RECENT TRENDS IN GEOMETRY

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Generally speaking, the subject of geometry falls into two main categories: the intrinsic theory of space and the theory of configurations existing in space. In this latter category we have the projective differential geometry of curves and surfaces which is largely devoted to the study of the differential invariants of these configurations which are left unaltered by transformations of the projective group and which, in this country, is usually associated with the names of Wilczynski and Lane. Usually however in dealing with problems in this category one is concerned with the purely metric theory, for example, the differential properties of configurations in a euclidean metric space which are unaltered by the group of orthogonal transformations of this space. In this connection one must think of the work of C. Burstin and W. Mayer who obtained for the first time the system of invariants for the general surface in a space of constant curvature by which the surface is characterized to within a rigid displacement in the imbedding space. Here, also, belong the recent researches of E. Kasner on conformal geometry, the investigations of W. C. Graustein, and the earlier work of L. P. Eisenhart.

But it is chiefly the intrinsic theory of spaces that has been the center of attention in recent years due to the influence of Einstein's theory of relativity and the attempts of many of the most distinguished mathematicians of the present time to devise a unified field theory of gravitation and electricity. While these investigations have so far fallen short of the physical requirements they have resulted in a number of elegant mathematical theories of abstract spaces and in this form they will undoubtedly survive. Owing to these and other researches we have today a rather extended abstract theory of various generalized spaces existing as a separate mathematical discipline and occupying a very definite place in the mathematical literature. It is this theory to which I have previously referred as the intrinsic theory of spaces or intrinsic geometry and with which I intend to occupy myself exclusively in this report. By thus restricting the domain of our considerations so as to permit a more detailed discussion and in view of the fact, also, that my own investigations have been primarily in the field of the intrinsic geometry I have hoped to make this report in some sense a scientific contribution.

1. By a geometry we shall understand the theory of a space and by a space a set of undefined objects, called points, on which a structure has been imposed by assuming that the points and certain subsets, usually said to be open, satisfy a prescribed number of conditions (structural

properties). For example, if we define the neighborhood of a point P of the space as any of the above open subsets containing P we may suppose that the points and the neighborhoods satisfy the four well known axioms of Hausdorff. In such a space one may define the concepts of limit point, complement, nowhere dense set, connected set, compact set, continuous function, and so on, which have become commonplace in the language of present day mathematics.

We arrive at the spaces of differential geometry by a fundamental restriction of the Hausdorff spaces, namely, by the requirement that the neighborhoods be homeomorphic to the n-dimensional number space, that is, the totality of ordered sets of n real numbers $x^1, \cdots, x^n$ with the ordinary arithmetic definition of neighborhood. Such a space is commonly called a topological manifold. Thus the differential geometer is provided with one of his essential tools, namely the use of coordinate systems which can be introduced into the various neighborhoods of the topoligical manifold owing to the above homeomorphisms. By a coordinate system is here meant a (1, 1) continuous correspondence between the points of a neighborhood of the manifold and a region (open point set) of the arithmetic space of n dimensions. Suppose now that it is possible to define in each of a set of neighborhoods covering the topological manifold a system of coordinates in such a way that the coordinate relationships which are defined in the intersections of any two coordinate neighborhoods are of class C^r, that is, are continuous and have all continuous partial derivatives to the order r inclusive where r is zero or a positive integer; these relations are regular if $r \geqq 1$, that is, if the jacobian is everywhere different from zero. We shall understand, moreover, that any point set U which is homeomorphic to a region of the n-dimensional number space and in which a system of coordinates $x^1, \cdots, x^n$ can be defined (one to one continuous correspondence between the coordinates of the system and points of the set) such that the coordinates of U are related by transformations of class C^r (with non-vanishing jacobians if $r \geqq 1$) to the coordinates of the above systems in the manifold is likewise admissible as a coordinate system in the topological manifold. We shall say that the topological manifold having as neighborhoods the coordinate regions so defined is an n-dimensional manifold of class C^r. In particular if the above coordinate relationships possess derivatives of all orders without restriction we shall speak of an n-dimensional manifold of class C^∞ and still more particularly if these relationships are analytic we have the analytic manifold or manifold of class C^ω, to use a notation suggested by Veblen and Whitehead in their Cambridge Tract on *The Foundations of Differential Geometry* [54].*

In the above paragraph we have given a hasty description of the manifold of class C^u (where $u = r \geqq 0$, ∞, or ω) by a process of specialization

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* Numbers in square brackets refer to the bibliography at the end of the paper.

having its beginning in the Hausdorff space. Veblen and Whitehead [54] have devised an axiomatic characterization of these manifolds in terms of an undefined class of allowable coordinate systems. Their axioms fall into three groups A, B, and C. The axioms A define the local structure completely, the axioms B define the class of allowable coordinate systems, while the axioms C impose certain general restrictions on the topology of the space of such a nature that A and C together characterized the space as a topological manifold. This tract by Veblen and Whitehead is worthy of serious study as the most complete and carefully thought out work that has appeared to date on the foundations of differential geometry defined as the general theory of manifolds of class C^u with $u \neq 0$. The influence of this work can easily be discerned in the present account of the intrinsic theory of differential geometry.

The above coordinate neighborhoods in a manifold $\mathfrak{M}$ of class C^u will also be called proper coordinate neighborhoods. An open set S which is the continuous map of a (proper) coordinate neighborhood of the n-dimensional number space will be said to be an improper coordinate neighborhood of $\mathfrak{M}$ with respect to the coordinates $x^1, \cdots, x^n$ which are thus introduced in S if any point $T \subset S$ is contained in a (proper) coordinate neighborhood $N(T)$ having $x^1, \cdots, x^n$ as coordinates. Thus the coordinates of an improper coordinate neighborhood S are *locally* in (1, 1) correspondence with points of $\mathfrak{M}$ but not necessarily over the entire neighborhood S.

It is often convenient to consider such coordinate systems the coordinates of which are not in (1, 1) correspondence with the points of the space. Indeed by a coordinate system we shall mean quite generally a correspondence $P \rightarrow x$ between a set of points $[P]$ of the space and a set of arithmetic points $[x]$, that is, points of the n-dimensional arithmetic space. Any arithmetic point x which corresponds to P in this correspondence may be called an image of P in $P \rightarrow x$ and P may be called an image of x. The numbers $x^1, \cdots, x^n$ which constitute an image of P are called the coordinates of P. An example of an important system of coordinates not in (1, 1) correspondence with the points of the space is the system of homogeneous coordinates used in the classical projective geometry.

The above discussion may be extended to the case where the set of coordinate relationships considered in the manifold is restricted to belong to a particular pseudo-group of transformations of class C^u (not identical with the set of all regular coordinate transformations of class C^u) and in the following such pseudo-groups will indeed appear (see, for example, the end of §11). In such cases the pseudo-group cannot be entirely arbitrary since there is evidently some connection between it and the topology of the manifold.

2. Before proceeding further let us pause to consider the beginning

which has already been made in the study of manifolds of class C^u where $u \neq 0$. I have in mind in particular the work of Hassler Whitney [63] who has shown, among other things, that an n-dimensional manifold of class C^r, where r is different from zero or ω, can be embedded in a euclidean space of $2n$ dimensions and that this embedding can be made without intersections in a euclidean space of $2n+1$ dimensions. The method here used by Whitney is closely related to one employed by Hurewicz [18] in purely topological embedding problems. An immediate consequence of this theorem is that there can be defined over the manifold a Riemann metric (§6). Generally speaking, the results which have been established for non-analytic manifolds cannot be extended to the case of the analytic manifold. Thus one has so far not succeeded in showing that an analytic manifold can be embedded analytically in a euclidean space, that any two points of an analytic manifold connected in the topological sense can be joined by an analytic arc, or that any analytic arc in an analytic manifold can be covered by an allowable coordinate system. Another problem in this connection would be that of showing whether or not one can define along any analytic arc in an analytic manifold a vector whose components are analytic functions of the parameter of the arc, other than the trivial case of the tangent vector. Of course such questions are primarily questions of analysis but they are likewise of interest from the standpoint of differential geometry.

Another contribution is the demonstration by S. S. Cairns [4] that any manifold of class C^1 can be triangulated, a result which has important applications, for example in the proof of the generalized theorem of Stokes [5]. However the methods employed by Cairns are rather complicated as are also, for that matter, those used by Whitney and it would be extremely desirable to obtain the results of these authors by simpler procedures.

3. In a sense the general manifold of class C^u is a rather amorphous affair and to arrive at the spaces which one habitually considers in the intrinsic theory one must add other elements of space structure. These additional elements of space structure are of primary importance in the theory of the space and on this account one usually speaks of this part of the structure briefly as the structure of the space. There are many ways of introducing this additional structure which lead to interesting spaces and we shall wish to examine some of these in detail. But first I should like to present an important definition applying quite generally to any space defined as a set of points with definitely specified structural properties. Two spaces U and V are said to be *equivalent* or *isomorphic* if there exists a (1, 1) correspondence between their points which carries the space U into the space V, that is, which carries the structural properties of U into those of V, and conversely. In particular, if $U=V$, the isomorphic correspondence is called an automorphism. Of course this general form of the concept of isomorphism must be somewhat vague without additional clarification.

Instead of attempting to enlarge on this idea in its general aspect we shall attempt to supply the needed clarification in connection with several particular but important spaces to which we next turn our attention.

4. One way in which additional elements of structure may be added to a manifold is by means of a special class of coordinate systems called *preferred coordinate systems*. Denoting by G a set of transformations of arithmetic points into arithmetic points, we define the structure of a space by means of the following set of axioms in which points and preferred coordinate systems enter as undefined terms.

A_1. *Each preferred coordinate system is a* (1, 1) *transformation of the space into the arithmetic space of n dimensions.*

A_2. *Any transformation of coordinates from one preferred coordinate system to another belongs to G.*

A_3. *Any coordinate system obtained from a preferred coordinate system by a transformation belonging to G is preferred.*

A_4. *There is at least one preferred coordinate system.*

It follows from the above axioms (we omit the details) that G is a group. Now let S and S' be two spaces satisfying the axioms A for the same group G. When we denote by $P\rightarrow x$ a preferred coordinate system for S and by $P'\rightarrow x'$ a preferred coordinate system for S' the relation $x=x'$ defines a (1, 1) correspondence C between the points of S and S'. It is easily seen that any preferred coordinate system for S is carried by C into a preferred coordinate system for S' and conversely. We have here an illustration of a (1, 1) point correspondence between two spaces which carries the structural properties (determined by the totality of preferred coordinate systems) of either one into the structural properties of the other as demanded by the general definition of isomorphism in the preceding section. Hence two spaces which satisfy the axioms A with the same group G are isomorphic.

If G is the affine group, any space satisfying the axioms A is an affine space and the preferred coordinate systems are then called cartesian coordinate systems. If G is the group of similarity transformations, a space satisfying the axioms A is a euclidean space. Also if G is the euclidean metric group, a space satisfying the axioms A is a euclidean metric space. In the last two cases the preferred coordinate systems are called rectangular cartesian coordinate systems.

