THE HISTORICAL BACKGROUND OF HARMONIC ANALYSIS

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This address deals with certain aspects of the theory of Fourier series and integrals during the nineteenth century. The subject of Fourier series has been a central one both in the region of pure mathematical analysis and in mathematical physics. It is just because this subject faces two ways that its history is of especial interest. At each period in the history of mathematics, the relation between mathematics and physics has been viewed from an aspect somewhat peculiar to the period; and in tracing the history of this subject, we shall have in miniature a history of the larger relations between the two disciplines.

The first beginnings of Fourier series theory may be regarded as very ancient, for the musical theories of Pythagoras have already in them elements which later investigation has shown to be of trigonometric nature. The whole classical theory of cycles and epicycles as found in Ptolemaic astronomy is also something which a modern point of view would interpret as harmonic analysis. These two motives, that of the acoustical-optical wave theory, and that of astronomical and geophysical periodicities, united by Plato in the music of the spheres, have ever since furnished the chief stimuli of the natural sciences to the study of harmonic analysis. The motions of the planets, the tides, and the irregular recurrences of the weather, with their hidden periodicities, form a counterpart to the vibrating string and the phenomena of light.

However, it is probably a mere picturesque feat of the imagination to push harmonic analysis further back than Huygens. In Huygens' principle, which by the way was not correctly used by Huygens himself, we resolve a wave front into a set of centers of instantaneous disturbances, and by continuing these disturbances over a small interval of time we are in some way able to determine the new wave front.

While we can not extract from Huygens the Huygens principle in rigorous modern form, we at least find in his work a preoccupation with phenomena continuous over space and with their transportation from point to point by a process of expansion. It is not fanciful to see an extension of this point of view in the philosophy of Leibniz, who is under a heavy intellectual debt to Huygens, here and elsewhere. Leibniz regards space as a plenum, and considers that disturbances are transported from one region to another by a point to point transfer of energy, and not by an actual action at a distance.

The tool for the discussion of such plena was also formulated by Leibniz, and is the differential calculus. It was indeed the Leibnizian school, rather than Leibniz himself, which developed the theory of partial differential equations, or, as we should put it in the jargon of the present day, field theory. In this we must mention above all other names those of the Bernoulli family and of Euler. It is with Clairaut, d'Alembert, and Daniel Bernoulli that trigonometric expansions in the strict sense first come into prominence. Clairaut introduced them in the astronomical theory of the perturbing function, but their most famous application lies elsewhere. Among the simplest of partial differential equations is that of the vibrating string. It is one-dimensional and conserves energy, and it was observed by d'Alembert and Euler to possess a solution in the form of the pulse moving with a certain fixed velocity either to the right or to the left. This holds for the infinite string; in the case of the finite string the fastening at the ends serves to reflect an arriving pulse either with or without reversal of phase according to the nature of the constraint, and send it back along the line to meet and cross a similar pulse reflected from the other end. The solution of the string problem, given the initial displacement in velocity, is thus possible by means of extraordinary simplicity.

Besides the wave solution of the problem of the vibrating string, it was early seen that another solution of what may be called a vibration type is possible. Instead of looking for solutions whose geometrical character remains unchanged by bodily motion along the string, we look for solutions which remain unchanged except in scale by the passage of time. These represent what are called stationary vibrations of the string. Their analytic representation is that of a function of position alone multiplied by a function of time alone. Both functions are trigonometric in character. It will be found that neither the one nor the other is completely arbitrary, but on the other hand, that each one can assume only one of a discrete set of admissible frequencies, the so-called characteristic frequencies of the string.

The problem thus arises of synthesizing a solution of general type from these particular solutions. As the differential equation is linear, the natural means of synthesis is addition. It was realized very early—by Daniel Bernoulli, in 1753, for instance, as well as indirectly by d'Alembert—that solutions additively compounded from stationary solutions have a high degree of generality, and that they may be used in many cases interchangeably with the formally simpler wave solutions of the string problem.

