \nu_1(\lambda)$, where $\nu_1(\lambda)$ is the greatest characteristic value of ν making the equations*

$$(\Delta' - \lambda^2)(\partial\theta/\partial x_\alpha - \lambda^2 f_\alpha) + (\lambda^2/\nu)U_{\alpha\beta}f_\beta = 0 \tag{6.26}$$

consistent with the boundary conditions (6.5).

7. **Couette motion with plane disturbance.** By a plane disturbance of Couette motion, we mean a disturbance in which the velocity is parallel to the plane $x_3 = \text{const.}$ (in the notation of §6) and independent of x_3. This is not a particular case of that considered in §6, because the condition of periodicity in x_3 is replaced by the condition of independence of x_3.

Limitations of space permit only a brief survey of this problem. The Couette motion (3.4) satisfies (2.27) with $\psi = \Psi$ where

$$\Psi = \tfrac{1}{2}Ar^2 + B\log r. \tag{7.1}$$

Putting first $\psi = \Psi + \epsilon\psi' + \cdots$ and then (introducing the azimuthal angle ϕ)

$$\psi' = f(r)e^{\sigma t + i\lambda\phi} \tag{7.2}$$

where λ is an integer (positive without loss of generality), we obtain the characteristic value problem for σ in the form

$$\begin{gathered} \nu LLf = (\sigma + i\lambda V/r)Lf, \\ L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\lambda^2}{r^2}, \end{gathered} \tag{7.3}$$

with the boundary conditions

$$f = f' = 0 \quad \text{for} \quad r = a_1 \quad \text{and for} \quad r = a_2. \tag{7.4}$$

The accent here indicates d/dr and V is given by (3.5). Multiplying (7.3) by $r\bar{f}dr$, integrating from $r = a_1$ to $r = a_2$, and adding the complex conjugate, we get (with $\sigma = \sigma_1 + i\sigma_2$)

$$\begin{aligned} &\sigma_1\left(\int rf'\bar{f}'dr + \lambda^2\int r^{-1}f\bar{f}dr\right)\Big/\nu \\ &\qquad = -\int rLfL\bar{f}dr - \tfrac{1}{2}ik\int r^{-2}(f\bar{f}' - \bar{f}f')dr, \end{aligned} \tag{7.5}$$

where the range of integration is (a_1, a_2) and

$$k = 2\lambda B/\nu. \tag{7.6}$$

Considering, as in §6, the upper bound of the right-hand side of (7.5), for f arbitrary save for (7.4) and a normalizing condition, we are led (as we were led to (6.26)) to the statement that *Couette motion is stable for plane disturbances, characterized with respect to periodicity in* ϕ *by the positive integer* λ, *provided that* $k < k_1(\lambda)$, *where* $k_1(\lambda)$ *is the smallest characteristic value of* k *making the equation*

$$\begin{aligned}(rf'')'' - (2\lambda^2 + 1)(r^{-1}f')' + (\lambda^4 - 4\lambda^2)r^{-3}f \\ - \tfrac{1}{2}ik\{(r^{-2}f)' + r^{-2}f'\} = 0\end{aligned} \tag{7.7}$$

consistent with (7.4).

Here, although derived in quite a different way, we have equation (19) of K. Tamaki and W. J. Harrison [18]. Since (7.7) is homogeneous, it is easy to get the general solution, but the subsequent calculations are intricate and the reader must be referred for them to a later paper by Harrison [19], where he found it possible to make use of the same type of argument as that used previously by Orr [20] for plane Couette motion.

8. **Couette motion with disturbance having rotational symmetry.** We proceed to discuss the one case where the method of the exponential time-factor has been used successfully to predict instability. For a disturbance with rotational symmetry, the first-order equations of disturbance of Couette motion are as in (3.8). Let us introduce dimensionless variables and constants as follows:

$$\begin{aligned} &t' = \nu t/a_1^2, & &r' = r/a_1, & &z' = z/a_1, \\ &\alpha = a_2/a_1 > 1, & &\beta = n_2/n_1, & &R = n_1a_1^2/\nu, \\ &A' = (\alpha^2\beta - 1)/(\alpha^2 - 1), & &B' = -\alpha^2(\beta - 1)/(\alpha^2 - 1); \end{aligned} \tag{8.1}$$

we recall that a_1, a_2 are the radii of the cylinders and n_1, n_2 their angular velocities, the subscript 1 referring to the inner cylinder. The constants α, β, R may be called, respectively, the *geometrical ratio*, the *kinematical ratio* and the *Reynolds number* of the steady motion. Substitution of (8.1) in (3.8) gives

$$\begin{aligned} \left(\frac{\partial}{\partial t'} - \Theta'\right)\Theta'\psi' &= -2R(A' + B'/r'^2)a_1\frac{\partial v'}{\partial z'}, \\ a_1\left(\frac{\partial}{\partial t'} - \Theta'\right)v' &= 2RA'\frac{\partial \psi'}{\partial z'}, \\ \Theta' &= \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{\partial^2}{\partial z'^2} - \frac{1}{r'^2}. \end{aligned} \tag{8.2}$$

The range of r' is $1 \leq r' \leq \alpha$ and the boundary conditions (3.9) are

$$(8.3) \qquad \frac{\partial \psi'}{\partial z'} = \frac{\partial}{\partial r'}(r'\psi') = v' = 0 \quad \text{for} \quad r' = 1 \quad \text{and for} \quad r' = \alpha.$$

We seek solutions of (8.2) of the form

$$(8.4) \qquad \psi' = a_1 f(r') e^{\sigma t' + i\lambda z'}, \qquad v' = i g(r') e^{\sigma t' + i\lambda z'},$$

where the functions f, g and the constant σ may be complex; the constant λ is real, and we may suppose it positive without loss of generality. The actual stream-function and azimuthal velocity are to be found by adding to (8.4) their complex conjugates. Substitution in (8.2) gives the two ordinary differential equations

$$(8.5) \qquad \begin{aligned} (L - \lambda^2 - \sigma)(L - \lambda^2) f &= -2\lambda R(A' + B'/r'^2) g, \\ (L - \lambda^2 - \sigma) g &= -2\lambda R A' f, \\ L = \frac{d^2}{dr'^2} &+ \frac{1}{r'}\frac{d}{dr'} - \frac{1}{r'^2}, \end{aligned}$$

with the boundary conditions

$$(8.6) \qquad f = df/dr' = g = 0 \quad \text{for} \quad r' = 1 \quad \text{and for} \quad r' = \alpha.$$

The system (8.5), (8.6) will be consistent only if the constants σ, λ, α, β, R satisfy a characteristic equation; it is natural to regard this as an equation for σ, the other (real) constants being assigned.

We have in (8.5), (8.6) a somewhat simpler form of the characteristic value problem than that given by G. I. Taylor [2]; the formal simplification arises from the use of the stream-function ψ. We proceed to give a modification of Taylor's method of obtaining the characteristic equation.*

The Bessel functions $J_1(\kappa r')$, $Y_1(\kappa r')$ are independent solutions of

$$(8.7) \qquad (L + \kappa^2)\phi(r') = 0.$$

Associating the boundary conditions $\phi(1) = \phi(\alpha) = 0$, we have as the characteristic values κ_n, $(n = 1, 2, \cdots)$, of the parameter κ the roots of

$$(8.8) \qquad J_1(\kappa) Y_1(\kappa\alpha) - Y_1(\kappa) J_1(\kappa\alpha) = 0;$$

the corresponding characteristic functions ϕ_n of (8.7) are†

$$(8.9) \qquad \phi_n(r') = C_n\{J_1(\kappa_n) Y_1(\kappa_n r') - Y_1(\kappa_n) J_1(\kappa_n r')\}, \qquad n = 1, 2, \cdots,$$

C_n being a normalizing factor, chosen to make

$$(8.10) \qquad \int_1^\alpha r' \phi_m \phi_n dr' = \delta_{mn}.$$

The constants κ_n, C_n and the functions ϕ_n are calculable in terms of α.