An automorphism $P\rightarrow\bar{P}$ of a space satisfying the axioms A is evidently represented in any preferred coordinate system $P\rightarrow x$ by a transformation $x\rightarrow\bar{x}$ belonging to the group G. Conversely any transformation $x\rightarrow\bar{x}$ belonging to G in a preferred coordinate system $P\rightarrow x$ defines an automorphism $P\rightarrow\bar{P}$ of the space. It is clear in fact that the group of automorphisms G^* of the space is isomorphic (in the group-theoretic sense) to the group G. Now the group G^* provides a method of classification of configurations (sets of points) of the space into congruent or non-congruent configurations. Two configurations are said to be *congruent* if one can be carried into the other by a transformation belonging to G^*. It is important to emphasize that the definition of congruence here introduced pertains to a space whose structure is defined by a group of coordinate transformations in accordance with the above axioms A.

The concept of congruence as determined by a group of point transformations was proposed by Klein [20] as the defining property of a space in his Erlanger Program (1872). When the structure of a space is defined by the totality of congruence relationships existing in the space, which is, strictly speaking, the point of view of Klein, the group of point transformations which enters into the determination of these congruence relationships will be a subgroup (and in certain cases a proper subgroup) of the group of automorphisms of the space. But this view point is of very restricted application since many of the spaces which have been the object of recent study have as their group of automorphisms the identity and this is evidently insufficient for their characterization. The idea of the coordinate transformation and not Klein's idea of the point transformation or automorphism of a space is fundamental in the modern space concept.

5. Projective and conformal spaces may be defined by sets of axioms analogous to the above axioms A. As a matter of interest in itself and also for future reference in this report we state the axioms for the projective space. The axioms are given in terms of homogeneous coordinate systems in which a point P corresponding to an arithmetic point $(x^1, \cdots, x^{n+1})$ also corresponds to the point $(\lambda x^1, \cdots, \lambda x^{n+1})$, where λ is any non-zero factor and no point of the space corresponds to the origin of the $(n+1)$-dimensional arithmetic space. The undefined terms are the point and the preferred homogeneous coordinate system.

B_1. *In a preferred homogeneous coordinate system each point is represented by at least one arithmetic point in the arithmetic space of $n+1$ dimensions, and each arithmetic point other than the origin represents just one point.*

B_2. *Two arithmetic points represent the same point if, and only if, they lie on the same arithmetic straight line through the origin.*

B_3. *Any preferred coordinate system can be transformed into any other by a linear homogeneous transformation.*

B_4. *Any homogeneous coordinate system obtained from a preferred coordinate system by a linear homogeneous transformation is a preferred coordinate system.*

B_5. *There is at least one preferred coordinate system.*

It follows from the axioms B_1 and B_2 that each preferred coordinate system is a (1, 1) correspondence between the points of the projective space and the arithmetic straight lines through the origin of the $(n+1)$-

dimensional arithmetic space. The group of automorphisms of a projective space is called the n-dimensional projective group.

6. One of the most important types of space is that known as the Riemann space which may be said to have had its origin in Riemann's *Habilitationsschrift* [30]. From the general point of view which we have adopted, a Riemann space is defined as a manifold of class C^u with an additional element of structure consisting of a positive definite quadratic differential form defined over the manifold and on the basis of which the metric relationships existing in the space are determined. The existence of the quadratic differential form implies that its coefficients, which are functions of the coordinates $x^1, \cdots, x^n$ in any particular coordinate neighborhood, have the tensor law of transformation in the transition from one coordinate neighborhood to another—these coefficients are the components of a tensor called the fundamental metric tensor of the space. We may suppose the components of the fundamental metric tensor to be of class C^r, where $r<u$ and $u\geqq 1$, since their class will then be preserved under allowable coordinate transformations in the manifold. Ordinarily we suppose the integer r to have its greatest value consistent with the value of u, that is, $r=u-1$. In such a case we may speak of a Riemann space of class C^r with the understanding that the underlying coordinate manifold is of class C^{r+1}; correspondingly, we may speak of an analytic Riemann space or Riemann space of class C^ω when the components of the fundamental metric tensor are analytic functions defined over an analytic manifold.

In the theory of the Riemann space one is usually not concerned with the particular quadratic differential form which enters in the definition of its structure. In other words one deals with properties which are common to all such spaces. Accordingly, by the Riemann space we frequently understand the class of all Riemann spaces whose structure is determined by particular quadratic differential forms. Whether the designation Riemann space refers to a (particular) Riemann space as above defined or to the class of all such spaces must usually be decided in the light of the context. Analogous remarks apply to the various generalized spaces which we shall consider in the following sections.

Much of the formal local theory of Riemann spaces into which we do not enter here is independent of the fact that the quadratic differential form is positive definite and for this part of the theory it suffices to assume merely that the discriminant of the form does not vanish. A direct incentive to the treatment of such indefinite quadratic differential forms has moreover been furnished by the theory of relativity. On this account many authors do not insist that the above quadratic differential form be positive definite (as originally formulated by Riemann) in their definition of the Riemann space. However, in certain investigations, account must be taken of the signature of this form and, in particular, this is the case when dealing with properties of the space in the large where the signature must of necessity play an important rôle. We would suggest, therefore, that the designation "Riemann space" be used in the strict sense here considered owing to the especial mathematical interest of these spaces; and that a coordinate manifold over which an indefinite but non-degenerate quadratic differential form is defined be called generally a pseudo-Riemann space or a Riemann space of signature $s(<n)$ when the actual value of the signature is of significance. Of course one may also consider spaces for which the discriminant of the quadratic differential form vanishes at exceptional points, or even becomes infinite, for example over a set of points nowhere dense in the underlying manifold, and such spaces would appear to offer the possibility of interesting investigations.

Two Riemann or pseudo-Riemann spaces of the same class C^u are isomorphic if, and only if, there exists a (1, 1) point correspondence between the spaces which is representable by coordinate relationships of class C^u and which transforms the quadratic differential form of one of the spaces into that of the other. An automorphism of such a space is sometimes referred to as a motion. Correspondingly, the group of automorphisms is referred to as the group of motions and under this heading has been extensively treated in the literature. Since the group of automorphisms of a Riemann or pseudo-Riemann space is in general the identity these spaces fall outside the scope of the Erlanger Program (see end of §4).

7. The concept of infinitesimal parallel displacement, formulated by Levi-Civita in 1917, was of the utmost importance in the development of the theory of generalized spaces and led directly to the discovery of spaces generalizing those of Riemann [22]. Starting with an n-dimensional surface S immersed in a euclidean space E of m dimensions, Levi-Civita so defined the parallel displacement of a vector ξ along a curve of class C^1 in S with reference to the enveloping space E that the change in the components of the vector ξ along C proceeded in accordance with the equations

$$\frac{d\xi^\alpha}{dt} + L^{\alpha}_{\beta\gamma}\xi^\beta \frac{dx^\gamma}{dt} = 0, \tag{1}$$

where t denotes the parameter of the curve and the L's are a set of symbols previously defined by Christoffel [8] and depending on the coefficients of the quadratic differential form which determines the element of arc length in S as a subspace of the euclidean space E. In the case of a two-dimensional surface S in a three-dimensional euclidean space, where the idea of Levi-Civita has its strongest intuitive appeal, the procedure, in brief, may be described as follows. Let F be a developable surface tangent to S along C (envelope of the one-parameter family of tangent planes to S along C). Let ξ_P be a vector in the tangent plane to S at a point P on C. Roll F on a plane so that C becomes a curve C' in the plane, P a point P' on C', and ξ_P a vector $\xi_{P'}'$ at P'. Displace $\xi_{P'}'$ along C' by parallel displacement in the ordinary sense (parallel displacement in a euclidean plane). We thus define vectors $\xi'(t)$ along C' parallel in the ordinary euclidean sense. Now wrap the plane about the surface S along C to secure the original developable surface F. Thereby the vectors $\xi'(t)$ go into vectors $\xi(t)$ tangent to S along C. We define the vectors $\xi(t)$ which are in the surface S, that is, in the tangent planes to S along C, to be parallel with respect to C and in fact to result from the original vector ξ_P at P by parallel displacement along C. Levi-Civita's definition of parallel displacement of a vector in a surface S generalizes the ordinary euclidean concept of parallel displacement in the sense that if S is a plane (in general an n-dimensional euclidean space) the parallel displacement is identical with the ordinary euclidean displacement.

Soon after the work of Levi-Civita, a significant contribution was made to the idea of parallel displacement by H. Weyl [55] in his search for a combined theory of gravitation and electricity. Weyl's contribution consisted primarily in removing the enveloping euclidean space. He furthermore generalized the functions $L^{\alpha}_{\beta\gamma}$ appearing in the above differential equations to functions of the coordinates $x^1, \cdots, x^n$ of class C^u with a definite law of transformation, namely,

$$\bar{L}^{\sigma}_{\beta\gamma}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} = \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\beta}\partial \bar{x}^{\gamma}} + L^{\alpha}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \bar{x}^{\beta}}\frac{\partial x^{\nu}}{\partial \bar{x}^{\gamma}}, \tag{2}$$

which is uniquely determined by the requirement of the invariance of the equations (1). Weyl called the space in which the concept of parallel displacement was so defined by means of the equations (1) an affinely connected space and spoke of the functions $L^{\alpha}_{\beta\gamma}$ appearing in these equations as the components of the affine connection. As Weyl assumed the functions $L^{\alpha}_{\beta\gamma}$ to be symmetric in their lower indices a further generalization, more or less trivial, remained; namely, the extension to a space with a nonsymmetric affine connection. We thus arrive at the concept of the general affinely connected space with structure that of a manifold of class C^u plus an affine connection L having definitely determined components $L^{\alpha}_{\beta\gamma}(x)$ of class C^r in the various coordinate neighborhoods which transform by the law (2) in the transition from one coordinate neighborhood to another and in which the parallel displacement of an arbitrary vector along a curve of class C^1 is defined by (1). It is here to be assumed in accordance with the transformation (2) that $u \geqq 2$ (or that the underlying manifold is analytic) and that $r \leqq u-2$ (that $r \geqq 0$ or $r = \omega$ in the analytic case) since the class C^r of the components $L^{\alpha}_{\beta\gamma}$ is then unaltered by coordinate transformations.

One may apply the calculus of Ricci in the extended and technically improved form in which it exists today in developing the theory of the affinely connected space. We shall speak of such investigations later in this report.

An automorphism of an affinely connected space is also called a collineation and the local theory of the collineation group (as well as the group of isomorphisms of certain other spaces which we shall later discuss) has been investigated by M. S. Knebelman [21]. In this connection we may mention an interesting result due to L. P. Eisenhart [12] who has shown that a non-symmetric affine connection can be defined in the group manifold of a simply transitive continuous group as well as a number of non-degenerate quadratic differential forms with respect to each of which the group manifold becomes a Riemann (or pseudo-Riemann) space admitting the given group as a transitive group of motions. Further developments of this idea of Eisenhart have been made by Cartan and Schouten [7].