However, the precise relations between these two types of solution still stood in need of elucidation. Series developments of particular numbers date well back into the seventeenth century, and the classical series development of a function—the power series of Taylor and Maclaurin—was known to Newton early in the eighteenth century. It was rediscovered about 1715 and soon was added to the repertory of any working mathematician. It is not surprising that certain highly special features of that development were taken as typical for series developments in general. In particular, the coefficients of a Taylor series are all obtained by successive differentiation in the neighborhood of a single point. It is thus a part of the very nature of such an expansion that it can represent only a function which is determined everywhere by its behavior in any single such neighborhood. The ordinary elementary functions of algebra and trigonometry, together with the logarithm and its inverse, the exponential function, are of this type. It was only natural that many mathematicians should expect that the series of stationary vibrations arising from the string problem might display a similar property.

On the other hand, the wave solution of the string problem is manifestly not confined to solutions with such a property of analyticity. A disturbance of any form, at least if it is smooth enough to have a slope, may be transported bodily along a string, reflected at the ends, transported back again, and so on; and the motion thus obtained is sufficiently perspicuous to make the verification of the fact that it satisfies the differential equation of the vibrating string a matter of trivial difficulty.

The early days of the Bernoulli vibration solution of the string problem were naturally stormy, even though neither Euler nor d'Alembert fully realized the degree of arbitrariness of the solution of the string problem which they had given, and in particular, did not believe the functions entering into their problem might coincide over one region and differ over another; they were not ready to believe that Bernoulli series could represent a motion of the degree of irregularity which fitted into their own solutions. Obviously no thoroughly satisfactory answer to this dilemma was possible until it was realized just how the coefficients of Bernoulli's trigonometric series depended on the functions represented. Lagrange made an attempt to arrive at this representation by a study of the vibrating string with equal concentrated masses distributed uniformly along it, and then proceeded to the limit as the points became sown more and more thickly. But his limit argument left much to be desired in rigor. The final formal representation of the coefficients of a trigonometric series is due to Fourier and, possibly independently, to Laplace.

Fourier's problem was closely analogous to that of the vibrating string, being, like it, the study of a partial differential equation with given initial conditions, although his equation was that of the flow of heat and was what we would now call of the parabolic type, as opposed to the equation of the string, which is of the hyperbolic type. Fourier represented his initial conditions by a trigonometric series in which the coefficients were given as definite integrals. This same representation of a function was employed by Laplace in connection with the use of the method of generating functions in his theory of probability. The moment this explicit representation was given, it became obvious that the method was formally applicable to a function made up of several discrete and independent pieces with angles or jumps between them. Thus a completely new idea of the nature of a function was originated, and perhaps the greatest step was taken in the theory which is now known as that of the functions of real variables.

We have mentioned that Laplace's study of trigonometric series arises in connection with the theory of generating functions. Generating functions were already known to Euler and applied by him to the theory of partitions and other similar combinatory problems. The basis of this whole theory is the formula which derives the coefficients of the product of two Taylor series from the coefficients of each series taken separately. It will be seen that the product coefficient is the sum of products of the coefficients from the two factor series taken in such a way that the sum of the exponents is fixed. Such a sum is very nearly what is known in modern terminology as the convolution or "Faltung" of the coefficients of the two series. This convolution is merely changed by a relabeling of its terms when one of the series is changed by multiplication by an integral power of a parametric variable. From another point of view, if we translate the coefficients of one series as a whole with respect to the coefficients of the other, we merely introduce a translation of the coefficients of the product series.

If we have a number of independent phenomena which combine additively, the distribution of the quantity representing the combined phenomena is derived by the successive convolution of the distribution functions of the components. This is the reason for introducing generating functions into the theory of probability. These functions represent a vital part of Laplace's contribution to the theory and dominate the theory at the present time, although the variables which we raise to a power in the series are generally now taken to be complex. In the light of this interpretation of the series a statement in a recent edition of the Encyclopedia Britannica, to the effect that Laplace's method of generating functions is obsolete, is not justifiable.

While the method of generating functions can be formulated for real Taylor or Laurent series, the problem of determining the coefficients of such a series is not physically natural enough to make this the most effective way of introducing generating functions. Laplace already knew that generating functions could be treated very effectively in the case in which the parametric variable is of absolute value 1. He was also aware of the advantages of replacing a discrete set of powers of the parametric variable combined in an ordinary sum by a continuum of such values combined in an integral. The so-called Laplace transform and its complex analogue now called the Fourier transform were both known to Laplace as well as to Fourier, who used them in connection with the problem of the conduction of heat in the half-plane, and so on.