* The analytical validity of the processes employed by Taylor has been discussed by Faxén [21], who also proposed alternative treatments.

† The summation convention does not operate in §8.

We assume the existence of solutions of (8.5), (8.6), expansible in series of the above characteristic functions,

$$(8.11)\qquad f = \sum_{n=1}^{\infty} b_n\phi_n, \qquad b_n = \int_1^{\alpha} r' f\phi_n dr',$$
$$g = \sum_{n=1}^{\infty} c_n\phi_n, \qquad c_n = \int_1^{\alpha} r' g\phi_n dr'.$$

Since

$$(8.12)\qquad \int_1^{\alpha} r'\phi_n Lg dr' = \int_1^{\alpha} r' g L\phi_n dr' = -\kappa_n^2 \int_1^{\alpha} r' g\phi_n dr' = -\kappa_n^2 c_n,$$

multiplication of the second of (8.5) by $r'\phi_n dr'$ and integration from 1 to α gives

$$(8.13)\qquad c_n = 2\lambda R A'(\kappa_n^2 + \lambda^2 + \sigma)^{-1} b_n.$$

We may expand the right-hand side of the first part of (8.5) in the form

$$(8.14)\qquad -2\lambda R(A' + B'/r'^2)g = \sum_{n=1}^{\infty} d_n\phi_n,$$

where

$$(8.15)\qquad d_n = -2\lambda R\left\{A' c_n + B' \int_1^{\alpha} (g\phi_n/r')dr'\right\};$$

by (8.13) we have

$$(8.16)\qquad d_m = -\sum_{n=1}^{\infty} D_{mn} b_n,$$
$$D_{mn} = 4\lambda^2 R^2 A'\left\{A'\delta_{mn} + B' \int_1^{\alpha} (\phi_m\phi_n/r')dr'\right\}(\kappa_n^2 + \lambda^2 + \sigma)^{-1}.$$

If we substitute from (8.14) in the first of (8.5), multiply across by $r'\phi_m dr'$ and integrate from 1 to α, we get

$$(8.17)\qquad -\left[\frac{d\phi_m}{dr'} r' \frac{d^2 f}{dr'^2}\right]_1^{\alpha} + (\kappa_m^2 + \lambda^2)(\kappa_m^2 + \lambda^2 + \sigma)b_m = -\sum_{n=1}^{\infty} D_{mn} b_n,$$
$$m = 1, 2, \cdots.$$

Let us define

$$(8.18)\qquad e_m = (\kappa_m^2 + \lambda^2 + \sigma)b_m,$$
$$E_{mn} = D_{mn}(\kappa_m^2 + \lambda^2)^{-1}(\kappa_n^2 + \lambda^2 + \sigma)^{-1},$$
$$\Phi_m^{(\alpha)} = (d\phi_m/dr')_{r'=\alpha}, \qquad \Phi_m^{(1)} = (d\phi_m/dr')_{r'=1}.$$

Then (8.17) may be written

$$(8.19)\qquad (\kappa_m^2 + \lambda^2)^{-1}\Phi_m^{(1)}(r' d^2 f/dr'^2)_{r'=1} - (\kappa_m^2 + \lambda^2)^{-1}\Phi_m^{(\alpha)}(r' d^2 f/dr'^2)_{r'=\alpha}$$
$$+ \sum_{n=1}^{\infty} (\delta_{mn} + E_{mn})e_n = 0, \qquad m = 1, 2, \cdots.$$

On account of the boundary conditions on f, the series for f in (8.11) may be differentiated term-by-term, giving

$$df/dr' = \sum_{n=1}^{\infty} b_n d\phi_n/dr', \qquad 1 \leqq r' \leqq \alpha, \tag{8.20}$$

and the vanishing of df/dr' at the ends of the range gives the two equations

$$\sum_{n=1}^{\infty} (\kappa_n^2 + \lambda^2 + \sigma)^{-1}\Phi_n^{(1)} e_n = 0, \qquad \sum_{n=1}^{\infty} (\kappa_n^2 + \lambda^2 + \sigma)^{-1}\Phi_n^{(\alpha)} e_n = 0. \tag{8.21}$$

The infinite set of equations (8.19), (8.21), linear in the quantities

$$(r'd^2f/dr'^2)_{r'=1}, \qquad (r'd^2f/dr'^2)_{r'=\alpha}, \qquad e_1, e_2, \cdots,$$

yields the characteristic equation

$$F(\sigma, \lambda, \alpha, \beta, R) = 0, \tag{8.22}$$

where F is the infinite determinant

$$\begin{vmatrix} 0 & 0 & (\kappa_1^2 + \lambda^2 + \sigma)^{-1}\Phi_1^{(1)} & (\kappa_2^2 + \lambda^2 + \sigma)^{-1}\Phi_2^{(1)} & \cdots \\ 0 & 0 & (\kappa_1^2 + \lambda^2 + \sigma)^{-1}\Phi_1^{(\alpha)} & (\kappa_2^2 + \lambda^2 + \sigma)^{-1}\Phi_2^{(\alpha)} & \cdots \\ (\kappa_1^2 + \lambda^2)^{-1}\Phi_1^{(1)} & (\kappa_1^2 + \lambda^2)^{-1}\Phi_1^{(\alpha)} & 1 + E_{11} & E_{12} & \cdots \\ (\kappa_2^2 + \lambda^2)^{-1}\Phi_2^{(1)} & (\kappa_2^2 + \lambda^2)^{-1}\Phi_2^{(\alpha)} & E_{21} & 1 + E_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} \tag{8.23}$$

The next step should be the evaluation of the roots σ of (8.22) corresponding to assigned values of λ, α, β, R. This would be almost impossibly laborious, however, and we follow Taylor in adopting a less direct attack. The work of §6 for a general disturbance is applicable in particular for a disturbance with rotational symmetry. Translating (6.23) into present dimensionless notation, we know that the real parts of all roots σ of (8.22) are negative provided that

$$R < \lambda^2 \frac{\alpha^2 - 1}{\alpha^2 | \beta - 1 |}. \tag{8.24}$$

Let us hold λ, α, β fixed and increase R. If instability appears, it will appear when R has such a value that (8.22) has a purely imaginary root. But this consideration does not effectively reduce the task of computation, so Taylor made the bold assumption that the roots σ of (8.22) are real. Then instability will appear when R passes through such a value that (8.22) has a root $\sigma = 0$. This critical value of R, marking the incidence of instability, will be the smallest positive root R of

$$F(0, \lambda, \alpha, \beta, R) = 0. \tag{8.25}$$

Regarding α, β as assigned once for all, this critical value of R is a function of λ, say $R(\lambda)$. The absolute critical value R_c will be the minimum of $R(\lambda)$ for arbitrary λ.

The task of solving (8.25) remains formidable, and Taylor found it necessary to limit his calculations to the case where the difference between the radii of the cylinders is small, that is, $(\alpha-1)$ small.* He was then able to substitute trigonometric expressions for the Bessel functions, the value of κ_n being approximately $n\pi/(\alpha-1)$. Even then the calculations are very complicated. The results for particular values of α are exhibited graphically in Taylor's paper. We may quote here in the present notation his approximate expression for R_c for the case where the cylinders rotate in the same sense $(\beta>0)$:

$$(8.26)\qquad \begin{aligned} R_c^2 &= \frac{\pi^4}{2}\,\frac{\alpha+1}{(\alpha-1)^3(1-\beta\alpha^2)(1-\beta)}\,(0.0571S+0.00056S^{-1})^{-1},\\ S &= \frac{1+\beta}{1-\beta} - 0.652(\alpha-1). \end{aligned}$$

The corresponding value of λ (minimizing $R(\lambda)$) is $\pi/(\alpha-1)$, giving a wave-length in the z-direction equal to twice the difference between the radii of the cylinders. The formula (8.26) and the corresponding calculations for $\beta<0$ are in very good agreement with experiment.

9. **Poiseuille motion in a tube of general section.** A problem as general as that of the stability of Poiseuille motion in a tube of arbitrary section does not appear to have been treated previously. It is, however, quite easy to establish some sufficient conditions for stability.