8. A curve of class C^1 which possesses the property that its tangents are parallel to the curve itself is given as a solution of the equations

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt} = 0, \tag{3}$$

where the $\Gamma^\alpha_{\beta\gamma}$ are the symmetric part of the components of the affine connection. Such curves are a generalization of the straight lines of a euclidean space and may be thought of as affording a means by which one may find his way about in a space with affine connection. A curve defined by (3) is called a path and on any path the parameter t is determined to within a linear (affine) transformation since the equations (3) are invariant under such, and only such, transformations of the parameter. When the structure of a space is defined by the totality of such paths it is called an affine space of paths. In a sense the affine space of paths so defined is but another name for a space with symmetric affine connection Γ since one can pass from this later space via (3) to an affine space of paths and, conversely, from (3) one can pass to the equations defining the parallel displacement of vectors, that is, to the equations (1) with $L^\alpha_{\beta\gamma}=\Gamma^\alpha_{\beta\gamma}$ or in other words to a space with symmetric affine connection. However the concept of the path as the fundamental structural element of a space which was initiated by Eisenhart and Veblen [11] has been the stimulus of several interesting investigations.

9. When the structure of a space is defined by the totality of paths given as solutions of (3) apart from any particular parametrization of the paths, the space is called a projective space of paths. The theory of this space or the projective geometry of paths thus consists in the body of theorems expressing properties of the paths as such, that is, independent of a particular parametric representation. As an illustration of a theorem of the projective geometry of paths we may mention Whitehead's convexity theorem which states that any point of a projective space of paths is contained in a simple convex neighborhood, that is, a neighborhood any two points of which can be joined by one and only one path lying in the neighborhood [60]. This theorem was later extended by Whitehead [61]

to the more general spaces considered in §13 and a new proof of this generalized theorem possessing some merit has recently been given by R. E. Traber [48].

The question arises as to whether or not there is more than one affine connection on the basis of which the paths of a projective space of paths are determined. The answer to this question is that two affine connections Γ and Λ will determine the same system of paths if, and only if, the components of these connections are related by the equations

$$\Lambda^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\beta\gamma} = \delta^{\alpha}_{\beta}\phi_{\gamma} + \delta^{\alpha}_{\gamma}\phi_{\beta}, \tag{4}$$

where the ϕ's are functions of the coordinates $x^1, \cdots, x^n$ to which the covariant vector law of transformation may be ascribed. The conditions (4) were originally found by Weyl [57] who also showed the possibility of defining a curvature tensor in the projective space of paths which is now usually referred to as the Weyl projective curvature tensor. As a matter of fact, the realization of the existence of the projective theory, the precise equations (4) and certain other results were discovered at a later date independently by Veblen [49] to whom Weyl's work was not available.

10. The fact that the structure of a projective space of paths may be defined by a multiplicity of affine connections, the components of which are related by (4), appears as an obstacle in the analytical development of the theory of this space. To overcome this difficulty Thomas [37] introduced the projective connection Π whose components $\Pi^{\alpha}_{\beta\gamma}$ defined by

$$\Pi^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \frac{\delta^{\alpha}_{\beta}}{n+1}\Gamma^{\sigma}_{\sigma\gamma} - \frac{\delta^{\alpha}_{\gamma}}{n+1}\Gamma^{\sigma}_{\sigma\beta} \tag{5}$$

have a unique determination in any coordinate neighborhood and in which, in fact, they appear as one of the above sets of functions $\Lambda^{\alpha}_{\beta\gamma}$ defining the system of paths. From (5) and the transformation law of the components $\Gamma^{\alpha}_{\beta\gamma}$ it follows that under changes of coordinates the quantities $\Pi^{\alpha}_{\beta\gamma}$ transform in accordance with the equations

$$\begin{aligned} \bar{\Pi}^{\sigma}_{\beta\gamma}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} &= \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\beta}\partial \bar{x}^{\gamma}} + \Pi^{\alpha}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \bar{x}^{\beta}}\frac{\partial x^{\nu}}{\partial \bar{x}^{\gamma}} \\ &\quad - \frac{1}{n+1}\frac{1}{(x\bar{x})}\left[\frac{\partial(x\bar{x})}{\partial \bar{x}^{\beta}}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\gamma}} + \frac{\partial(x\bar{x})}{\partial \bar{x}^{\gamma}}\frac{\partial \bar{x}^{\alpha}}{\partial \bar{x}^{\beta}}\right], \end{aligned} \tag{6}$$

where $(x\bar{x})$ stands for the jacobian of the coordinate transformation. Thus the system of paths may be defined by

$$\frac{d^2x^{\alpha}}{dp^2} + \Pi^{\alpha}_{\beta\gamma}\frac{dx^{\beta}}{dp}\frac{dx^{\gamma}}{dp} = 0,$$

in which the parameter p, called the projective parameter, is determined in any coordinate neighborhood to within a linear transformation and changes, when the coordinates are changed, by a definitely determined law. The projective space of paths may therefore be regarded as a space whose structure is defined by a projective connection and the theory of such a projectively connected space is the projective geometry of paths.

11. The possibility of applying the elegant methods of the tensor analysis to the treatment of the theory of an affinely connected space depends essentially on the existence of a set of equations of the form (2) for the transformation of the components of the affine connection. This happy state of affairs may be realized if we restrict the structure of the projective space by a set of coordinate neighborhoods such that $(x\bar{x})=1$ and then limit ourselves to coordinate transformations for which the jacobian is equal to unity since then in (6) the last two sets of terms will disappear (equi-projective geometry of paths). To obtain an analogous result without this restriction on the structure of the space Thomas [38] introduced a fictitious coordinate x^0 and augmented correspondingly the set of components of the projective connection. Suppose that Greek indices take on the values $1, \cdots, n$ and Latin indices the values $0, 1, \cdots, n$ in the remainder of this section. Define a set of quantities $\Pi^i_{jk}(x)$ in the coordinate neighborhoods of the projective space of paths by the requirement that the Π^i_{jk} are identical with the components $\Pi^\alpha_{\beta\gamma}$ of the projective connection when the indices i, j, k have the values $1, \cdots, n$ and for the remaining quantities Π^i_{jk} put

$$\Pi^i_{j0} = \Pi^i_{0j} = -\frac{\delta^i_j}{n+1}, \qquad \Pi^0_{\beta\gamma} = \Pi^0_{\gamma\beta} = \left(\frac{n+1}{n-1}\right)\mathfrak{P}^\alpha_{\beta\gamma\alpha},$$

where $\mathfrak{P}^\alpha_{\beta\gamma\delta}$ are the components of the curvature tensor of the equi-projective theory. Then

$$\bar{\Pi}^l_{jk}\frac{\partial x^i}{\partial \bar{x}^l} = \frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k} + \Pi^i_{pq}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}, \tag{7}$$

where the derivatives in these equations are calculated from the relations

$$x^\alpha = \phi^\alpha(x^1, \cdots, x^n), \quad x^0 = \bar{x}^0 + \log(x\bar{x}) + \text{const.}, \tag{8}$$

the first set of which denotes the coordinate transformation between the coordinates of two neighborhoods of the projective space of paths. In case the indices i, j, k in (7) have values $1, \cdots, n$ these equations are identical with the equations of transformation of the components of the projective connection given by (6). In case j or k has the value zero the equations (7) reduce immediately to an identity. The remaining case $i=0$, and $j, k\neq 0$ gives the equations of transformation of the components of the contracted equi-projective curvature tensor. As only the derivatives of $\log(x\bar{x})$ occur in (7) it is immaterial whether or not the jacobian $(x\bar{x})$ is positive or negative; however, in order that the relations (8) shall be strictly real, we shall now suppose that $(x\bar{x})$ stands for the absolute value of the Jacobian of the coordinate transformation in the projective space of paths.

The projective space of paths may now be defined as a space whose structure is given by the above extended connection Π, the components of which transform by (7), under transformations of the coordinates in the underlying n-dimensional manifold. Owing to the form of the relations (7), the formal theory of the affinely connected space may be carried over without modification to the projective space of paths. Thus we may construct the curvature tensor B by the ordinary method and we may define the repeated covariant derivatives of the curvature tensor. It is interesting to observe that the above curvature tensor B yields the Weyl curvature tensor when the indices appearing on the symbol of its components are restricted to the range $1, \cdots, n$ and that in the two dimensional case, where Weyl's curvature tensor vanishes identically, it provides us with a tensor analogous to the curvature tensor of Weyl. It should be emphasized here that the components Π^i_{jk} are independent of the variable x^0 and the same is therefore true of the curvature tensor B and its covariant derivatives. Defining the components of a tensor field (for example, T^i_j) to depend only on the coordinates $x^1, \cdots, x^n$ of the underlying manifold and to transform by the ordinary tensor law under transformations (8) the covariant derivatives of the tensor will have the same property. Although the extra coordinate x^0 is in some sense analogous to the factor of proportionality in the classical projective geometry, the vanishing of this coordinate in the above theory, contrary to the situation in the classical projective geometry, is the occasion of no especial concern since it does not enter explicitly in any of the invariant relationships of the projective geometry.

The generalized n-dimensional projective space is defined in the following manner. We generalize the relations (8) to the relations of the form

$$(9) \qquad x^\alpha = \phi^\alpha(x^1, \cdots, x^n), \quad x^0 = \bar{x}^0 + \log (x\bar{x})^m + \text{const.},$$

where m is an arbitrary (but fixed) constant and the remaining designations in these relations have their previous significance. We furthermore consider a connection, which we likewise denote by Π, having components Π^i_{jk} which may be arbitrary functions of the coordinates $x^1, \cdots, x^n$ of the coordinate neighborhoods of a manifold of class C^u subject to conditions on their class of the sort specified in §8 and which transform by the equations (7) in which the derivatives are determined by (9) when the coordinates $x^1, \cdots, x^n$ of the underlying manifold undergo a transformation given by the first set of these equations. By a generalized n dimensional projective space we mean an n dimensional manifold of class C^u whose (additional) structure is defined by the above connection Π. This space is of course a generalization of the projective space of paths and its theory is the generalized n-dimensional projective geometry.

Two points of view are now possible: We may either adopt the above point of view in which we adhere to the underlying n-dimensional space of the projective geometry or we may associate with this space an $(n+1)$-dimensional affinely connected space having coordinates $x^0, x^1, \cdots, x^n$ and affine connection Π the components of which transform by (7) under transformations (9) of the coordinates. This latter viewpoint provides a direct $(n+1)$-dimensional affine representation of the generalized projective space and lends itself very readily to interesting geometrical interpretations, some of which have been given by Whitehead [59].

12. We shall now consider a generalization of the classical projective space due to Veblen [51]. While the point of view of Veblen was purely local his work can be extended without serious difficulty so as to apply to spaces in the general sense here adopted. We shall present the theory in its extended form and remark that the details of this extension into which we do not enter may be found in the planographed *Lectures on Differential Geometry* [45].