Thus, at the very beginning of the theory of Fourier series, ideas belonging to a large number of related fields made their appearance. We have vibration problems, problems for the theory of the conduction of heat, combinatorial and other theory of number problems, and problems of probability. Nevertheless, the orthodox line of development of Fourier series theory went through the work of Fourier and, in so far as it was practically applied, it was chiefly to the boundary-value problems of the theory of partial differential equations as they occur in mathematical physics. Fourier himself, following along lines already laid out by Laplace, studied developments in spherical harmonics and Bessel functions and laid the groundwork for the general theory of orthogonal functions. The immediate followers of Fourier, such as Poisson, Sturm, Liouville, on the one hand, and Hamilton and Dirichlet on the other, partly devoted their efforts to a further exploration of this direction and partly to a more rigorous justification of the formulas which Fourier had attacked in a rough heuristic manner. This was a period when the exuberant fertility of the eighteenth century had already begun to give way to a more critical spirit, and when the influence of Cauchy had led mathematicians to make a much more careful examination of the convergence of their expansions. Thus it was no more than natural that an overwhelming part of the efforts of the students of Fourier series during the last century should have been devoted to questions of convergence.

We have already mentioned Cauchy. Besides his indirect influence on the theory of Fourier series from the point of view of rigor, his study of functions of a complex variable led to a better understanding of the relation between the Taylor series and the differential method of determining its coefficients on the one hand, and the Fourier series and the integral method of determining its coefficients on the other. It was seen that these two theories, widely discrepant as they seem to be when we remain in the real domain, form part of the same theory in the complex plane. The real part of a function of a complex variable is a harmonic function; this fact also led to a study of the relation between Fourier series and harmonic functions. This, it is true, may equally well be studied from the point of view of Fourier himself and the boundary-value problem on the circle. Both of these theories have a definite relation to the work of Poisson.

At the same time that the Fourier series theory was receiving this development either from the point of view of pure mathematics or from that of a rather rigidly formalized branch of applied mathematics in the case of boundary-value problems, other physical problems were familiarizing the physicists with the notion of trigonometric developments in a totally different direction. The old string problem received an enormously broad extension with the increasing popularity of wave theories, both in the domain of sound and in that of optics. This was particularly due to Young and to Fresnel. These theories borrowed much from the older string theory, but in view of the greater number of dimensions to which they applied, went far beyond it. The idea of white light as the additive combination of monochromatic vibrations was, of course, along the same lines as the theory of the Fourier integral, but there seems to be no evidence that any of the earlier writers saw that it demanded far more than had been developed explicitly by Fourier and Laplace.

To this same order of ideas belong questions of coherency and interference, of the different types of polarization, and so on. The astronomical motive in Fourier analysis, which had already appeared in the formal investigations of stability by Laplace, Lagrange, and Poisson, became the foundation of a new lunar theory in the hands of Hill and Brown. In all of this work, one can only wonder at the ability of the physicist working with the crudest of mathematical tools and total lack of rigor, to develop an intuition which, almost without exception, saves him from the apparently inevitable consequences of his mathematical sins.

It is a real tragedy of the mathematics of the nineteenth century that, on the one hand, its physical stimulus was continually schematized, reduced in breadth, thrown into the background, or even forgotten, while on the other the experimental work of the physicists gradually sundered itself from the fertilizing influence of the development of mathematical theory. For a long period the theory of Fourier series tended to contract itself to the scope of a mathematical game whose applications, whatever they might have been, were regarded as of little interest outside of mathematics. On the other hand, the harmonic analysis of the physicist gropingly developed its own technique as if the physicist had believed that the mathematical difficulties of principle had all been swept away somewhere obscurely in the literature by the pure mathematician, and that all he needed to do was to recreate crudely his own working formulas.

Within the limited field which the mathematicians have taken for their own, the first efforts of Poisson, Dirichlet, and their followers were directed to the purpose of finding some field, no matter how limited, in which the theory of the Fourier development could be carried out rigorously. The theory of functions of limited total variation and of uniform convergence are to a large extent by-products of this effort. Dirichlet's conditions represent the first important satisfactory achievement of the sort. The works of Riemann on the unicity of trigonometric developments attach themselves closely to this order of ideas, and foreshadow in certain respects the later theory of summable series.