The steady motion is as in (3.11) and the first-order equations of disturbance as in (3.14). We substitute

$$(9.1)\qquad u_i' = f_i(x_1, x_2)e^{\sigma t+i\lambda x_3}, \qquad p' = g(x_1, x_2)e^{\sigma t+i\lambda x_3},$$

the functions f_i, g and the constant σ being in general complex and the constant λ real; we assume it positive without loss of generality. On elimination of g, we obtain the two partial differential equations for f_α

$$(9.2)\qquad \begin{aligned} \sigma\left(\frac{\partial\theta}{\partial x_\alpha} - \lambda^2 f_\alpha\right) &= \nu(\Delta'-\lambda^2)\left(\frac{\partial\theta}{\partial x_\alpha} - \lambda^2 f_\alpha\right)\\ &\quad + i\lambda\left\{\frac{\partial}{\partial x_\alpha}\left(f_\beta\frac{\partial U}{\partial x_\beta}\right) - \theta\frac{\partial U}{\partial x_\alpha} - U\left(\frac{\partial\theta}{\partial x_\alpha} - \lambda^2 f_\alpha\right)\right\}, \end{aligned}$$

where

$$(9.3)\qquad \theta = \partial f_\beta/\partial x_\beta = -i\lambda f_3, \qquad \Delta' = \partial^2/\partial x_\beta\partial x_\beta.$$

* Without introducing this restriction, one can prove stability if $n_2a_2^2>n_1a_1^2>0$, or, equivalently, $\alpha^2\beta>1$; cf. [40].

We have written U instead of U_3 for simplicity in notation. The boundary conditions are

$$f_\alpha = \theta = 0 \quad \text{on} \quad C, \tag{9.4}$$

C being the boundary of the section. The consistency of (9.2) and (9.4) demands the satisfaction of a characteristic equation by σ, λ, ν.

One may observe the similarity of (9.2) to the analogous equation (6.3) for Couette motion. This suggests the application of similar methods. One may, however, note that i does not appear explicitly in (6.3), whereas it does appear in (9.2).

We shall now establish conditions under which the Poiseuille motion is stable, in the sense that all characteristic values of σ have negative real parts. Let us multiply (9.2) by $\bar{f}_\alpha dS$ and integrate over the section. On integration by parts this gives, by virtue of (9.4)

$$\begin{aligned}\sigma(I_2^2 + \lambda^2 I_1^2) &= -\nu(I_3^2 + \lambda^2 I_2^2 + \lambda^2 J_2^2 + \lambda^4 I_1^2) \\ &\quad + i\lambda\left\{\int (\partial U/\partial x_\alpha) f_\alpha \bar{\theta}\, dS - \int U\theta\bar{\theta}\, dS - \lambda^2 \int U f_\alpha \bar{f}_\alpha\, dS\right\},\end{aligned} \tag{9.5}$$

where we have used for certain positive-definite integrals the notation (6.7). Putting

$$\sigma = \sigma_1 + i\sigma_2, \tag{9.6}$$

and adding to (9.5) its complex conjugate, we obtain

$$\begin{aligned}\sigma_1(I_2^2 + \lambda^2 I_1^2)/\nu &= -(I_3^2 + \lambda^2 I_2^2 + \lambda^2 J_2^2 + \lambda^4 I_1^2) \\ &\quad + \tfrac{1}{2} i(\lambda/\nu) \int (\partial U/\partial x_\alpha)(f_\alpha \bar{\theta} - \bar{f}_\alpha \theta)\, dS,\end{aligned} \tag{9.7}$$

which may be compared with (6.10), the difference between the forms of the last terms being noted.

With a view to establishing conditions under which the right-hand side of (9.7) is negative for f_α arbitrary save for the restriction (9.4), we shall now establish some inequalities. We shall use χ to denote an arbitrary real number.

We have

$$\int (f_\alpha + i\chi\theta\partial U/\partial x_\alpha)(\bar{f}_\alpha - i\chi\bar{\theta}\partial U/\partial x_\alpha)\, dS \geqq 0, \tag{9.8}$$

and so

$$\left| \int \frac{\partial U}{\partial x_\alpha}(f_\alpha \bar{\theta} - \bar{f}_\alpha \theta)\, dS \right| \leqq 2I_1\left(\int \frac{\partial U}{\partial x_\alpha}\frac{\partial U}{\partial x_\alpha}\theta\bar{\theta}\, dS\right)^{1/2} < 2GI_1I_2, \tag{9.9}$$

where G is the maximum of the gradient of U,

$$G = \max\left(\frac{\partial U}{\partial x_\alpha}\frac{\partial U}{\partial x_\alpha}\right)^{1/2}. \tag{9.10}$$

By virtue of equations (3.11) and (3.12) this maximum is attained on the boundary.

Also

$$\int\left(\frac{\partial f_\alpha}{\partial x_\beta} + \chi\frac{\partial f_\beta}{\partial x_\alpha}\right)\left(\frac{\partial \bar{f}_\alpha}{\partial x_\beta} + \chi\frac{\partial \bar{f}_\beta}{\partial x_\alpha}\right)dS \geqq 0, \tag{9.11}$$

or

$$J_2^2 + \chi\int\left(\frac{\partial f_\alpha}{\partial x_\beta}\frac{\partial \bar{f}_\beta}{\partial x_\alpha} + \frac{\partial \bar{f}_\alpha}{\partial x_\beta}\frac{\partial f_\beta}{\partial x_\alpha}\right)dS + \chi^2 J_2^2 \geqq 0. \tag{9.12}$$

But

$$\begin{aligned}\int\left(\frac{\partial f_\alpha}{\partial x_\beta}\frac{\partial \bar{f}_\beta}{\partial x_\alpha} + \frac{\partial \bar{f}_\alpha}{\partial x_\beta}\frac{\partial f_\beta}{\partial x_\alpha}\right)dS &= -\int\left(\bar{f}_\beta\frac{\partial \theta}{\partial x_\beta} + f_\beta\frac{\partial \bar{\theta}}{\partial x_\beta}\right)dS \\ &= 2\int \theta\bar{\theta}dS = 2I_2^2,\end{aligned} \tag{9.13}$$

and hence (9.12) gives

$$J_2 \geqq I_2. \tag{9.14}$$

Also

$$\int\left(\frac{\partial U}{\partial x_\alpha}\theta + \chi\frac{\partial \theta}{\partial x_\alpha}\right)\left(\frac{\partial U}{\partial x_\alpha}\bar{\theta} + \chi\frac{\partial \bar{\theta}}{\partial x_\alpha}\right)dS \geqq 0, \tag{9.15}$$

or

$$\int\frac{\partial U}{\partial x_\alpha}\frac{\partial U}{\partial x_\alpha}\theta\bar{\theta}dS + \chi\int\frac{\partial U}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}(\theta\bar{\theta})dS + \chi^2 I_3^2 \geqq 0. \tag{9.16}$$

But by (3.11), (3.12)

$$\int\frac{\partial U}{\partial x_\alpha}\frac{\partial}{\partial x_\alpha}(\theta\bar{\theta})dS = -\int \Delta' U\cdot\theta\bar{\theta}dS = AI_2^2, \tag{9.17}$$

A being the constant in (3.11), which we may assume positive without loss of generality. We note that

$$A = -\mu^{-1}dP/dx_3, \tag{9.18}$$

where dP/dx_3 is the pressure gradient in the steady motion. From (9.16), (9.17) we have

$$AI_2^2 < 2GI_2I_3, \qquad I_3 > \tfrac{1}{2}I_2A/G. \tag{9.19}$$

Substituting from (9.9), (9.14), (9.19) in (9.7), we have

$$\sigma_1(I_2^2 + \lambda^2 I_1^2)/\nu < (G\lambda/\nu)I_1I_2 - I_2^2(2\lambda^2 + \tfrac{1}{4}A^2/G^2) - \lambda^4 I_1^2. \tag{9.20}$$