Consider the n-dimensional projective space as defined in §5 by the axioms B. As we have already remarked, the points of this space are in (1, 1) correspondence with the straight lines through the origin of the $(n+1)$-dimensional arithmetic space. If we denote by $z^0, z^1, \cdots, z^n$ the coordinates of points in the arithmetic space, the points of the projective space are in (1, 1) correspondence with the points of the sphere $z^\alpha z^\alpha = 1$, where points at opposite extremities of a diameter are considered as identical. Denoting by $\mathfrak{P}$ the point set consisting of the points of this sphere with the above identifications, we shall use the concrete representation of the points of the projective space which is given by the set $\mathfrak{P}$ as an aid in the following discussion. The n-dimensional projective space under consideration may be regarded (to within an isomorphism) as the point set $\mathfrak{P}$ having the structural properties defined by the axioms B. It is possible to cover the point set $\mathfrak{P}$ by coordinate neighborhoods having coordinates $x^1, \cdots, x^n$ related analytically to one another and in consequence of which the point set $\mathfrak{P}$ becomes an analytic manifold $\mathfrak{M}$. We can now define over $\mathfrak{M}$ a set of $n+1$ analytic scalars $f^0, f^1, \cdots, f^n$ such that the determinant

$$\Delta = \begin{vmatrix} f^0 & f^1 & \cdots & f^n \\ \dfrac{\partial f^0}{\partial x^1} & \dfrac{\partial f^1}{\partial x^1} & \cdots & \dfrac{\partial f^n}{\partial x^1} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial f^0}{\partial x^n} & \dfrac{\partial f^1}{\partial x^n} & \cdots & \dfrac{\partial f^n}{\partial x^n} \end{vmatrix}$$

vanishes at no point of the manifold and possessing the property that in a definite preferred homogeneous coordinate system the coordinates z^α of any point of the projective space are given by $z^\alpha = f^\alpha(x^1, \cdots, x^n)\sigma$, where the factor σ is different from zero but otherwise arbitrary. In accordance with the axioms B the coordinates of points of the projective space in any preferred homogeneous coordinate system are then $z^\alpha = p_\beta{}^\alpha f^\beta \sigma$ where the $p_\beta{}^\alpha$ are constants such that the determinant $|p_\beta{}^\alpha|$ is different from zero. For the analytical requirements of the theory it is convenient to represent this last set of relations in the form $z^\alpha = \pm p_\beta{}^\alpha e^{x^0} f^\beta$ which automatically takes account of the fact that the above factor σ is different from zero.

To free ourselves from the special rôle played by the above scalars f^α we now put

$$z^\alpha = p_\beta^\alpha A^\beta, \qquad (A^\beta = \pm e^{x^0} f^\beta), \tag{10}$$

and eliminate the constants $p_\beta{}^\alpha$ from these equations. When we observe that quantities $a_\beta{}^\gamma$ may be defined by writing $a_\beta{}^\gamma \partial A^\alpha / \partial x^\gamma = \xi_\beta{}^\alpha$, since $\Delta \neq 0$, this elimination leads to the equations

$$\frac{\partial^2 z^\alpha}{\partial x^\beta \partial x^\gamma} = \overset{\sigma}{\Pi}_{\beta\gamma} \frac{\partial z^\alpha}{\partial x^\sigma}, \qquad \left(\overset{\sigma}{\Pi}_{\beta\gamma} = \overset{\sigma}{a}_\tau \frac{\partial A^\tau}{\partial x^\beta \partial x^\gamma} \right). \tag{11}$$

The $\Pi^\sigma_{\gamma\beta}$ are functions of the coordinates $x^1, \cdots, x^n$ having a unique determination in each of the coordinate neighborhoods covering the manifold $\mathfrak{M}$ if we suppose that the identity transformation $x^0 = \bar{x}^0$ is combined with the transformation $x \to \bar{x}$ of the coordinates of the neighborhoods. Assuming the scalar character of the z^α and the invariance of the above equations (11) and making use of the fact that the determinant $|\partial z^\alpha / \partial x^\beta|$ does not vanish, which is a consequence of $\Delta \neq 0$, it follows that the functions $\Pi^\alpha_{\beta\gamma}$ transform by the equations

$$\overset{\sigma}{\overline{\Pi}}_{\beta\gamma} \frac{\partial x^\alpha}{\partial \bar{x}^\sigma} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} + \overset{\alpha}{\Pi}_{\mu\nu} \frac{\partial x^\mu}{\partial \bar{x}^\beta} \frac{\partial x^\nu}{\partial \bar{x}^\gamma} \tag{12}$$

under an arbitrary regular analytic transformation of all the variables $x^0, x^1, \cdots, x^n$. We shall not, however, consider such arbitrary transformations of these variables but shall limit ourselves to transformations of the form

$$x^i = \phi^i(\bar{x}^1, \cdots, \bar{x}^n), \quad x^0 = \bar{x}^0 + \log \rho(\bar{x}^1, \cdots, \bar{x}^n), \tag{13}$$

where the first set of these equations represents an allowable analytic transformation of the coordinates of our manifold and where $\rho(\bar{x})$ is a positive analytic function, in consequence of the fact that under such transformations the form of the scalars z^α as given by (10) is unaltered. In particular, we speak of the transformation $x^i = \phi^i(\bar{x})$, $x^0 = \bar{x}^0$ as a coordinate transformation and of the transformation $x^i = \bar{x}^i$, $x^0 = \bar{x}^0 + \log \rho(\bar{x})$ as a gauge transformation. The general transformation (13) is called a representation (Darstellungs) transformation. Owing to the possibility of gauge transformations there exists an infinite number of sets of the functions $\Pi^{\alpha}_{\beta\gamma}$ in any coordinate neighborhood of the projective space.

The above considerations lead to the generalized projective space in the sense of Veblen which may be defined as follows: The *generalized n-dimensional projective space* is a space whose structure is defined by a (projective) connection Π having components $\Pi^{\alpha}_{\beta\gamma}$ which are functions of the coordinates $x^1, \cdots, x^n$ of class C^r and which transform in accordance with the equations (12) under representation transformations (13). The above form of the functions A^{α} gives the motivation of Veblen's definition of projective tensor in the generalized projective space. Veblen defines the projective scalar as a geometrical object (see §14) having components $e^{x^0}f(x^1, \cdots, x^n)$ in any representation and which is a scalar in the ordinary sense under transformations of the representation; more generally a projective scalar of index M is the object having the components $e^{Mx^0}f(x^1, \cdots, x^n)$ in any representation where M is a constant. Similarly the contravariant vector of index M is defined as the object having components $e^{Mx^0}\xi^{\alpha}(x^1, \cdots, x^n)$ in any representation and transforming under transformations of the representation as the components of an ordinary contravariant vector. The general projective tensor of index M is defined in an analogous manner and its (repeated) covariant derivatives can be constructed by the usual procedure. Likewise a projective curvature tensor can be defined as well as its successive covariant derivatives on the basis of the equations (12). The equations of transformation of the components of vectors and tensors have an interesting decomposition in view of the special form of the representation transformation (13) but it seems inadvisable to enter here into such detailed considerations.

In the case of the classical projective space having the (analytic) projective connection Π defined by the above equations (11) the following three conditions are satisfied:

(a) The connection is symmetric, that is, $\Pi^{\alpha}_{\beta\gamma} = \Pi^{\alpha}_{\gamma\beta}$;

(b) The conditions $\Pi^{\alpha}_{\beta 0} = \Pi^{\alpha}_{0\beta} = \delta_{\beta}{}^{\alpha}$ hold;

(c) The curvature tensor vanishes identically.

Conversely, it can undoubtedly be shown that a generalized n-dimensional analytic projective space, the points of which are in (1, 1) correspondence with the straight lines through the origin of the $(n+1)$-dimensional arithmetic space, and for which the above conditions (a), (b), (c) hold is a realization of the classical projective space of n-dimensions in the sense that preferred homogeneous coordinate systems can be defined such that the axioms B in §5 are satisfied. The corresponding local result was established by O. Veblen [52]. If we use the procedure employed by Veblen, it should be possible to work out the remaining details necessary to prove the above converse theorem and so obtain the precise conditions under which the generalized projective space becomes the space of the classical projective geometry.

It is immediately clear from its formulation on the basis of the axioms B of §5 that the generalized projective space of Veblen is indeed a generalization of the classical projective space. That the generalized projective space of §11 is likewise a generalization of the classical projective space is perhaps not so immediately evident but that such is the case is easily seen by the following very simple consideration. In the case of the classical projective space, where the components $\Pi^{\alpha}_{\beta\gamma}$ have the form given by (11) and the above conditions (a), (b), (c) are satisfied, the transformation relations (12) yield, for gauge transformations, the following conditions

$$\overline{\Pi}^{i}_{ik} = \frac{\partial \log D}{\partial x^k} + (n+1)\frac{\partial \log \rho}{\partial x^k}, \qquad \left(D = \text{absolute value of } \left|\frac{\partial A^{\alpha}}{\partial x^{\beta}}\right|\right).$$

Hence $\overline{\Pi}^{i}_{ik}$ vanishes if ρ is taken equal to $1/D^{n+1}$. It is therefore possible to cover the projective space with coordinate neighborhoods in each of which the condition $\Pi^{i}_{ik}=0$ is satisfied. It now follows readily from the equations (12) which give the relations between the components of the connection Π in the intersection of any two such coordinate neighborhoods (we omit the details of the calculation) that $\rho=c(x\bar{x})^{-1/n+1}$, where c is an arbitrary positive constant and, as above, $(x\bar{x})$ denotes the absolute value of the jacobian of the coordinate transformation. Hence for the above covering by coordinate neighborhoods (13) becomes

$$x^i = \phi^i(\bar{x}^1, \cdots, \bar{x}^n), \quad x^0 = \bar{x}^0 + \log (x\bar{x})^{-1/n+1} + \text{const.},$$

and is therefore identical with (9) when $m=-1/n+1$. In other words, the classical projective space may be thought of as a special case of the generalized projective space defined in §11 and this latter projective space becomes the classical projective space under the same conditions as the generalized projective space in the sense of Veblen.

The above consideration indicates clearly that there is no essential difference between the generalized projective space defined in §11 and the generalized projective space as defined by Veblen in this section. It might appear at first sight that the definition of the generalized projective space of §11 is preferable owing to the uniqueness of the determination of the components of the projective connection and the components of tensors in any coordinate neighborhood but on the other hand the multiplicity of these determinations in Veblen's formulation of the generalized projective space arises from the gauge transformation, and this transformation appears as an important element in the projective theory of relativity [52].

It may likewise be pointed out here that the later formulations of generalized projective spaces by the Dutch school and in particular the generalized projective space of van Dantzig based on the use of homogeneous coordinates has led to nothing essentially different from the projective spaces above discussed.

13. We have seen that the parameter of any path in the affine space of paths was determined to within a linear or affine transformation (§8) and that in the projective space of paths there was no such limitation on the allowable parameter transformations (§9). The significance of such parameter transformations in the determination of the structure of a space was clearly brought out by Jesse Douglas [9]. A general space of paths in the sense of Douglas is a space whose structure is defined by a system of curves or paths such that through each point P of the space there is one and only one path having an arbitrarily prescribed direction at P. We understand here, following Douglas, that the vectors $\lambda\xi$, where ξ is a particular non-zero vector at P and λ is an arbitrary non-zero constant, define the same direction at P and that any direction at P can be so defined (an extension of this definition of direction will result if we require λ to be a positive constant). We shall understand also that if the underlying manifold is of class C^u the paths are of class C^r, where $r \leqq u(\neq\omega)$ and where r is completely arbitrary if $u=\omega$, since under these conditions the class C^r of the paths is preserved under allowable coordinate transformations in the manifold. The requirement $u \geqq 2$ or $u=\omega$ is furthermore necessitated by the following analytical procedure.