On the other hand, once Fourier developments had been established on a sound basis in any region, no matter how narrow, mathematicians took upon themselves the problem of delimiting the exact bounds of this region. Inasmuch as the Fourier coefficients are defined by integration, this demanded a reinvestigation and a revision of the notions of measure and integral. It was soon seen that functions much more general than those of limited total variation could be given at least formal Fourier development, and what is much more important, that sets of coefficients much more general than those of functions satisfying the Dirichlet conditions determined, in some sense or other, the Fourier series of well defined functions.

In the course of time these activities led to numerous by-products. Parseval's relation between the sum of the squares of the coefficients of a Fourier series and the integral of the square of the function represented was a powerful stimulus to the investigation of the general question: under what circumstances can a function be determined by an arbitrary set of coefficients such that the sum of their squares converges. Far more recently, Hurwitz also studied this region of ideas. Again, the theory of the summability of series owes much of the interest which it has excited to its Fourier series applications. Here the work of Fejér is to be mentioned as the fountainhead of an important school of research. It became quite clear that the convergence theory, useful as it was in giving rigor to a limited region in Fourier analysis, was quite incapable of delimiting a sharp and well defined region in the theory of such series, and a great deal of Euler's work which had been buried under the wave of rigorism in the time of Cauchy was brought back to the light of day. We must mention Borel as one of the pioneers in this direction, as in so many that have helped to form the modern theory of harmonic analysis.

The ultimate triumph of this movement is to be found in the Riesz-Fischer theorem. Before we come to this theorem in detail, however, it is necessary to go back to the domain of mathematical physics and see what some of the leading ideas of statistical mechanics were in the last half of the nineteenth century. We have already seen the early connection of Fourier analysis and statistical theory in the work of Laplace. This connection is intimate and reappears on several different levels. Besides Laplace's problem of the composition of probabilities, the theory of light brought in the germs of a statistical theory of harmonic analysis itself. No one has ever seen an oscillograph of a ray of visible light. The evidence that this light is of a wave character is and can be only such evidence as is applicable to a statistical assemblage of vibrations, rather than to a single pure vibration. Thus the founders of wave optics were driven despite themselves, to invent a statistical harmonic analysis. Young, Fresnel, Goüy, and Schuster all did important work in this direction, as well as in the related field of partially polarized light. In the hands of Schuster, this range of ideas, together with ideas from meteorology and astronomy, led to the discovery of the periodogram, a theory for which the pure mathematical basis was defective for many decades. It was only in 1924 that a paper by Wiener established the precise connection between this and the rigorous modern theory of the Fourier integral. Another spectrum theory, earlier than that of Wiener and covering radically different aspects of singular distributions, was developed by Hellinger in order to analyze the unitary invariants of Hermitian transformations of Hilbert space.

Besides these two relations between harmonic analysis and statistics, the autonomous development of the theory of statistical mechanics has introduced yet a third, and this the most important. Statistical mechanics had its inception in Maxwell's theory of the distribution of the velocities of the particles in a gas. Its more mature form is due to Gibbs and concerns, instead of the distribution of quantities attached to the different particles of a single system, the distribution of all systems where the laws of force are given and only the initial conditions are allowed to vary. In any case, statistical mechanics makes no assertion about the individual particle or the individual system, but merely about what will happen except for a fraction of the particles or systems, which is so small that in some sense or other it may be neglected. If the Maxwell gas is taken as filling an infinite space, or if the Gibbs ensemble of systems is really taken as complete, these sets of negligible likelihood must be taken as in some sense or other of zero likelihood. We thus need for the complete justification of statistical mechanics a rigorous theory of measure in which sets of zero measure have a legitimate place. It is, of course, clear that a theory of sets of zero measure in rigorous form was not reached in a day. The fundamental assumption of Gibbs, for example, that an ensemble of dynamic systems in some way traces in the course of time a distribution of parameters which is identical with the distribution of parameters of all systems at a given time, was first stated in a form not merely inadequate, but impossible. This is the famous ergodic hypothesis. It is one of the greatest triumphs of recent mathematics in America, or elsewhere, that the correct formulation of the ergodic hypothesis and the proof of the theorem on which it depends have both been found by the elder Birkhoff of Harvard.