The expression on the left is negative-definite if

$$G\lambda/\nu < \lambda^2(8\lambda^2 + A^2/G^2)^{1/2}. \tag{9.21}$$

To agree with (1.1) for a circular section, we may define the Reynolds number R for Poiseuille flow through a general section by

$$R = GS/(2\pi\nu), \tag{9.22}$$

where S is the area of the section. Then, from (9.20), we may say that *Poiseuille motion in a tube of any section is stable for disturbances of wave-length* $l(=2\pi/\lambda)$ *if*

$$R < (S/l)(32\pi^2/l^2 + A^2/G^2)^{1/2}, \tag{9.23}$$

where A is related to the pressure-gradient by (9.18) *and G is as in* (9.10). *For a tube of circular section of radius a this condition, R being as in* (1.1), *is*

$$R < (2\pi a/l)(1 + 8\pi^2 a^2/l^2)^{1/2}. \tag{9.24}$$

To obtain stronger conditions for stability, we may return to (9.7) and pursue the same line of reasoning as that which led us from (6.10) to (6.26). Thus we may say that *Poiseuille motion in a tube of any section is stable for disturbances of wave-length* $2\pi/\lambda$ *provided that* $\nu > \nu_1(\lambda)$, *where* $\nu_1(\lambda)$ *is the greatest characteristic value of* ν *making the following equations consistent with the boundary conditions* (9.4):

$$(\Delta' - \lambda^2)\left(\frac{\partial\theta}{\partial x_\alpha} - \lambda^2 f_\alpha\right) + \tfrac{1}{2}i(\lambda/\nu)\left\{\frac{\partial}{\partial x_\alpha}\left(\frac{\partial U}{\partial x_\beta} f_\beta\right) + \frac{\partial U}{\partial x_\alpha}\theta\right\} = 0. \tag{9.25}$$

10. **Poiseuille motion in a tube of circular section with disturbance having rotational or axial symmetry.** We have in (9.24) a sufficient condition for stability for Poiseuille motion through a tube of circular section. Since the right-hand side has zero for minimum with respect to variable l, the condition is a very small step indeed towards the theoretical establishment of the experimental fact (1.2), which remains an outstanding challenge to mathematicians. The condition (9.24) has at least the merit that the disturbance is of a general type.

Taylor's success (§8) in handling the problem of the stability of Couette motion by means of a disturbance with rotational symmetry leads us to try the same plan for Poiseuille motion in a tube of circular section. The results are however disappointing, as we shall now see.

In cylindrical coördinates the first-order equations of disturbance are as in (3.19). We note that the variables ψ', v' are separated. Putting, as usual,

$$v' = g(r)e^{\sigma t + i\lambda z}, \tag{10.1}$$

the second equation of (3.19) gives

$$(L - \lambda^2 - \sigma/\nu - i\lambda W/\nu)g = 0,$$

$$(10.2) \qquad L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2},$$

$$W = W_0\left(1 - \frac{r^2}{a^2}\right),$$

with the boundary conditions

$$(10.3) \qquad g = 0 \quad \text{for} \quad r = 0 \quad \text{and for} \quad r = a.$$

Multiplying (10.2) by $r\bar{g}dr$, integrating from 0 to a, and adding the complex conjugate equation, we see immediately that the real part of σ is negative; thus there is stability as far as v' is concerned. Instability, if it occurs, must arise from the first of (3.19). But this is the single equation (3.21) for axially symmetric disturbances. *Hence the problem of stability for disturbances with rotational symmetry is identical with the problem of stability for disturbances with axial symmetry.*

The axially symmetric disturbance of Poiseuille motion through a tube of circular section was discussed by Orr [20, p. 135]. His fundamental equation may be obtained from (9.25): since the disturbance is axially symmetric, we have

$$(10.4) \qquad \begin{aligned} f_\alpha &= x_\alpha f(r), & \theta &= rdf/dr + 2f, \\ U &= W_0(1 - r^2/a^2), & f_\beta \partial U/\partial x_\beta &= -2W_0 r^2 f/a^2, \end{aligned}$$

and substitution in (9.25) gives

$$(L - \lambda^2)^2 f - \frac{2i\lambda W_0}{\nu a^2 r}\frac{d}{dr}(r^2 f) = 0,$$

$$(10.5) \qquad L = \frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr},$$

an equation immediately identified with Orr's (84) on putting $r^2 f = \psi$. The boundary conditions are $f = df/dr = 0$ for $r = a$ and regularity for $r = 0$. Solving his equation by a power series and hence calculating the largest characteristic $\nu_1(\lambda)$, Orr was led, by application of the type of argument associated with equation (6.26), to the conclusion that *Poiseuille motion in a tube of circular section is stable for axially symmetric disturbances of arbitrary wave-length if*

$$(10.6) \qquad R \equiv W_0 a/\nu < 180.$$

(This definition of R is easily seen to be the same as that given by equation (1.1).) Since this is only a sufficient condition for stability, it does not conflict with (1.2).

The problem of axially symmetric disturbances has also been discussed by Sexl [22, 23]. We shall indicate briefly the nature of his argument, using a slightly different approach. Introducing in (3.21) the dimensionless quantities

$$t' = \nu t/a^2, \qquad r' = r/a, \qquad z' = z/a, \qquad R = W_0 a/\nu, \tag{10.7}$$

and then putting

$$\psi' = f(r')e^{\sigma t' + i\lambda z'}, \tag{10.8}$$

we obtain for f the differential system

$$\begin{aligned} \{L - \lambda^2 - \sigma - i\lambda R(1 - r'^2)\}\, F &= 0, \\ (L - \lambda^2)f &= F, \\ L = \frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'} - \frac{1}{r'^2}&, \end{aligned} \tag{10.9}$$

with the boundary conditions

$$\begin{aligned} f = df/dr' = 0 \quad &\text{for} \quad r' = 1, \\ f = 0 \quad &\text{for} \quad r' = 0. \end{aligned} \tag{10.10}$$

Here we have a characteristic value problem for σ, λ, R. Rejecting one of the solutions of the first of (10.9) on account of a singularity at $r'=0$, we have

$$F = F_1(r', \sigma, \lambda, R), \tag{10.11}$$

a power series in r'. With the usual notation for Bessel functions, let us multiply the second of (10.9) by $r'J_1(i\lambda r')dr'$ and integrate over the range $(0, 1)$. We get

$$\int_0^1 r'J_1(i\lambda r')F_1(r', \sigma, \lambda, R)dr' = 0, \tag{10.12}$$

equivalent to Sexl's transcendental characteristic equation. It is difficult to make general deductions from this equation, and Sexl confines his attention to the cases where λR is either very small or very large, and in the latter case an asymptotic form for F_1 is used. There is further restriction to the cases where λ is either very small or very large. Sexl found in all cases considered that the real value of σ is negative, and this appears to be generally accepted (cf. Müller [24, p. 320]) as a proof that Poiseuille motion in a tube of circular section is stable for axially symmetric disturbances of any wave-length for any value of the Reynolds number.

11. **Plane Couette and plane Poiseuille motions (P.C.M. and P.P.M.).** As remarked earlier, the mathematical complexity of the three-dimensional problems physically most interesting has led to a concentration of attention on analogous plane problems, namely, the stability of plane Couette

motion (*P.C.M.*) and plane Poiseuille motion (*P.P.M.*), described in §3(c). The first-order equations of disturbance (for disturbances in the plane of the motion) are given in (3.28), (3.29). It might appear from the analogy of G. I. Taylor's work (§8) that instability should be sought in a disturbance not confined to the plane of the motion, but H. B. Squire [25] has shown that stability is increased by the introduction of such more general disturbances. We shall here confine ourselves to disturbances in the plane of the motion.