By the specification of a definite group of allowable parameter transformations Douglas imposes an additional structural property on the general space of paths and treats in detail three cases which are thus distinguished by their group of parameter transformations, namely:

(a) *The affine space of paths*: on each path the parameter is determined to within a linear or affine transformation, that is, only transformations of the type $\bar{t}=at+b$ are allowed where $a(\neq 0)$ and b are arbitrary constants.

(b) *The descriptive space of paths*: any parameter transformation of class C^r is allowed on a path.

(c) *The metric space of paths*: only parameter transformations of the form $\bar{t}=t+b$ are allowed, where b is an arbitrary constant.

In the case of the affine space of paths a set of n functions $x^\alpha(at+b)$ will represent the path through an arbitrary point P of the space and having at P a specified direction. If we choose a particular non-zero vector ξ to represent the direction at P, the requirement that the first derivatives $(dx^\alpha/dt)_P=\xi^\alpha$ will fix the value of the constant a and determine uniquely the second derivatives at the point P. Hence in any coordinate neighborhood of the affine space of paths there exists a unique set of functions $H^\alpha(x, \xi)$ such that the paths are solutions of the equations

$$\frac{d^2x^\alpha}{dt^2} = H^\alpha(x, \xi), \qquad \left(\xi^\alpha = \frac{dx^\alpha}{dt}\right). \tag{14}$$

We refer to (14) as the differential equations of the paths of an affine space of paths.

By the above consideration the functions $H^\alpha(x, \xi)$ are defined only for non-zero vectors ξ. Let us therefore put $H^\alpha(x, 0)=0$ to complete the definition of these functions. Evidently if we replace the vector ξ by $\lambda\xi$, where λ is not equal to zero but otherwise arbitrary, we must change the parameter from t to t' where $t=\lambda t'+b$ in the definition of the functions H^α. This leads to the relations $H^\alpha(x, \lambda\xi)=\lambda^2 H^\alpha(x, \xi)$ which are thus seen to hold for all values of λ including zero in view of the above complete definition of the functions H^α. Hence the functions $H^\alpha(x, \xi)$ are homogeneous of the second degree in the variables ξ^α (in case the above extended definition of direction and corresponding modification of the group of parameter transformations are adopted, the functions H^α to which we are led will be positively homogeneous of the second degree in these variables). The invariance of (14) under affine transformations of the parameter is an immediate consequence of this homogeneity property.

To derive the relation between the H^α in different coordinate systems we have only to differentiate the transformation equations $x^\alpha=\phi^\alpha(\bar{x})$ with respect to the parameter of a path. We thus obtain

$$\xi^\alpha = \bar{\xi}^\beta \frac{\partial x^\alpha}{\partial \bar{x}^\beta}, \quad H^\alpha(x, \xi) = \bar{H}^\beta(\bar{x}, \bar{\xi}) \frac{\partial x^\alpha}{\partial \bar{x}^\beta} + \bar{\xi}^\beta \bar{\xi}^\gamma \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}, \tag{15}$$

and on the basis of these equations we can give an immediate formal proof of the invariance of (14) under coordinate transformations.

With the existence of the functions $H^\alpha(x, \xi)$ established we may now reverse our point of view and consider these functions as the fundamental structural element of the space. From this point of view, we consider a space whose structure is defined by a unique set of n functions $H^\alpha(x, \xi)$ in each coordinate neighborhood, these functions being homogeneous of the second degree in the variables ξ^α and transforming by the law (15) under coordinate transformations. For the requirements of the differential geometry it is furthermore desirable to assume certain differentiability properties of the functions $H^\alpha(x, \xi)$, for example, that these functions are of class C^1 for values of the x's in any coordinate neighborhood and for unrestricted values of the ξ's. Under these conditions the differential equations (14) have a solution $x^\alpha(t)$ uniquely determined in any coordinate neighborhood by arbitrary initial values t_0, x_0^α, and ξ_0^α of these variables. Also $x^\alpha(at+b)$ will be a solution of (14) on account of the homogeneity property of the functions $H^\alpha(x, \xi)$ and this solution corresponds in fact to the initial values t_0', x_0^α, $a\xi_0^\alpha$, the value of t_0' (which may be arbitrary) being determined by the value of the constant b. Hence the parameter on any curve given as a solution of (14) is determined to within an affine transformation and it follows that if we take these curves as our paths the space is an affine space of paths in the sense of Douglas.

If we assume that the $H^\alpha(x, \xi)$ have a sufficient number of derivatives with respect to the x's and ξ's, we can develop the theory of the space in a manner which corresponds closely to the theory of the affine space of paths as defined in §8. In fact we can define functions

$$\Gamma^\alpha_{\beta\gamma}(x, \xi) = \frac{1}{2}\frac{\partial^2 H^\alpha(x, \xi)}{\partial\xi^\beta\partial\xi^\gamma}$$

which, because of the relations (15), have the same transformation law as the components of the affine connection, namely (2). We can thus define a curvature tensor and we can develop a theory of covariant differentiation analogous to the theory of covariant differentiation in the ordinary affine space of paths (§8). We can also define the parallel displacement of a vector along a curve by means of equations analogous to (1) and those curves possessing the property that their tangent vectors are parallel with respect to the curves are easily seen to be the paths of the space. It is an interesting and easily demonstrated fact in this connection that if the above differentiability properties of the functions $H^\alpha(x, \xi)$ are such that these functions are, in the neighborhood of the values $\xi^\alpha = 0$, of class C^2 in the ξ's, then the $H^\alpha(x, \xi)$ will be homogeneous polynomials of the second degree in the ξ's, in consequence of which the above functions $\Gamma^\alpha_{\beta\gamma}$ will be independent of the ξ's and the space will revert to the ordinary affine space of paths.

Just as the affine space of paths in the sense of Douglas is a generalization of the ordinary affine space of paths so is Douglas' descriptive space of paths a generalization of the projective space of paths (§9). On the other hand the metric space of paths when the parameter t is taken as the arc length becomes identical with the space of the calculus of variations or Finsler space as it is sometimes called [14]. While the treatment of these spaces by Douglas is very interesting and worthy of a more detailed account we must pass on to a discussion of other topics. We should, however, mention that our treatment of the affine space of paths in the sense of Douglas is not, strictly speaking, the same as that given by Douglas who has, for example, assumed the existence of a definite number of parameters $a^1, \cdots, a^{2n-2}$ determinative of the paths of the space and has then attempted to deduce the differential equations (14) as the result of an elimination of these parameters from a system of equations in which they enter instead of deriving these equations, as we have done, as a direct logical consequence of the fundamental postulate on the determination of the paths of the space. The procedure which we have here employed for the case of the affine space of paths can of course be applied likewise to the descriptive and metric spaces of paths, and for further details in this connection reference may be made to *Lectures on Differential Geometry* [45].

14. In developing the ideas leading to the theories of the generalized affine, projective, and other spaces it came to be realized that one was in

each case concerned essentially with the theory of an invariant of the space and that in this sense the theory of each of these spaces was the theory of an appropriate invariant. For example, in the case of the Riemann space this invariant is a quadratic differential form (or what amounts to the same thing a symmetric covariant tensor of the second order), in the general affinely connected space (§7) it is the affine connection L and in the projective space of paths it is the projective connection Π defined in §10, and so on.

The idea of a geometry as the theory of an invariant was first stated in its full generality by O. Veblen in his Bologna address on *Differential invariants and geometry* [50]. As an alternative to the word invariant, the term *geometric object*, which appears likely to come into general use, was introduced by J. A. Schouten and E. R. van Kampen [36] and we shall likewise adopt their terminology. By a geometric object we shall mean precisely (regardless of possible variations in the use of this term by other writers) an abstract object having a *unique* set of components, depending on the coordinates and their differentials to a specified order, in any coordinate neighborhood of a topological manifold. In this definition of the geometric object the topological manifold may be a manifold of class C^u or the set of coordinate transformations considered in the manifold may be restricted further by the condition that they belong to a specified pseudo-group of transformations, for example, transformations of class C^u such that the jacobian is equal to unity or again transformations of class C^u belonging to the family (9), and so on. A geometric object such that its components in any coordinate neighborhood of the manifold depend only on the coordinates will be called a simple geometric object. The Riemann space is an example of a space with structure defined either by a geometric object which is not a simple object (quadratic differential form) or by a simple geometric object (fundamental metric tensor). Strictly speaking the projective connection Π of the generalized n-dimensional projective space in the sense of Veblen (§12) is not a geometric object since it admits a multiplicity of components in any coordinate neighborhood but this circumstance can be avoided (if we desire) by considering the $(n+1)$-dimensional representation obtained after the manner mentioned at the end of §11 by adjoining the variable x^0 to the n coordinates of the generalized n-dimensional projective space. The structure of each of the other generalized spaces which we have discussed can be described directly in terms of an appropriate geometric object (at least under suitable continuity and differentiability conditions) and in each case this object has been clearly exhibited.

In the case of the classical projective space the structure of the space can be defined by means of a particular geometric object as follows from the result given in §12 on the characterization of this space. The classical affine, euclidean, and euclidean metric spaces defined by means of the axioms A in §4 may likewise be defined by suitable geometric objects (see §17).

It must be emphasized that we do *not* say *any* geometry is the theory of a geometric object but rather that we *may* say that the geometry is the theory of a geometric object under certain circumstances, namely, when the structure of the space in question can be shown to be defined by an appropriate geometric object. For example the general affine space of paths of Douglas, when the paths are curves of class C^0, cannot, apparently, be defined by a geometric object in the above sense but the geometric object having the components $H^\alpha(x, \xi)$ may define the structure of such an affine space of paths under certain conditions as was shown in §13. Again, it is not immediately clear that the structure of the projective space of paths as defined in §9 can likewise be defined by a geometric object but it was later shown (§10) that the projective connection Π was capable of defining the structure of this space and in this sense the theory of this geometric object, that is, the projective connection Π, can be identified with the projective geometry of paths.

When one has discovered a geometric object in terms of which the structure of a given space can be defined (in case the structure of the space is not defined directly by a geometric object) one has obtained an unexcelled means for the exploitation of the theory of the space and this is especially true when the object is of the general nature of an affine connection as determined by the transformation law of its components under coordinate transformations in the space (for example the projective connections Π defined in §11 and §12). It is conceivable that a generalization of the definition of the geometric object may be desirable for certain purposes but it is likewise evident that if this generalization is pushed too far, the geometric object will lose its above significance in the development of the theory of a space defined as a point set with definitely specified structural properties.