However, long before a really adequate formulation of the ergodic hypothesis was found, the notion of "almost all" had become an accepted part of the equipment of every physicist. It was under the influence of ideas belonging to this domain that Poincaré at the end of the last century developed the philosophy of questions in the theory of probability which marked the first really great progress in that theory since the days of Laplace. The ideas of statistical randomness and phenomena of zero probability were current among the physicists and mathematicians in Paris around 1900, and it was in a medium heavily ionized by these ideas that Borel and Lebesgue solved the mathematical problem of measure.

On the mathematical side, the problem of measure was absolutely vital to the foundation of a symmetrical theory of Fourier series. The formulas connecting a function and its Fourier coefficients are so symmetrical that it was early obvious that the problem of determining the coefficients of a function and the problem of determining a function from its coefficients were closely parallel. In any earlier theories of measure and integration, on the other hand, the problem of proceeding to a function from a set of coefficients enjoying some simple and very general property had no natural solution. The question of putting the theory of integration on a broad enough basis to allow a solution of these problems was attacked simultaneously by many mathematicians in many places. Besides Borel and Lebesgue in Paris and W. H. Young in England, we must mention the anticipatory work of Osgood and Pierpont in America which, if it did not lead completely to the goal of the new integration theory, at least did much to point to the direction in which that goal would be found. After the Lebesgue integral had been discovered, Daniell gave it an abstract form independent of dimensionality.

The Lebesgue theory of measure and integration produced an immediate and explosive expansion in the theory of harmonic analysis. The chief result was the Riesz-Fischer theorem, to the effect that there exists a perfect equivalence between the class of Lebesgue measurable functions with Lebesgue integrable square moduli, and the class of functions which can be represented by Fourier series for which the sum of the squares of the moduli of the Fourier coefficients converges. It is this theory which gives the motivation of the infinitely-many dimensional space of Hilbert. To this realm of ideas belong Weyl's lemma, to the effect that a sequence of functions of this class converging in the mean converges in the mean to a limit defined almost everywhere, and the Plancherel theorem as to the existence of Fourier transforms for an analogous class.

Both in the case of Fourier series and in that of the Fourier integral, there is an L^p theory analogous to the L^2 theory but less perfect. Here the chief names are Hölder, Hausdorff, Minkowski, Titchmarsh, and Paley.

Another block of related theorems includes the Lebesgue theorem that an integrable function is almost everywhere the derivative of its integral, and Lebesgue's other theorem that the Fourier series of an integrable function is almost everywhere Cesàro summable of the first order to the function. The whole theory of the convergence and summability of Fourier series was recast by a group of mathematicians working largely in England, among whom must be mentioned W. H. Young, Hobson, Hardy, Littlewood, and Chapman, together with the brothers Riesz, Szász, and Zygmund on the continent. The theory of summability in connection with trigonometric series, while anticipated by Abel and Poisson, was first consciously employed by Riemann. The study of convergence factors and exponents of convergence for Fourier series and even for general orthogonal series developed a literature of its own, particularly in the hands of the Polish school. Although no natural necessary and sufficient condition for the convergence of a Fourier series was forthcoming, such a condition was found by Hardy and Littlewood for summability of unspecified order.

It is impossible to follow the classical theory of Fourier series into all its ramifications, nor is it any easier to do so with the related theory of the Lebesgue integral. This latter theory, it is important to note, soon established relations with allied statistical theories. The work of Einstein and Smoluchowski on the Brownian motion had led in 1909 to the important investigations of Perrin in Paris, and it was not long before he was consulting his colleague Borel on the matter. Besides his work on measure, Borel was known for his investigations in the theory of probability, and was in close touch with the ideas of Poincaré. Perrin had noticed that the paths of particles subjected to the Brownian motion were suggestive of the nondifferentiable continuous curves of Weierstrass. It is perhaps not too far-fetched to see in this suggestion a strong reinforcement of the motives which led Borel to apply the methods of the theory of probability—that is, of measure—to analytical problems. This is his theory of "probabilités denombrables," later studied by Steinhaus, Rademacher, Wiener, and others. In the hands of Paley and Zygmund, it became the nucleus of a fertile method for the devising of Gegenbeispiele in the theory of Fourier series. Wintner, first by himself and then in collaboration with Jessen, developed a parallel theory which yielded much information on the distribution of the values of the Riemann zeta function and other related functions. The Wiener theory of differential space is a direct development of the mathematical ideas behind the theory of the Brownian motion. It has led to a very general theory of continuous randomness, and is related to the work of Khintchine and Kolmogoroff, and to the theories of stochastic processes developed by these authors, Cramér, Lévy, and others. Through the work of Cramér, ideas of this order have been introduced into certain generalizations of number theory. Lastly, it now seems likely that a systematic study of such notions may lead to a really constructive systematization of the study of random ensembles in statistical mechanics, as, for example, in the theory of turbulence or in that of the statistical mechanics of liquids.