We substitute in (3.28), (3.29)

$$\psi' = f(y)e^{\sigma\tau + i\lambda x}, \tag{11.1}$$

where the function f and the constant σ are in general complex and the constant λ real and positive without loss of generality. Thus we obtain for f the ordinary differential equation of the fourth order

$$\begin{gathered} LLf = \sigma Lf + i\lambda RMf, \\ L = d^2/dy^2 - \lambda^2, \\ M = yL \text{ for } P.C.M.; \quad M = (1 - y^2)L + 2 \text{ for } P.P.M. \end{gathered} \tag{11.2}$$

By (3.31) the boundary conditions are

$$f = df/dy = 0 \quad \text{for} \quad y = \pm 1. \tag{11.3}$$

The system (11.2), (11.3) defines the characteristic value problem: for consistency σ, λ, R must satisfy a characteristic equation. We recall that R is the Reynolds number (3.27). It is natural to regard λ, R as assigned, and σ as the unknown characteristic value. If

$$\sigma = \sigma_1 + i\sigma_2, \tag{11.4}$$

then $\sigma_1 \leqq 0$ (for all characteristic values) is a sufficient condition for stability.

Doubts would be set at rest and much arduous labor saved, if a simple proof were forthcoming for the following theorem: *For any positive values of the real constants* λ, R, *no characteristic value* σ *of the system* (11.2), (11.3) *has a positive real part.* Such a theorem would establish the stability of *P.C.M.* and *P.P.M.* under all conditions. This result appears contrary to physical intuition (because we believe such motions to be unstable for large R), but it is a result towards which the theory appears to be slowly moving, and proofs have already been offered for the case of *P.C.M.* Before proceeding to discuss these complicated proofs, let us develop some simple results regarding the characteristic values.

Denoting the complex conjugate as usual by a bar and d/dy by an accent, we obtain, on multiplying (11.2) by $\bar{f}dy$ and integrating for the range $(-1, 1)$,

$$\sigma(I_1^2 + \lambda^2 I_0^2) = - i\lambda RQ - (I_2^2 + 2\lambda^2 I_1^2 + \lambda^4 I_0^2), \tag{11.5}$$

where

$$I_0^2 = \int_{-1}^{1} f\bar{f}dy, \qquad I_1^2 = \int_{-1}^{1} f'\bar{f}'dy,$$

(11.6)

$$I_2^2 = \int_{-1}^{1} f''\bar{f}''dy, \qquad I_3^2 = \int_{-1}^{1} f'''\bar{f}'''dy,$$

and

$$Q = \int_{-1}^{1} y(f'\bar{f}' + \lambda^2 f\bar{f})dy + \int_{-1}^{1} f'\bar{f}dy \quad \text{for} \quad P.C.M.,$$

(11.7)
$$Q = \int_{-1}^{1} [(1 - y^2)f'\bar{f}' + \{\lambda^2(1 - y^2) - 2\}f\bar{f}]dy$$

$$- 2\int_{-1}^{1} yf'\bar{f}dy \quad \text{for} \quad P.P.M.$$

Subtracting its complex conjugate from (11.5), we get

(11.8)
$$\sigma_2(I_1^2 + \lambda^2 I_0^2) = -\tfrac{1}{2}\lambda R(Q + \bar{Q}).$$

For *P.C.M.* this gives

(11.9)
$$\int_{-1}^{1} (\sigma_2 + \lambda R y)(f'\bar{f}' + \lambda^2 f\bar{f})dy = 0,$$

and hence (cf. Orr [20, p. 117], Solberg [26, p. 389]),

(11.10)
$$-\lambda R < \sigma_2 < \lambda R.$$

For *P.P.M.* (11.8) gives

(11.11)
$$\int_{-1}^{1} \{\sigma_2 + \lambda R(1 - y^2)\}(f'\bar{f}' + \lambda^2 f\bar{f})dy = \lambda R\int_{-1}^{1} f\bar{f}dy;$$

thus the integrand on the left must be positive somewhere in the range, and hence $\sigma_2 > -\lambda R$. We can also write (11.11) in the form

(11.12)
$$\int_{-1}^{1} \{\sigma_2 + \lambda R(1 - y^2)\}f'\bar{f}'dy + \lambda^2\int_{-1}^{1} \{\sigma_2 + \lambda R(1 - y^2) - R/\lambda\}f\bar{f}dy = 0.$$

If $\sigma_2 > 0$, so that the first integral is positive, the second integrand must be negative somewhere in the range, and hence, with the previous result, we have (cf. Solberg [26, p. 389])

(11.13)
$$-\lambda R < \sigma_2 < R/\lambda.$$

The inequalities (11.10), (11.13) are interesting inasmuch as they limit σ to strips in the complex plane, but they tell us nothing directly about stability, since they concern σ_2, not σ_1.

If we add to (11.5) its complex conjugate, we get

$$\sigma_1(I_1^2 + \lambda^2 I_0^2) = -\tfrac{1}{2}i\lambda R(Q - \bar{Q}) - (I_2^2 + 2\lambda^2 I_1^2 + \lambda^4 I_0^2). \tag{11.14}$$

It is easily seen that

$$|Q - \bar{Q}| < 2qI_0I_1, \qquad q = 1 \text{ for } P.C.M.;\ q = 2 \text{ for } P.P.M., \tag{11.15}$$

and hence

$$\sigma_1(I_1^2 + \lambda^2 I_0^2) < q\lambda R I_0 I_1 - (I_2^2 + 2\lambda^2 I_1^2 + \lambda^4 I_0^2), \tag{11.16}$$

an inequality of the same general type as (6.17), (6.22) ,(9.20).

We can deduce from (11.16) some simple conditions for stability, valid both for *P.C.M.* and *P.P.M.* For any real constants α, β,

$$\int_{-1}^{1}(f + \alpha y f' + \beta f'')(\bar{f} + \alpha y \bar{f}' + \beta \bar{f}'')dy > 0, \tag{11.17}$$

from which we deduce

$$\beta^2 I_2^2 > I_1^2(\alpha\beta - \alpha^2 + 2\beta) + I_0^2(\alpha - 1), \tag{11.18}$$

and hence, from (11.16),

$$\sigma_1\beta^2(I_1^2 + \lambda^2 I_0^2) < \beta^2 q\lambda R I_0 I_1 - I_1^2(2\lambda^2\beta^2 + \alpha\beta - \alpha^2 + 2\beta) - I_0^2(\lambda^4\beta^2 + \alpha - 1). \tag{11.19}$$

Therefore $\sigma_1 < 0$ if

$$(\beta^2 q\lambda R)^2 < 4(2\lambda^2\beta^2 + \alpha\beta - \alpha^2 + 2\beta)(\lambda^4\beta^2 + \alpha - 1), \tag{11.20}$$

where α, β are any real constants satisfying

$$2\lambda^2\beta^2 + \alpha\beta - \alpha^2 + 2\beta > 0, \qquad \lambda^4\beta^2 + \alpha - 1 > 0. \tag{11.21}$$

We now make certain choices of α, β, satisfying (11.21).

For $\alpha = \beta = 1$, we have $\sigma_1 < 0$ if

$$(qR)^2 < 8\lambda^2(\lambda^2 + 1). \tag{11.22}$$

For *P.P.M.* this reads $R^2 < 2\lambda^2(\lambda^2 + 1)$, and improves a condition given by Pekeris [27, p. 66] and also the conditions (7′), (8′) of Solberg [26].

For $\alpha = \beta = 2$, we have $\sigma_1 < 0$ if

$$(qR)^2 < (2\lambda^2 + 1)(4\lambda^4 + 1)/\lambda^2. \tag{11.23}$$

We note that, given R, there is stability for very great or very small λ, and that there is stability for arbitrary λ if $(qR)^2$ is less than the minimum of the right-hand side of (11.23).