15. We have now discussed a fair number of the more important spaces and we have developed a point of view which would appear to be adequate to describe those theories which are nowadays grouped under the heading of differential geometry. It remains, largely as a matter of record, to mention certain other spaces which have been the subject of more or less recent investigation. Chief among those is, perhaps, the conformal Riemann space which may be defined as the theory of a relative symmetric covariant tensor of the second order and weight $-2/n$ (geometric object). A very complete formal theory of this space is to be found in the various notes on this subject which have appeared between the years 1925 and 1935, in the Proceedings of the National Academy of Science. It would be interesting however to develop this theory along more geometrical lines starting perhaps with the set of axioms which define the classical conformal space analogous to the axioms A (§4) and B (§5) and to proceed to the con-

formal Riemann space or indeed to a generalization of this space. In this connection there is an interesting result on conformal spaces (which we shall not state here explicitly) by J. A. Schouten and J. Haantjes [35] which will undoubtedly find its place in such a development of the conformal space.

There is also the space originated by Weyl in his theory of gravitation and electricity which may be described roughly as a cross between an affine and a conformal space. Mention may also be made of the space of distant parallelism (or teleparallelism) in terms of which Einstein at one time hoped to find the solution to the space-time problem. This is a space where the structure is given by an ennuple of n independent vectors and in which it is possible, as the name implies, to define parallel displacement in such a way that the parallelism of vectors at distant points of the space is independent of the route of displacement. Of various other spaces which might be mentioned we shall say only a word concerning the Finsler space which we already encountered in connection with the work of Douglas (§13). The formal theory of the Finsler space is closely analogous to that of the Riemann space and has been considerably over-emphasized in the literature; on the other hand, those investigations which are not of a formal nature and which are undeniably of interest are usually treated in a special branch of differential geometry called the calculus of variations. Finally we signalize without further comment the analogues involving functional and Banach spaces which have been developed principally by the Pasadena school under the leadership of A. D. Michal.

16. The importance of the characterization of a space by means of its invariants by which one arrives at the idea of a complete set of invariants of the space seems to be recognized by the topologists but the significance of the corresponding characterization of the spaces of differential geometry does not appear to be so clearly recognized by the differential geometers. We shall say that a set of differential invariants of a generalized space is complete if conditions, algebraic in the components of the invariants, can be found by means of which the space is characterized to within an isomorphism. In dealing with the question of equivalence or isomorphism of the spaces of differential geometry attention has so far been confined to the local aspect of the problem and it has been shown that the various spaces which we have discussed can all be characterized locally by sets of tensor differential invariants in the above sense.

As an illustration we take the Riemann space. Let S and $\bar{S}$ denote n-dimensional Riemann spaces of class C^r, the value of the integer r to be determined by the requirements of the following discussion. Then S and $\bar{S}$ are isomorphic if, and only if, there exists a (1, 1) point correspondence between these spaces which is representable by coordinate relationships $x^\alpha = \phi^\alpha(\bar{x})$ of class C^{r+1} and transforms the fundamental quadratic differential form of one of the spaces into that of the other, that is, such that

$$\bar{g}_{\alpha\beta}(\bar{x}) = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta} \tag{16}$$

over S or $\bar{S}$ (cf. §6). The integrability conditions of (16) when combined with the equations (16) themselves give the following sequence

$$\bar{g}_{\alpha\beta} = g_{\mu\nu} u^{\mu}_{\alpha} u^{\nu}_{\beta}\, ;\ \bar{B}_{\alpha\beta\gamma\delta} = B_{\mu\nu\sigma\tau} u^{\mu}_{\alpha} \cdots u^{\tau}_{\delta}\, ;\ \bar{B}_{\alpha\beta\gamma\delta,\epsilon} = B_{\mu\nu\sigma\tau,\eta} u^{\mu}_{\alpha} \cdots u^{\eta}_{\epsilon}\, ; \cdots ,$$

where the $u_\beta{}^\alpha$ stand for the corresponding derivatives of the above coordinate relationships. Suppose that there exists an integer M such that the first M sets of the above equations are algebraically consistent as equations for the determination of the x^α and $u_\beta{}^\alpha$ as functions of the coordinates $\bar{x}^\alpha$ of a neighborhood $\bar{N}$ of the space $\bar{S}$. It can be proved that this condition is necessary and sufficient for the isomorphism of a neighborhood N of the space S and some neighborhood $N' \subset \bar{N}$ of the space $\bar{S}$ (theorem of local isomorphism). Under the conditions of this theorem the equations (16) will admit a solution $x^\alpha = \phi^\alpha(\bar{x})$ of class C^{r+1} defined in some neighborhood $\tilde{N} \subset \bar{N}$ in which the jacobian will not vanish in view of the fact that the determinants $|g_{\mu\nu}|$ and $|\bar{g}_{\alpha\beta}|$ are different from zero by hypothesis. Hence there exists a regular (1, 1) correspondence of class C^{r+1} between neighborhoods N and $N' \subset \tilde{N}$ of the spaces S and $\bar{S}$, respectively, satisfying (16) by which the isomorphism of these neighborhoods is established. It is evident that the maximum value which can be assumed by the integer M in the above theorem is given by the sum of the variables x^α and $u_\beta{}^\alpha$ or, in other words, is equal to $n(n+1)$. Hence the above theorem gives necessary and sufficient conditions for the local isomorphism of n-dimensional Riemann spaces of class C^r where $r \geqq n(n+1)+1$ but may likewise furnish a test of isomorphism for smaller values of r in case the above maximum value of M is not attained in the process of attempting to satisfy the conditions of the theorem. The above theorem can be extended to analytic Riemann spaces and analogous theorems of isomorphism can be given for the various generalized spaces.

E. B. Christoffel [8] was the first to give an equivalence theorem of the above type. He considered, however, the case when the first M sets of the above equations admit a single independent solution and so arrived only at sufficient conditions for the equivalence of two quadratic differential forms. The fundamental existence theorem for systems of differential equations which is now commonly used in the proof of the above theorem of local isomorphism seems to have had a rather lengthy history. It is, for example, essentially the same as a theorem given by J. E. Wright [64] and T. Levi-Civita [23], but the form of this theorem which is best adapted to our purpose is due to Veblen and J. M. Thomas [53] who used it to prove a theorem of local isomorphism for the projective space of paths. The question of the extent to which such theorems of local isomorphism can be carried to furnish necessary and sufficient conditions for the isomorphism of generalized spaces in the large has, as far as I am aware, never been considered.

Whitehead [59] has proved that under certain restrictions the general affine equivalence of the $(n+1)$-dimensional affine representations of two generalized projective spaces in the sense of Veblen implies the equivalence of the projective spaces. The nature of the restrictions imposed by Whitehead are sufficiently weak to permit one to say that the equivalence of two projective spaces of paths is always a consequence of the general affine equivalence of their $(n+1)$-dimensional affine representations [39].

It follows from the definition of the complete set of invariants of a space that any local condition on the space can be expressed directly in terms of the invariants of the complete set without the necessity of further differentiation. Moreover, it would appear that these invariants furnish the simplest possible means of expressing such conditions when they are of tensor character. It is evident that such is the case in dealing with the problem of the local isomorphism of spaces. As a further illustration of the advantages attending the use of the tensor differential invariants of a space one may point to the researches on the problem of the determination of the absolute scalar differential invariants of a Riemann space by means of the integration of complete systems of differential equations. Earlier writers beginning with Sophus Lie (1884) and K. Zorawski (1892) made direct use of the components $g_{\alpha\beta}$ of the fundamental metric tensor and their ordinary partial derivatives as the basis for expressing the scalar invariants. All this work has been greatly simplified and extended by Michal and Thomas [26] by use of the tensor differential invariants of the space. Now the quantities $g_{\alpha\beta}$ and their partial derivatives to a specified order constitute the components of a spatial invariant (geometric object) and in fact a non-tensor invariant the components of which have a linear homogeneous transformation law. The use of invariants whose components have the general linear homogeneous law of transformation should be discarded whenever possible in favor of invariants of simple tensor character.

17. If the structure of a space S can be regarded as a component part of the structure of a space S' of lesser generality, the space S will under certain conditions *reduce* to the space S'. Thus the generalized projective space may reduce to the classical projective space and the conditions for this reduction have in fact already been given in §12. Similarly the general n-dimensional affine space with symmetric affine connection Γ will reduce to the classical affine space if, and only if, it is homeomorphic to the n-dimensional number space and the curvature tensor vanishes identically. Under corresponding conditions the Riemann space reduces to the euclidean metric space and the conformal Riemann space to the classical or euclidean conformal space. Likewise the euclidean space is a special case of the generalized conformal-affine space whose structure is defined by a

composite geometric object consisting of a positive definite relative covariant tensor K of the second order and weight $-2/n$ and a symmetric affine connection Γ. In fact the n-dimensional conformal-affine space reduces to the euclidean space if, and only if, it is homeomorphic to the n-dimensional number space and two invariants of the space vanish, namely, the covariant derivative of the tensor K and the curvature tensor; that is, under these conditions preferred coordinate systems can be defined for the conformal-affine space which satisfy the axioms A of §4 where G is the group of euclidean similarity transformations. As far as I am aware, the proof of these theorems is not to be found in the literature but their demonstration should be a comparatively easy matter.

Usually one considers the question of the local reducibility of spaces. For example, a general affinely connected space with symmetric affine connection, but without regard to its topological properties, is said to reduce locally to the classical affine space or to be locally flat if the curvature tensor vanishes identically. In this case any point P of the space is contained in a coordinate neighborhood $N(P)$ with respect to which the components $\Gamma^{\alpha}_{\beta\gamma}$ of the affine connection vanish and in consequence of this the neighborhood $N(P)$ can be mapped upon a neighborhood of the classical affine space in such a way that the paths in $N(P)$ go into the straight lines of this latter space. Similar theorems of local reducibility with corresponding geometrical interpretations can be stated for the various generalized spaces for which curvature tensors can be constructed and indeed any such theorem expresses the characteristic property of the curvature tensor for the space under consideration.