Besides the classical theories of Fourier series and integrals, a number of investigations have been made of trigonometric developments not falling strictly under either of these heads. Non-harmonic Fourier series are at least as old as Fourier, and the Dirichlet series of the student of analytic number theory give rise to trigonometric series, where the frequencies do not form an arithmetic progression. It was these number-theoretic series on the one hand, and the dynamical nonharmonic Fourier series of Bohl, on the other, which led Harold Bohr, in the early twenties, to formulate the notion of the almost periodic function. His original papers became the nucleus for a considerable literature. Bohr's fundamental theorems, to the effect that the Parseval theorem holds in a certain "mean" form for almost periodic functions, and that these functions are precisely those which admit a uniform approximation by trigonometric polynomials, were proved in widely different ways by Weyl, de la Vallée Poussin, and Wiener. The theory of Wiener was sufficiently general in scope to cover periodograms and the theory of white light. The class of almost periodic functions was generalized by Stepanoff, Weyl, Besicovitch, and Wiener. The definition of almost periodic functions was recast by Bochner into the form, "the class of all continuous functions, such that among any infinite sequence of translations of one of them, a uniformly convergent subset can be chosen." Bohr himself studied almost periodic functions in the complex domain, and was followed by Jessen, Favard, and Cameron in the investigation of the distribution of the values of an analytic almost periodic function in strips.

The more general question of the harmonic analysis of the orbits of particles in a dynamical system owes much to the work of Birkhoff. Birkhoff's work represents an essential refinement of the ideas of Poincaré on Poisson stability. Here we must mention the related physical theory of adiabatic invariants, as developed by Ehrenfest and Levi-Civita. Birkhoff's ergodic theory furnishes the necessary *point d'appui* of Wiener's generalized harmonic analysis. Indeed, it was for the purpose of such a harmonic analysis that the ergodic theorem was first applied. A concrete achievement in this direction is the theory of mean planetary motion of Wintner, von Kampen, Weyl, and others. Other forms of generalized harmonic analysis go back to the work of Hahn, and concern themselves with what should now be called the theory of Stieltjes-Fourier transforms being pursued so successfully by Wintner and his collaborators. Other writers who have contributed to the theory of generalized harmonic analysis are Bochner and Berry.

Almost periodic functions are characterized by a certain property of their translations, which is a property invariant under translation. This is related to the fact that the functions e^{iux} are the characteristic functions of the translation group. Harmonic analysis is thus the appropriate tool for the study of linear problems formally invariant under this group. As most physical phenomena are not attached to a fixed origin in time, and as indeed the existence of such an attachment would make it impossible to repeat an experiment, and to develop a consistent physics, harmonic analysis is indicated over a wide domain of practical problems. These group-theoretical aspects of harmonic analysis are foreshadowed in the work of Lagrange and Laplace on ordinary difference and partial differential equations with constant coefficients. The calculus of operations associated with the names of Arbogast and Boole, as well as the work of Heaviside on the electric circuit, represent further developments of this point of view. The example of Heaviside shows how directly it has remained in contact with physical and engineering practice. After Heaviside, obscure and intuitive as he was, an extensive interpretative literature developed, in which the names of Bromwich, Doetsch, Jeffries, and others are to be mentioned. The methods of Heaviside, as well as the alternative approach of Carson, have been pushed into the background by the conscious, explicit use of the Fourier integral, which has been much facilitated by the recent tables of Fourier transforms published by Campbell and Foster, of the Bell Telephone Laboratories.