For $\alpha=\beta=1/\lambda$, we have $\sigma_1<0$ if

$$(qR)^2 < 8(1 - \lambda^2 + \lambda^3 + \lambda^4). \tag{11.24}$$

The above conditions are very weak, but they possess the merit of simplicity. To strengthen the conditions, we return to (11.14) and consider the maximum value of the right-hand side, f being arbitrary save for the boundary conditions (11.3) and a normalizing condition. It is unnecessary to repeat the same type of reasoning as that which led us to (6.26). Applying the calculus of variations to the right-hand side of (11.14) and equating the Lagrange factor to zero, we obtain this result: *P.C.M. or P.P.M. is stable for disturbances of wave-length* $2\pi/\lambda$ *provided that* $R<R_1(\lambda)$, *where* $R_1(\lambda)$ *is the smallest characteristic value of* R *for which the system*

$$f'''' - 2\lambda^2 f'' + \lambda^4 f = i\lambda R\Phi, \qquad f = f' = 0 \quad \text{for} \quad y = \pm 1, \tag{11.25}$$

is consistent, the function Φ *being*

$$\Phi = -f' \quad \text{for} \quad P.C.M.; \qquad \Phi = 2yf' + f \quad \text{for} \quad P.P.M. \tag{11.26}$$

We observe that we have passed from the original characteristic value problem (11.2), (11.3) to a new characteristic value problem (11.25). But whereas a complete consideration of the former might conceivably give a sufficient condition for instability (some $\sigma_1>0$), conclusions drawn from (11.25) can only be sufficient conditions for stability.

Although derived in a different way, (11.25) contains the fundamental equations of Orr [20, pp. 125, 131]. The treatments for *P.C.M.* and *P.P.M.* are very different, because for *P.C.M.* (11.25) is an equation with constant coefficients and the general solution can be obtained in finite form. In the case of *P.P.M.*, Orr had recourse to a development of the solution in power series. Space only permits quotation of Orr's results: *the steady motion is stable for disturbances of arbitrary wave-length if*

$$R < 44.3 \quad \text{for} \quad P.C.M.; \qquad R < 88 \quad \text{for} \quad P.P.M., \tag{11.27}$$

R being defined as in (3.27). Orr's result for *P.P.M.* was confirmed by MacCreadie [28] with greater accuracy.

There is however another mode of attack. Instead of multiplying (11.2) by $\bar{f}dy$, as we did to obtain (11.5), let us multiply by $L\bar{f}dy$ and integrate over the range $(-1, 1)$. Adding the complex conjugate equation, we obtain in the notation of (11.6)

$$\begin{aligned} \sigma_1(I_2^2 + 2\lambda^2 I_1^2 + \lambda^4 I_0^2) = \tfrac{1}{2}\left[f''\bar{f}''' + \bar{f}''f'''\right]_{y=-1}^{y=1} \\ - (I_3^2 + 3\lambda^2 I_2^2 + 3\lambda^4 I_1^2 + \lambda^6 I_0^2), \end{aligned} \tag{11.28}$$

the same form for both *P.C.M.* and *P.P.M.* Multiplying (11.2) by $\exp(\epsilon\lambda y)dy$, where $\epsilon=\pm 1$, and integrating we obtain

(11.29) $$\left[e^{\epsilon\lambda y}(f''' - \epsilon\lambda f'')\right]_{y=-1}^{y=1} = i\lambda R \int_{-1}^{1} e^{\epsilon\lambda y} M f dy.$$

Solving these two equations for $f'''(1)$, $f'''(-1)$ and substituting in (11.28), the right-hand side becomes a function ϕ of λ, R, $f''(1)$, $f''(-1)$ and of certain integrals involving f, f', f'', f''' taken over the range $(-1, 1)$, the complex conjugates of these quantities occurring also, since the expression is real. We now seek the maximum of ϕ, when f is arbitrary save for the boundary conditions (11.3) *and for assigned values of* $f''(1)$, $f''(-1)$. The calculus of variations gives for the maximizing f a differential equation of the sixth order

(11.30) $$f^{(6)} - 3\lambda^2 f^{(4)} + 3\lambda^4 f'' - \lambda^6 f = RF(y, \lambda, f''(1), f''(-1)),$$

with an obvious notation for derivatives, where

(11.31) $$\begin{aligned} F &= i\lambda^2 \operatorname{cosech} 2\lambda \{f''(1) \cosh \lambda(y+1) - f''(-1) \cosh \lambda(y-1)\} \\ &\qquad\qquad \text{for } P.C.M., \\ F &= -2i\lambda^2 y \operatorname{cosech} 2\lambda \{f''(1) \cosh \lambda(y+1) - f''(-1) \cosh \lambda(y-1)\}, \\ &\qquad\qquad \text{for } P.P.M. \end{aligned}$$

The equation (11.30) is easily solved, and the boundary conditions are such as to make the solution unique; it is a function of y, λ, R, $f''(1)$, $f''(-1)$. Substituting it in ϕ, and referring to (11.28), we find that for $P.P.M.$

(11.32) $$\begin{aligned} \sigma_1(I_2^2 + 2\lambda^2 I_1^2 + \lambda^4 I_0^2) \leqq (A + BR^2)\{f''(1)\bar{f}''(1) + f''(-1)\bar{f}''(-1)\} \\ + (C + DR^2)\{f''(1)\bar{f}''(-1) + \bar{f}''(1)f''(-1)\}, \end{aligned}$$

where A, B, C, D are complicated but explicit functions of λ alone and $f''(1)$, $f''(-1)$ are evaluated for that characteristic function to which σ corresponds. Hence it follows at once that $\sigma_1 < 0$ and *there is stability if* λ, R *satisfy*

(11.33) $$A + BR^2 < 0, \qquad |A + BR^2| > |C + DR^2|.$$

This method has not been worked out in detail for $P.C.M.$, but the results for $P.P.M.$ will be reported at the International Congress for Applied Mechanics, Cambridge, Mass., in 1938. It is found that $P.P.M.$ *is stable for disturbances of arbitrary wave-length if*

(11.34) $$R < 155.$$

This improves considerably Orr's result, 88, which was quoted in equation (11.27).

When approached as above, it is natural to attack $P.C.M.$ and $P.P.M.$ by a single method. But this disregards an important difference between the forms of (11.2) for the two cases. This equation is undoubtedly simpler for *P.C.M.*, for then f occurs only in the form Lf, and the equation may be written

$$\phi'' - \lambda^2\phi = (\sigma + i\lambda Ry)\phi, \qquad f'' - \lambda^2 f = \phi, \tag{11.35}$$

with the boundary conditions $f = f' = 0$ for $y = \pm 1$.

This characteristic value problem has been treated by R. von Mises [3, 4] and L. Hopf [5] in quite different ways. They both reach the same conclusion, namely, that the real part of σ is negative for all real values of λ and R, so that *P.C.M.* is stable under all conditions, a surprising result from a physical point of view. We shall now describe these methods briefly.

In the method of von Mises, we multiply the second of (11.35) by $\exp(\pm\lambda y)dy$ and integrate, obtaining

$$\int_{-1}^{1} e^{\lambda y}\phi dy = 0, \qquad \int_{-1}^{1} e^{-\lambda y}\phi dy = 0. \tag{11.36}$$

Instead of considering the original differential system of the fourth order with four homogeneous boundary conditions, we now consider a system of the second order, namely, the first of (11.35), with (11.36) instead of boundary conditions. We already know that the real part of σ is negative if λ is given and R is small enough. Hence, if instability occurs, it will occur when σ passes through a purely imaginary value, $\sigma = i\sigma_2$. In this critical state we have

$$\phi'' - \lambda^2\phi = i\sigma_2(1 + Ky)\phi, \tag{11.37}$$

where $K(= \lambda R/\sigma_2)$ is real. Hence *if P.C.M. is to be unstable under any circumstances, the system*

$$\begin{gathered} \phi'' - \lambda^2\phi = \sigma(1 + Ky)\phi, \\ \int_{-1}^{1} e^{\lambda y}\phi dy = 0, \qquad \int_{-1}^{1} e^{-\lambda y}\phi dy = 0, \end{gathered} \tag{11.38}$$

must possess a purely imaginary characteristic value σ for some choice of the real constants λ, K. The aim of the argument of von Mises is to prove that all the characteristic values σ of (11.38) are real. He regards (11.38) as the limit of a set of difference equations. For details, the reader must refer to the original papers. The essential point is that the characteristic values of the difference system are finite in number, and if we can account for them all (with real values) at any stage, then no imaginary characteristic values can occur in the differential system, since characteristic values for the differential system must be limit points of characteristic values for the difference system. Although the general theory of the method is closely developed, the application to our particular hydrodynamical problem is less full, and it is possible to entertain a doubt as to whether the stability of $P.C.M.$ under all circumstances is fully established.*