As another illustration, selected from among many which might be given, of a theorem on the local reducibility of a space let us consider the problem of finding conditions for an affine space of paths to reduce to a Riemann space. In other words, we seek necessary and sufficient conditions for the paths to be the extremals of an integral $\int (g_{\alpha\beta}(x)dx^{\alpha}dx^{\beta})^{1/2}$, where the expression in parentheses is a positive definite quadratic differential form. This is one of the inverse problems of the calculus of variations. The problem is equivalent to the problem of finding a solution $g_{\alpha\beta}(x)$, such that the form $g_{\alpha\beta}(x)dx^{\alpha}dx^{\beta}$ is positive definite, of the system of equations

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} = g_{\sigma\beta}\Gamma^{\sigma}_{\alpha\gamma} + g_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma}, \tag{17}$$

in which the $\Gamma^{\alpha}_{\beta\gamma}$ are the components of the affine connection of the given space. To solve this problem we consider the following sequence of integrability conditions of (17), namely,

$$g_{\sigma\beta}B^{\sigma}_{\alpha\gamma\delta} + g_{\alpha\sigma}B^{\sigma}_{\beta\gamma\delta} = 0;\ g_{\sigma\beta}B^{\sigma}_{\alpha\gamma\delta,\epsilon} + g_{\alpha\sigma}B^{\sigma}_{\beta\gamma\delta,\epsilon} = 0;\ \cdots .$$

Suppose that (a) there exists an integer M such that the first M sets of equations of the above sequence are algebraically consistent as equations for the determination of the symmetric quantities $g_{\alpha\beta}$ in a neighborhood N of the affinely connected space, (b) all the above solutions satisfy the $(M+1)$th set of equations of the sequence, and (c) there exists one solution $g_{\alpha\beta}(x)$ of the first M sets of equations defined in N such that the form $g_{\alpha\beta}(x)dx^\alpha dx^\beta$ is positive definite. The following theorem may be proved: *A necessary and sufficient condition for the affinely connected space to reduce to a Riemann space in some neighborhood $N' \subset N$ is that the above conditions* (a), (b), *and* (c) *be satisfied.* Since the maximum value which it is necessary to take for the integer M is easily seen to be $n(n+1)/2$, the above theorem gives the conditions for the local reducibility of an affine space of paths for which the components of the affine connection are of class C^r with $r \geqq n(n+1)/2+1$. A theorem of this type giving sufficient conditions only for the reducibility was first proved by Eisenhart and Veblen [11]. The basis of the proof of the above stated theorem is the existence theorem on systems of differential equations mentioned in the preceding section. In the usual statement of this theorem the above condition (c) is omitted and this is a rather serious omission since the equations (17) may have solutions and only such solutions $g_{\alpha\beta}(x)$ for which the determinant $|g_{\alpha\beta}(x)|$ vanishes identically. We remark also that the statement of the above theorem has been carefully phrased so as to avoid the difficulties occasioned by possible singular points in the affine space (that is, points analogous to the singular points defined in §19). For a detailed proof in which these questions are considered reference may be made to the *Lectures on Differential Geometry* [45].

18. When one deals with problems of differential geometry having to do with the existence of specified properties of a space, one is invariably confronted with the integration of a system of differential equations. It is sometimes possible to reduce this system—frequently the defining equations of the property in question—to an equivalent system of differential equations exhibiting greater simplicity in certain respects. When this has been done, the reduced system is usually said to furnish a solution of the problem although in no fundamental sense is this correct since these latter conditions are likewise of differential character. Now it can be shown under very general conditions that the question of the existence of a solution of a system of differential equations can be reduced to the question of the existence of a solution of a system of algebraic equations to which can be applied the highly developed theory of algebraic elimination. This procedure of algebraic elimination will lead to a set of conditions involving polynomials in the fundamental structural functions of the space (in case the structure of the space is defined by one or more geometric objects) which will be necessary and possibly sufficient for the existence of the property under consideration. We embody precise types of such conditions

which are of especial interest in the following definition of the algebraic characterization.

We shall say that the conditions $F_1=0$, $F_2\neq 0$, $F_3>0$, $F_4\geqq 0$ constitute an algebraic characterization of a property P of a space, where the F's denote polynomials in the fundamental structural functions of the space and their derivatives to a definite order, provided that these conditions are necessary and sufficient for the existence of the property P. A simple example of an algebraic characterization of the type $F_1=0$ is afforded by the equations expressing the vanishing of the curvature tensor of any one of the various generalized spaces for which such tensors have been found, these equations giving in fact necessary and sufficient conditions for the space in question to be locally flat. Other examples only slightly more complicated have been given in a paper by J. Levine and the present writer [46]. In a Riemann space of class one the rank of the matrix of the second fundamental quadratic differential form can be shown to be an intrinsic invariant of the space and this invariant has been called the type number of the space. As an intrinsic invariant the type number of a Riemann space is definable regardless of whether the space is of class one or not. It has been shown by Thomas [42] that an $n(\geqq 3)$ dimensional Riemann space of class C^3 and type number $\tau\geqq 3$ is of class one if, and only if, conditions of the form $F_1=0$, $F_3>0$, and $F_4\geqq 0$ are satisfied, the inequalities expressing the conditions of reality involved in the solution of the problem. This algebraic characterization of Riemann spaces of class one has been extended to Riemann spaces of class p by C. B. Allendoerfer [2].

The conditions given in the preceding section for the local reducibility of affinely connected spaces to Riemann spaces constitute what may be called an *algebraic test* of reducibility rather than a true algebraic characterization. As a matter of fact it can be proved that there exists no algebraic characterization of the class of affinely connected spaces which reduce to Riemann spaces [41, 43]. These facts, namely, the existence of the algebraic test and non-existence of the algebraic characterization, suggest the desirability of a critical discussion of the possible types of conditions for the various properties of spaces that have been treated.

It is therefore of significance to inquire concerning the existence or non-existence of an algebraic characterization for any specified property P of a space and this question gives rise to a host of interesting and difficult problems in differential geometry. For example, one may consider the question of the algebraic characterization of Riemann spaces of class one without the above supplementary condition on the type number. One might also consider the question of the existence of an algebraic characterization of Riemann spaces admitting an n-tuply orthogonal system of hypersurfaces [47].

19. Since the structure of an affinely connected space is determined

essentially by the definition of parallel displacement a fundamental class of problems in the theory of this space has to do with questions concerning the parallel displacement of vectors. We may mention here the well known theorem on the parallel displacement of a vector around an infinitesimal closed circuit. This problem was first considered by J. A. Schouten [34] and Schouten's work was later extended and perfected by H. Weyl [56].

A field of vectors in a domain D of an affinely connected space is said to be *parallel* if the vectors of the field are parallel in the sense of §7 with respect to any curve of class C^1 in D. The necessary and sufficient condition for a field of contravariant vectors $\xi(x)$ to be parallel in the domain D is that the equations

$$\frac{\partial \xi^\alpha}{\partial x^\beta} + L^{\alpha}_{\mu\beta}\xi^{\mu} = 0 \tag{18}$$

be satisfied in this domain. A rather general investigation of the existence of fields of parallel vectors defined over open point sets in an affinely connected space was made by Mayer and Thomas [24] under the hypothesis that the components $L^\alpha_{\beta\gamma}$ (not necessarily symmetric) of the affine connection were of class C^s, where $s \geqq n+1$ and n is the dimensionality of the space. In consequence of this hypothesis the equations (18) admit the following sets of integrability conditions

$$(E_0)\xi^{\mu} B^{\alpha}_{\mu\beta\gamma} = 0, \quad (E_1)\xi^{\mu} B^{\alpha}_{\mu\beta\gamma,\delta_1} = 0, \cdots, \quad (E_n)\xi^{\mu} B^{\alpha}_{\mu\beta\gamma,\delta_1,\cdots,\delta_n} = 0,$$

where the B's denote the components of the curvature tensor and its successive covariant derivatives. A point P of the space will be said to be regular (called regular or non-essential singular in above paper) if there exists a neighborhood $U(P)$ in which the rank of the matrix M formed from the coefficients B of the system $(E_0)+\cdots+(E_{n-1})$ is constant. Other points of the space will be said to be singular (called essential singular in above paper). It can be shown that the set R of regular points is open and the set S of singular points is nowhere dense. By the component of a point $P \subset R$ is meant the greatest open connected point set $K(P)$ in R which contains the point P. Thus the set R is divided into a finite or infinite number of components $K(P)$ with boundaries composed of singular points. It can be proved that the rank of the above matrix M is constant in any component $K(P)$ and that this rank determines the number of independent fields of parallel vectors in $K(P)$. In fact, in any connected and simply connected open point set O contained in any component $K(P)$ the parallel displacement of any solution vector ξ of the system $(E_0)+\cdots+(E_{n-1})$ at a point Q to any other point Q' of O will be independent of the path of the displacement and hence will give rise to a field of parallel vectors in O.

A necessary condition for the existence of a field of parallel vectors

$\xi(x)$ over the above affinely connected space $\mathfrak{M}$ is that the system $(E_0) + \cdots + (E_{n-1})$ shall possess a non-trivial solution at any point of $\mathfrak{M}$ and this condition can be expressed by the vanishing of the resultant system $\mathcal{R}$ of $(E_0) + \cdots + (E_{n-1})$ over $\mathfrak{M}$. If now, conversely, $\mathcal{R}=0$ over $\mathfrak{M}$, a field of parallel vectors will exist in the open point sets O in any component $K(P)$. In particular, if all the components $K(P)$ are simply connected, a field of parallel vectors will exist in each of these components, but it may not be possible to choose these fields so that discontinuities will not arise at the singular points in $\mathfrak{M}$, that is, at the boundaries of the various components $K(P)$. Whether or not the space $\mathfrak{M}$ is itself simply connected appears to be without especial significance in this connection. Here arises one of the essential differences in the problem of characterizing spaces admitting a field of parallel vectors under the non-analytic and analytic hypotheses. For in the analytic case the equations $\mathcal{R}=0$ give an algebraic characterization of those (topologically) simply connected spaces $\mathfrak{M}$ over which a field of parallel vectors can be defined [44]. In a non-analytic affinely connected space the various components $K(P)$ play the same role as that of the entire space under the analytic hypothesis.

It is known that if p independent fields of parallel vectors can be defined in a neighborhood of a Riemann space, it is possible to introduce in this neighborhood a system of coordinates with respect to which the fundamental quadratic differential form becomes $(dx^1)^2 + \cdots + (dx^p)^2 + g_{\mu\nu}dx^\mu dx^\nu$, where the last set of terms is independent of the variables $x^1, \cdots, x^p$. This result admits a natural extension when the fields of parallel vectors are replaced by the more general concept of the parallel vector space as has been shown by W. Mayer [10]. The corresponding reduction of the fundamental form of a pseudo-Riemann space in which null parallel vector fields may arise has recently been investigated by L. P. Eisenhart [13].

20. This report would not be complete without some mention of the theory of the tangent or associated spaces which can be defined at the points of a manifold of class $C^u (u \neq 0)$. Consider the totality of contravariant vectors ξ at any point P of such a manifold. These vectors, regarded as abstract objects, constitute the points of a space called the tangent space to the manifold at the point P. We can introduce a coordinate system in this tangent space in the following manner. If $U(P)$ is a coordinate neighborhood of the manifold containing the point P, we define the components ξ^α with respect to the coordinates of $U(P)$ of any contravariant vector ξ at P to be the coordinates of the corresponding point ξ of the tangent space. As so defined, a coordinate system in the tangent space is determined to within a linear homogeneous transformation. With these coordinate systems as preferred coordinate systems the axioms A of §4 are satisfied, where G is the group of linear homogeneous transformation, that is, the tangent space is a *centered* affine space, the center being determined by the zero vector at P. This tangent space may be thought

of as having contact with the manifold at P, the center of the tangent space coinciding with the point P of the manifold. Any centered affine space determines uniquely an affine space in which the center is regarded as equivalent to any other point. The affine space determined in this way by the above tangent centered affine space at any point P of the manifold is called the tangent affine space at P.