The considerations which lie back of the operational calculus of Heaviside are clearly of group-theoretical nature, as we have seen. Other groups besides the translation-group possess characteristic functions, and a theory of expansions in terms of these functions. The characteristic functions belonging to the denumerable abelian group, all of whose operators are of period two, are the so-called Walsh functions, and they have as a basis the well known Rademacher functions. The functions of Haar and Franklin, while not characteristic, are closely related. It was a heuristic remark by Paley that the Walsh functions have expansion properties very nearly identical with those of the trigonometric functions, of which they form a good working model. This suggested to Paley and the author the formation of a bridge between the theory of Fourier developments and the theory of group measure of Haar and Pontrjagin. The work of von Neumann on almost periodic functions on groups represents a further development of this line of research, and in particular, the extension to the non-abelian case is an important advance. It has its inspiration in Weyl's approach to the theory of almost periodic functions, and in the earlier work of Hilbert and E. Schmidt.

While the theory of Tauberian theorems is not a Fourier theory in its origins, and while the original method of Hardy and Littlewood has nothing to do with trigonometric considerations, an early use of Tauberian methods was to establish the Poisson approach to Fourier series on a rigorous basis. However, there are other more intimate relations between the Tauberian theorems and harmonic analysis. Many of the most common theorems of Tauberian type or, in other words, inverse theorems concerning limit processes, deal with averages of a function which have a certain sort of invariance with respect to a multiplicative change in scale. The logarithmic transformation reduces this multiplicative change of scale to an additive change, and hence transforms Tauberian problems into a form suitable for the application of the methods of harmonic analysis. I repeat, the base of this application consists in the group-theoretical properties of the multiplication group. The earlier work of Hardy and Littlewood on these problems had no such group-theoretical justification, and it is in a paper of Wiener that a rational basis was found for introducing Fourier methods. Later on the author saw that the key theorem was one concerning the reciprocal of an absolutely convergent Fourier series. With the aid of these methods the solution of Tauberian problems has been reduced to a routine. The author's work was stimulated by the earlier work and personal suggestions of Robert Schmidt. The Toeplitz-Schmidt theory of gestrahlte Mittelbildungen bears the same relation to the author's work that the Laplace integral does to the Fourier transform.

Another closely related piece of work is that of Karamata, which, however, did not have in its inception any Fourier aspect whatever. With the aid of Tauberian theorems the author's pupil Ikehara and he have introduced simplifications into the proof of the prime number theorem, and these simplifications have led to subsequent work by Heilbronn, Landau, and others. This Tauberian work has found direct application in the theory of Fourier series and in other branches of harmonic analysis, and has also been turned by Paley and the author to the study of the behavior of entire functions of exponential type in the complex domain. Here their work joins with that of Pólya, Miss Cartwright, and others, and in its application to the study of gap series, with that of Mandelbrojt. A closely related field of harmonic analysis work is the theory of quasi-analytic functions, where the original introduction of trigonometric methods is due to de la Vallée Poussin, and where Paley, Mandelbrojt, and the author have carried the theory further. The recent very refined work of Levinson develops the theory in all these directions.

It is impossible in the time allotted me to take up all branches of harmonic analysis, and the motives which have played a rôle in forming the theories. The theory of the Laplace integral has been developed by Widder, Doetsch, S. Bernstein, Hille, Tamarkin, and others. In Japan, Izumi and his school have done much noteworthy work. The general theory of Watson transforms is also to be mentioned here. It will be seen that recent work in harmonic analysis is highly international, and that no inconsiderable part of it has been performed in this country.

While the historical facts in any concrete situation rarely point a clear-cut moral, it is worth while noting that the recent fertility of harmonic analysis has followed a refertilization of the field with physical ideas. It is a falsification of the history of mathematics to represent pure mathematics as a self-contained science drawing inspiration from itself alone and morally taking in its own washing. Even the most abstract ideas of the present time have something of a physical history. It is quite a tenable point of view to urge this even in such fields as that of the calculus of assemblages, whose exponents, Cantor and Zermelo, have been deeply interested in problems of statistical mechanics. Not even the influence of this theory on the theory of integration, and indirectly on the theory of Fourier series, is entirely foreign to physics. The somewhat snobbish point of view of the purely abstract mathematician would draw but little support from mathematical history. On the other hand, whenever applied mathematics has been merely a technical employment of methods already traditional and jejune, it has been very poor applied mathematics. The desideratum in mathematical as well as physical work is an attitude which is not indifferent to the extremely instructive nature of actual physical situations, yet which is not dominated by these to the dwarfing and paralyzing of its intellectual originality. Viewed as a whole, the theory of harmonic analysis has a very fine record of this sort. It is not a young theory, but neither is it yet in its dotage. There is much more to be learned and much more to be proved.

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