The method of Hopf, arising out a paper by A. Sommerfeld [29], is quite different. On putting

$$z = (\lambda^2 + \sigma + i\lambda R y)/(\lambda R)^{2/3}, \tag{11.39}$$

we transform the first of (11.35) into

$$d^2\phi/dz^2 + z\phi = 0, \tag{11.40}$$

of which independent solutions are

$$\phi_1(z) = z^{1/2}H_{1/3}{}^{(1)}(\tfrac{2}{3}z^{3/2}), \qquad \phi_2(z) = z^{1/2}H_{1/3}{}^{(2)}(\tfrac{2}{3}z^{3/2}), \tag{11.41}$$

where the H's are Hankel functions. The second equation of (11.35) then reads

$$d^2f/dz^2 + \kappa^2 f = A\phi_1 + B\phi_2, \tag{11.42}$$

where $\kappa^3 = \lambda^2/R$ and A, B are constants; multiplying by $\sin \kappa z dz$, $\cos \kappa z dz$ and integrating from z_1 to z_2 (the values of z corresponding to $y=-1$, $y=1$), the left-hand sides vanish, and elimination of A, B from the right-hand sides gives the characteristic equation

$$\int_{z_1}^{z_2}\int_{z_1}^{z_2} \sin \kappa(z' - z'')\phi_1(z')\phi_2(z'')dz'dz'' = 0, \tag{11.43}$$

which involves σ, λ, R in the limits of integration and in κ. Obviously the deduction of general results from (11.43) is well-nigh impossible; Hopf found it necessary to limit himself to cases where the argument of the Hankel functions is either very small or very large. In all cases amenable to calculation, he found the real part of σ negative, and concluded that $P.C.M.$ is stable under all circumstances, confirming the conclusion of von Mises.

In view of the theoretical interest of the problem and the complexity and somewhat incomplete nature of the methods described above, it is to be hoped that mathematicians will not regard the problem of the stability of $P.C.M.$ as closed. A simple general proof of stability is greatly to be desired.

The problem of the stability of $P.P.M.$ has not been so fully treated. We can only mention in passing the work of H. Solberg [26], W. Heisenberg [30], and F. Noether [32]. The methods of the last two are asymptotic for large R. Heisenberg's work indicates instability for a range of λ and R, but Noether's indicates only stability. Recently papers have appeared by S. Goldstein [33] and C. L. Pekeris [27], the latter attempting

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* Professor von Mises has informed the writer that he does not regard his own proof of the stability of $P.C.M.$ as adequate, nor does he accept the proof of Hopf.

a solution in the form of a power series in R; these papers indicate stability as far as the calculations go.*

PART III. THE METHOD OF DECREASING POSITIVE-DEFINITE INTEGRALS

12. **The method of energy.** The method of energy has been more important historically than might appear from the small space devoted to it here. Some investigations actually conducted under this head lead to the same mathematical problems as have already been discussed in Part II, and it is unnecessary to reconsider them here. We shall show below how the method of energy is included in the methods of Part II.

We accept as basic the first-order equations of disturbance of §3. For disturbed motion given by (3.2), we define the *energy of the disturbance* as $\epsilon^2 T'$ where

$$T' = \tfrac{1}{2}\rho \int u_i' u_i' \, d\tau, \tag{12.1}$$

where $d\tau$ is an element of volume. As in Part II, we shall consider only disturbances spatially periodic; then the region of integration in (12.1) is the cell of periodicity, a region fixed in space. In plane problems, we replace (12.1) by an integral over an area.

The essential feature of (12.1) is the positive-definite character of the integrand: if $T'=0$, then $u_i'=0$ everywhere, and the first-order disturbance vanishes. If T' remains bounded for all positive values of the time t, then u_i' remain bounded almost everywhere, and thus we are led to accept the boundedness of T' as a sufficient condition for stability; for distinction we may call this *stability in the mean*.

Some writers demand a more stringent condition for stability, namely, $T'\to 0$ as $t\to\infty$. In the applications the actual condition we shall employ is

$$dT'/dt \leqq 0 \quad \text{for} \quad t \geqq 0, \tag{12.2}$$

which certainly implies the boundedness of T', but not necessarily $T'\to 0$ as $t\to\infty$. We shall accept (12.2) as a sufficient condition for stability in the mean.

To apply the method we do not have to integrate the equations of disturbance (3.3). By virtue of these equations we can express dT'/dt in terms of u_i', $\partial u_i'/\partial x_j$, $\partial p'/\partial x_i$, $\Delta u_i'$; if the resulting expression is zero or negative for arbitrary u_i' (satisfying the boundary conditions and the equation of continuity), then (12.2) is satisfied and there is stability in the mean. It is important to note that this method may give a sufficient condition for stability, but never a sufficient condition for instability (that is, unbounded

* In a later paper by Pekeris [41] approximate characteristic values are found by replacing the differential equation (11.2) for *P.P.M.* by a difference equation. All the approximate values so found indicate stability.

T'). We are never able to prove even that $dT'/dt > 0$ for arbitrary u_i', because a sufficiently rapid spatial oscillation in u_i' will reverse this inequality.

We shall now consider the application of the method to Couette motion and to Poiseuille motion, using (with slight modification) the notation of §§6, 9. We shall assume spatial periodicity in the direction of the x_3-axis, with wave-length $2\pi/\lambda$.

By (3.3) we have

$$
\begin{aligned}
\rho^{-1}dT'/dt &= \int u_i'(\partial u_i'/\partial t)d\tau \\
&= -\int u_i' U_j(\partial u_i'/\partial x_j)d\tau - \int u_i' u_j'(\partial U_i/\partial x_j)d\tau \\
&\quad - \rho^{-1}\int u_i'(\partial p'/\partial x_i)d\tau + \nu\int u_i'\Delta u_i' d\tau,
\end{aligned}
\tag{12.3}
$$

U_i being the velocity in steady motion. The integrals extend over the portion of the fluid between the planes $x_3 = 0$, $x_3 = 2\pi/\lambda$. Using Green's theorem, with the condition $u_i' = 0$ on the walls, and also the condition of spatial periodicity, we find that the first and third integrals vanish, and so

$$
\rho^{-1}dT'/dt = -\int u_i' u_j'(\partial U_i/\partial x_j)d\tau - \nu\int (\partial u_i'/\partial x_j)(\partial u_i'/\partial x_j)d\tau. \tag{12.4}
$$

Let us put

$$
u_i' = f_i(x_1, x_2, t)e^{i\lambda x_3} + \bar{f}_i(x_1, x_2, t)e^{-i\lambda x_3}, \tag{12.5}
$$

the functions f_i being complex and λ real and positive. Defining

$$
\theta = \partial f_\beta/\partial x_\beta, \tag{12.6}
$$

we have by the last of (3.3) $f_3 = i\theta/\lambda$, and since $\partial U_i/\partial x_3 = 0$ in both Couette and Poiseuille motions, we obtain from (12.4)

$$
\begin{aligned}
CdT'/dt &= -\tfrac{1}{2}(\lambda^2/\nu)\int (\partial U_\alpha/\partial x_\beta)(f_\alpha\bar{f}_\beta + \bar{f}_\alpha f_\beta)dS \\
&\quad + \tfrac{1}{2}i(\lambda/\nu)\int (\partial U_3/\partial x_\alpha)(f_\alpha\bar{\theta} - \bar{f}_\alpha\theta)dS \\
&\quad - (I_3^2 + \lambda^2 I_2^2 + \lambda^2 J_2^2 + \lambda^4 I_1^2),
\end{aligned}
\tag{12.7}
$$

where C is a positive constant and I_1, I_2, J_2, I_3 are defined as in (6.7), all integrals being taken over the section $x_3 = \text{const}$. The boundary conditions on the walls are

$$
f_\alpha = \theta = 0. \tag{12.8}
$$

In the case of Couette motion, the second integral on the right-hand side of (12.7) vanishes since $U_3 = 0$, and we are left with an expression for

CdT'/dt formally the same as the right-hand side of (6.10). Hence the problem of finding conditions under which dT'/dt is zero or negative for arbitrary f_α, satisfying (12.8), is the same as that of finding conditions under which the σ_1 of (6.10) is zero or negative for arbitrary f_α, satisfying (6.5). Thus for Couette motion nothing new is to be learned from the substitution of the method of energy for that of the exponential time-factor, except of course the fact that under those conditions of §6 which make $\sigma_1 \leqq 0$, we have stability in the mean.