The definition of parallel displacement in a (topologically connected) space with affine connection L determines a mapping of the tangent affine space at any point P on the tangent affine space at any other point Q. Join P to Q by a curve $C(t)$ of class C^1 where $t_0 \leqq t \leqq t_1$ with $P = C(t_0)$ and $Q = C(t_1)$. By the existence theorem for systems of differential equations, the general solution of (1) along the curve C has the form $\xi^\alpha = a_\beta{}^\alpha(t, t_0)\xi^\beta(t_0)$, where the $\xi^\beta(t_0)$ are arbitrary constants, $a_\beta{}^\alpha(t_0, t_0) = \delta_\beta{}^\alpha$ and the determinant $|a_\beta{}^\alpha(t, t_0)| \neq 0$ for $t_0 \leqq t \leqq t_1$. It is easily seen that these equations define a geometrical or point correspondence between the tangent affine space at P, referred to coordinates $\xi^\beta(t_0)$, and the tangent affine space at Q, referred to coordinates $\xi^\alpha(t_1)$, and that this correspondence is independent of the parametrization of the curve C. By this correspondence, the zero vector in the tangent space at P is carried into the zero vector in the tangent space at Q. But this property may be lost under a more general definition of the mapping between tangent spaces. For example, this will be the case if we determine the correspondence between tangent spaces by integration of the system of equations obtained by adding the quantities $B_\beta{}^\alpha dx^\beta/dt$ to the left-hand members of (1) as we may do in a space whose structure is defined by a composite geometric object consisting of an affine connection L and a mixed tensor B of the second order. Various modifications can be made in the system of differential equations used to define the correspondences between tangent spaces, and some of these may be worthy of serious study [54]. This process of mapping tangent spaces on one another may be expected to apply whenever the underlying space contains a geometric object having the general properties of an affine connection. In particular, it has been used in the theory of the generalized projective space by Veblen [52].

A general theory of the correspondences between associated spaces may be based on the following abstract formulation: With each point P of a manifold of class C^u there is associated a space $S(P)$, all these spaces being isomorphic. By a family of correspondences we shall mean a set of transformations $S(P) \rightarrow S(Q)$, such that:

(a) Any correspondence $S(P) \rightarrow S(Q)$ is an isomorphic transformation of $S(P)$ into $S(Q)$;

(b) If P and Q are any two points of the manifold, there exists at least one correspondence $S(P) \rightarrow S(Q)$;

(c) The resultant of a correspondence $S(P) \rightarrow S(Q)$ followed by a correspondence $S(Q) \rightarrow S(R)$ is a correspondence $S(P) \rightarrow S(R)$;

(d) The inverse of any correspondence $S(P) \rightarrow S(Q)$ is a correspondence $S(Q) \rightarrow S(P)$.

A family of correspondences is therefore a pseudo-group of isomorphisms between associated spaces. If P is any point in the manifold, the set of correspondences which carries $S(P)$ into itself is obviously a group contained in this pseudo-group. It is a sub-group of the group of automorphisms of $S(P)$ and is called the *holonomic* group at P [6]. It is easily seen that the holonomic groups at any two points P and Q are (simply) isomorphic. Hence, if the holonomic group reduces to the identity at any point, it reduces to the identity at every point of the manifold. In such a case the pseudo-group of correspondences between the associated spaces is said to be *holonomic*. The pseudo-group of correspondences is necessarily holonomic if the group of automorphisms of $S(P)$ is the identity. For a further discussion of this general theory see Veblen and Whitehead [54].

E. Cartan has attempted, in a succession of papers extending from 1923 to 1937, to use the associated spaces and their correspondences in a definition of generalized affine, conformal, and projective geometries. Apparently his procedures require some sort of relation between the associated spaces and the underlying manifold and the lack of such a relation has been referred to by H. Weyl as a blemish in the theory [58]. Cartan has likewise failed to set up an isomorphic correspondence between the pseudo-group of coordinate transformations relative to coordinate neighborhoods $U(P)$, with P fixed, and the group of coordinate transformations in the associated space $S(P)$ in consequence of which any coordinate neighborhood $U(P)$ determines a unique coordinate system in $S(P)$. Such an isomorphic correspondence was established at the beginning of this section for the case of the tangent vector spaces and its importance has been emphasized by H. P. Robertson [32] and H. Weyl [33]. Briefly stated, the general problem with which one is here concerned is the problem of finding a geometric object defined in a manifold of class C^u which can be used to establish a pseudo-group of isomorphic correspondences between a given set of isomorphic associated spaces A by the process of integrating a system of differential equations along curves in the manifold. The space S whose structure is defined by this geometric object may be thought of as constituting a generalization of the associated spaces A if, in the holonomic case, S is isomorphic to A. I am not sure whether this is an accurate description of the point of view which Cartan has adopted in his treatment of generalized spaces and so it is best that I present this formulation of the problem on my own responsibility.

21. There is an interesting and difficult class of problems which have to do with the relation between the structure of a space and its topology. One of these is the Clifford-Klein space problem. Historically, this problem had its beginning in the discovery by Clifford in 1873 of a surface having zero curvature in three dimensional elliptic space. Klein in 1890 drew

attention to Clifford's result and formulated the general problem of the topology of Riemann spaces of constant curvature K. This problem has since come to be known as the Clifford-Klein space problem and was studied by Killing in his book *Grundlagen der Geometrie* (1893) where it was shown that the determination of all Clifford-Klein space forms can be carried back to the determination of the discontinuous groups of motions without fixed points of the euclidean, spherical and hyperbolic spaces. A recent investigation of this problem from the standpoint of modern topological theory has been made by H. Hopf [15].

As a significant concept in the investigation of Riemann spaces in the large, H. Hopf and W. Rinow [17] defined the non-continuable analytic Riemann space and this concept has an immediate extension to the case of any generalized space whose structure is determined by an analytic geometric object. An analytic Riemann space S is said to be continuable if there exists an analytic Riemann space S' such that S is isomorphic to a proper part of S'; otherwise S is non-continuable. Evidently the continuability or non-continuability of a space S is an intrinsic property of this space, although Hopf and Rinow did not succeed in finding an intrinsic characterization of this property. However, any one of the following four intrinsic conditions is sufficient for the space S to be non-continuable and each of these can be proved to be equivalent to the other three conditions:

(a) Every geodesic can be continued to infinite length.

(b) Every divergent line is infinitely long (a single valued continuous image in S of a euclidean ray is called a divergent line in S if to every divergent sequence of points on the ray corresponds a divergent sequence of points in S).

(c) Every Cauchy sequence converges.

(d) Every bounded set of points has a limit point.

An analytic Riemann space S satisfying any one of the above conditions is said to be *complete*. It can be shown, for example, that any two points of a complete space can be joined by a curve of shortest length (geodesic) and this property does not persist in non-continuable Riemann spaces. The complete spaces constitute a rather large subclass of the non-continuable Riemann spaces which is worthy of separate investigation.

The general problem of the relation between the metric structure and the topology of an analytic Riemann space was divided by H. Hopf [16] into two separate problems, namely, the *metrization* problem and the *continuation* problem (evidently corresponding problems can be framed for any generalized space with structure defined by an analytic geometric object). The metrization problem has to do with the extent to which the metric structure is determined or conditioned by the underlying topological manifold. In a sense, the metrization problem is the converse of the continuation problem since in the latter, one is concerned with the question of the possibility of continuing an arithmetic n-cell in which an analytic positive definite quadratic differential form (Riemann element) is defined to a non-continuable or more particularly a complete space. Going out from this general point of view, Hopf has proved a number of very interesting special results and has, moreover, mentioned as many problems for investigation, some of which have recently been solved by S. B. Myers [27]. A fundamental result in the theory of spatial continuation has been proved by W. Rinow [31], namely, that two simply connected and complete spaces S and S', each of which is the continuation of the same Riemann element, are isomorphic (isometric).

A number of new results have recently been obtained by S. B. Myers and J. H. C. Whitehead on the minimum point locus with respect to an arbitrary point P of a Riemann or Finsler space. A point M on a geodesic ray g issuing from P is said to be a minimum point with respect to P on g if M is the last point on g such that PM furnishes an absolute minimum to the arc length of curves joining P to M. This locus was originally introduced by Poincaré [29] who considered it only for closed simply connected surfaces of positive curvature. Whitehead [62] treats this locus, which he calls a cut locus, for an n-dimensional Finsler space subject to a completeness condition. He obtains the theorem that such a space can be decomposed into an n-cell and the cut locus which forms the singular boundary of the n-cell. A study of the topology of the minimum locus was made by Myers [28] for two-dimensional compact Riemann spaces. Myers showed that in the case of the analytic Riemann space the minimum locus is a linear graph and obtained certain additional properties of this locus. He also made a brief study of the minimum locus for non-analytic Riemann spaces and showed that for such a space this locus turns out to be a continuous curve (not necessarily a linear graph) under the assumption that the space is compact and possesses certain properties of regularity. One could proceed quite naturally from these results to many others which are usually treated under the heading of the calculus of variations.

It is known that over certain types of topological manifolds it is not possible to have a continuous vector field without singular points. For example, Brouwer [3] and J. W. Alexander [1] have shown that such a field cannot be defined over an n-dimensional spherical surface if n is an even integer. It follows from this fact and the theorem on the algebraic characterization of spaces admitting a field of parallel vectors in §19 that it is not possible to define an affine connection L with analytic components over a spherical surface of even dimensionality such that the conditions $\mathfrak{R}=0$ are satisfied; in particular the affine connection cannot be such that the curvature tensor will vanish at all points of this surface. Quite generally, one may consider the topological character of any generalized space over which is satisfied a specified system of invariant differential equations in the fundamental structural functions of the space (components of the geometric object defining the space). One extreme of this class of problems is reached when no such structural conditions are imposed on the space. Here one must expect little or no restriction on the underlying topological manifold as indicated by the researches of H. Whitney (§2). At the other extreme the structural conditions are so restrictive as to characterize the space locally to within an isomorphism. As an example of this extreme we may mention the Riemann space of constant curvature (Clifford-Klein space problem). Between these two extremes there is a large class of exceptionally difficult problems. A particularly interesting problem belonging to this last category and one which is, in fact, a direct extension of the Clifford-Klein space problem is the problem of the topological nature of Riemann spaces of constant *mean* curvature. In this connection one might investigate a question corresponding to Hopf's metrization problem with supplementary conditions; namely, to what extent is the metric of a Riemann space of constant mean curvature zero and dimensionality $n \geqq 4$ determined by the requirement that its manifold be homeomorphic to the n-dimensional number space and that it be flat at infinity? The euclidean metric space satisfies the above conditions and, although it seems likely that the euclidean metric space is the only such Riemann space of constant mean curvature zero, this has never been established. With regard to the general problem a result has been proved by Thomas [40] which can roughly be described by saying that a compact Riemann space S of positive constant mean curvature, but not of constant curvature, cannot differ infinitely little in its metric relationships from a space of constant curvature S' when the two spaces S and S' are considered in their entirety. In other words, the transition from S to S' corresponds to a jump or discontinuous process. While this is a rather special result it would appear to indicate that a thorough investigation of Riemann spaces of constant mean curvature in the large would lead to some interesting discoveries.

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