In the case of Poiseuille motion, the first integral on the right-hand side of (12.7) vanishes, since $U_\alpha = 0$, and we are left with an expression formally the same as the right-hand side of (9.7). The remarks made above apply equally here; the method of energy coalesces with the method of the exponential time-factor, insofar as the latter makes use of (9.7).

Actually the method of the exponential time-factor seems to be the more powerful method: first, it is capable of establishing instability, which the method of energy can never do; secondly, it admits deductions other than (6.10) and (9.7), which represent the only products of the method of energy; thirdly, it is conceivable that in (6.10) and (9.7) we might make use of the fact that the f_α are characteristic functions, whereas in (12.7) the f_α are arbitrary functions. The only compensating disadvantage in the method of the exponential time-factor lies in the question of expansions in terms of characteristic functions. The method of energy leads without complication to conditions for stability in the mean.

Through lack of appreciation of the true situation, illegitimate deductions have been made from (12.4) or (12.7) or an equivalent equation. Suppose we take some definite u_i', satisfying the boundary and periodicity conditions and the equation of continuity, and substitute in (12.4). A negative value for dT'/dt so obtained is of no significance whatever as far as stability is concerned. It is only when $dT'/dt \leqq 0$ for *arbitrary* u_i' (subject to the boundary and periodicity conditions and the equation of continuity) that we can deduce stability in the mean. On this point H. A. Lorentz [34] (for *P.C.M.*) and F. R. Sharpe [35] (for *P.P.M.* and for Poiseuille motion in a tube of circular section) were not clear, and stated results which cannot be maintained.

We shall now briefly consider the method of energy as applied to plane motions, using the notation of §3(c) and §11. Instead of (12.4) we have

$$(12.9)\quad \rho^{-1}dT'/dt = -\int u_\alpha' u_\beta' (\partial U_\alpha/\partial x_\beta)dS - \nu \int (\partial u_\alpha'/\partial x_\beta)(\partial u_\alpha'/\partial x_\beta)dS,$$

the integrals being taken over a rectangle of periodicity in the plane of the motion. We have for both *P.C.M.* and *P.P.M.*

$$(12.10)\qquad U_1 = U(x_2),\quad U_2 = 0,\quad u_1' = -\partial\psi'/\partial x_2,\quad u_2' = \partial\psi'/\partial x_1,$$

and so, putting

(12.11) $$x = x_1/h, \quad y = x_2/h, \quad \psi' = f(y, t)e^{i\lambda x} + \bar{f}(y, t)e^{-i\lambda x},$$

where λ is real and positive, we obtain

(12.12) $$CdT'/dt = \tfrac{1}{2}i(\lambda h/\nu)\int_{-1}^{1}(dU/dy)(f\bar{f}' - \bar{f}f')dy - (I_2^2 + 2\lambda^2 I_1^2 + \lambda^4 I_0^2)$$

in the notation of (11.6), the accents on $f, \bar{f}$ denoting $\partial/\partial y$ and C being a positive constant. Since $U = U_0 y$ for $P.C.M.$ and $U = U_0(1 - y^2)$ for $P.P.M.$, it follows that

(12.13) $$CdT'/dt = -\tfrac{1}{2}i\lambda R(Q - \bar{Q}) - (I_2^2 + 2\lambda^2 I_1^2 + \lambda^4 I_0^2),$$

where Q is as in (11.7) and R as in (3.27). The right-hand side of (12.13) is the same formally as the right-hand side of (11.14), and hence the method of energy coalesces with the method of the exponential time-factor, insofar as the latter employs (11.14).

13. **The method of vorticity.** The integral of energy (12.1) is not the only positive-definite integral which may be used to give sufficient conditions for stability in the mean. An irrotational disturbance u_i' is necessarily inconsistent with the stringent boundary conditions ($u_i' = 0$). Let us write ξ_i' for the vorticity vector of the disturbance, so that

(13.1) $$\xi_i' = \tfrac{1}{2}\epsilon_{ijk}\partial u_k'/\partial x_j$$

(ϵ_{ijk} being the permutation symbol); then the conditions $\xi_i' = 0$ imply $u_i' = 0$. Hence it is fitting to consider the *integral of vorticity*

(13.2) $$V = \int \xi_i' \xi_i' d\tau,$$

integrated over the cell of spatial periodicity, and to assert that a steady motion is stable in the mean if V is bounded, $t > 0$, or more particularly, if

(13.3) $$dV/dt \leqq 0 \quad \text{for} \quad t \geqq 0.$$

We shall here confine our attention to the plane disturbances of $P.C.M.$ and $P.P.M.$, so that the vorticity integral is

(13.4) $$V = \frac{1}{4}\int (\partial u_2'/\partial x_1 - \partial u_1'/\partial x_2)^2 dS,$$

integrated over the rectangle of periodicity.* Introducing the notation of (12.10), (12.11) we obtain

(13.5) $$2CV = \int_{-1}^{1} LfL\bar{f}dy, \qquad L = \partial^2/\partial y^2 - \lambda^2,$$

* This integral has been used by Southwell and Chitty [36]; it may be remarked that their statement (equation (17), p. 230) that this integral is constant for the disturbance of an inviscid liquid flowing with a general velocity profile is not correct; cf. [37, equation (16)].

where C is a positive constant. If we substitute for ψ' from (12.11) in (3.28), (3.29), we get

$$\partial Lf/\partial\tau = LLf - i\lambda RMf, \tag{13.6}$$

where M is as in (11.2) and τ as in (3.26). Hence

$$\begin{aligned} CdV/d\tau &= \frac{1}{2}\int_{-1}^{1}\{Lf(LL\bar{f} + i\lambda RM\bar{f}) + L\bar{f}(LLf - i\lambda RMf)\}dy \\ &= \frac{1}{2}[f''\bar{f}''' + \bar{f}''f''']_{y=-1}^{y=1} - (I_3^2 + 3\lambda^2 I_2^2 + 3\lambda^4 I_1^2 + \lambda^6 I_0^2), \end{aligned} \tag{13.7}$$

where the accents on $f, \bar{f}$ denote $\partial/\partial y$ and I_0, I_1, I_2, I_3 are as in (11.6). But we have here on the right precisely the right-hand side of (11.28), and so apparently the method of vorticity coalesces with the method of the exponential time-factor. However, to use (11.28) we required (11.29), which followed from the fact that f is a characteristic function. If in (13.7) f is an arbitrary function of y (save for $f=f'=0$ for $y=\pm 1$), then (11.29) is not available. However, (11.29) may be established in connection with (13.7) by considerations based on the regularity of the motion (cf. G. Hamel [38], and also [39]), so that the method of vorticity does coalesce with the method of the exponential time-factor.

Conclusion

In conclusion we may pick out what appear to be the outstanding challenges to mathematicians in the field of hydrodynamical stability:

(i) A simple proof, not involving elaborate computations, that plane Couette motion is stable under all circumstances.

(ii) A similar treatment for plane Poiseuille motion, if in fact it is stable under all circumstances.

(iii) The establishment of some inequality defining a condition under which Poiseuille motion in a tube of circular section is unstable. Any attempt to fix a precise critical value for the Reynolds number must inevitably involve elaborate calculation. But we might hope for a simple method to establish that under certain circumstances at least one of a set of characteristic values has a positive real part.

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UNIVERSITY OF TORONTO,
TORONTO, ONTARIO, CANADA