THE SPHERE IN TOPOLOGY

BY

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INTRODUCTION

Probably no branch of mathematics has experienced a more surprising growth than has, during the past two decades, that field known variously as Topology or Analysis Situs. Originating in the work of many mathematicians of the past century, including Cantor, Riemann, and Kronecker, it won recognition as a distinct branch of mathematics largely through the writings of Poincaré about the beginning of the present century. Although having many ramifications, it has progressively become a unified subject, and due to its foundation in the theory of abstract spaces, has come to collaborate with abstract group theory as a unifying force in mathematics as a whole. It has provided a tool for classification and unification, as well as for extension and generalization, in algebra, analysis, and geometry. Considered as a most specialized and abstract subject in the early 1920's, it is today almost an indispensable equipment for the investigator in modern mathematical theories.

It is with pardonable pride that one can point to the part which American mathematicians have played in this development. Although his name will not appear again in this monograph, we may well ponder how much this was due to that great American mathematician, E. H. Moore, by whose students, particularly R. L. Moore and O. Veblen, the actual beginnings of Topology in this country were made.

The writer was tempted to give the chief rôle in this monograph to the historical aspects of Topology. More careful reflection, based on limitations of space and ideals of coherence, seemed to point to a different task. There exist several modern treatises in Topology, as well as numerous special monographs; and besides these an almost overwhelming number of original papers, many of which are very lengthy. To the average mathematician who would like to gain some acquaintance with the subject, from either cultural or utilitarian motives, the literature probably presents a confusing aspect; he may select two books with identical titles, but with almost no overlapping of material—indeed, a casual glance at their contents may make him wonder if the printer has not erred in imprinting of titles on the jackets. It is perhaps not strange, under these circumstances, that the writer decided to subordinate the historical to the expository. More specifically, it is the intention of this monograph to furnish, first of all, a brief introduction to Topology in all its aspects, abstract, set-theoretic, and combinatorial. There seems no reason, a priori, why such an introduction cannot be given which will at the same time incorporate to some extent, in its development from the simple to the more complex, the various stages of historical progress. This is the key to what the writer has tried to accomplish. The choice of subject, *The sphere in topology*, was dictated by a consideration of what material could best achieve these objectives.

It may as well be frankly stated that what appears herein has been hastily conceived and executed for the purposes of this semicentennial celebration. The writer makes no apologies for what in some sections, may seem a rather naive presentation; nor for errors or omissions unintentionally made. The first three sections are quite elementary, and should be readily handled by a first-year graduate student. The subject matter is necessarily limited, but is at least self-sufficient, and with the accompanying bibliography will, it is hoped, prove a reasonable introduction to the huge literature which it is supposed to represent. The only extraneous material required of the reader is some elementary knowledge of abstract group theory.

No more has been included in Chap. I than is needed in the later development. The axiomatic set-up of Chap. II and the statements of theorems without proof were motivated by the feeling that the doing of mathematics is the only safe road to real appreciation and rigor. Although the material in Chap. III is limited to the 2-sphere, symbols and terminology have been so arranged as to adapt themselves to immediate generalization later on. Chap. V is admittedly a sop to the writer's own predilections in Topology; but possibly we can plead that the justification for inclusion here lies in its nature as a sample of the direction indicated by Schoenflies. Chaps. IV, VI, and VII are intended to develop some of the more recent tendencies, and will possibly give an indication of the present status of Topology.

I. The space concept

1.1. **Set-theoretic remarks.** If M is any collection or *set* of elements, then $x \varepsilon M$ will be read "x is an element of M." More generally, $x_1, x_2, \cdots, x_n \varepsilon M$ will be read "$x_1, x_2, \cdots, x_n$ are elements of M." A set A will be called a *subset* of a set M, in symbols $A \subset M$, or $M \supset A$, if $x \varepsilon A$ implies $x \varepsilon M$. Negations of these relationships are indicated in the usual way, thus $\not\supset$ and $\not\varepsilon$, respectively. If $A \subset M$ and there exists x such that $x \not\varepsilon A$ and $x \varepsilon M$, then A will be called a *proper* subset of M. It is convenient and customary to introduce the *null* (empty or *vacuous*) set 0, which is defined as being the set such that for no x does the relation $x \varepsilon 0$ hold. As a logical consequence of its definition, 0 is an element of most classes of sets; for example, the class of all subsets of a given set, of all connected sets (§1.3), of all open sets (§1.2), etc.

If A and B are sets, then $A+B$ (*sum* of the sets A and B) is the set of all elements x such that at least one of the relations $x \varepsilon A$, $x \varepsilon B$ holds. The set $A \cdot B$ (*common part* of the sets A and B) is the set of all elements x such that both of the relations $x \in A$, $x \in B$ hold. If $A \cdot B = 0$, then A and B are called *mutually exclusive*. Finally, the set $A - B$ is the set of all elements x such that $x \in A$ and $x \notin B$.

1.2. **Spaces.** Generally speaking, any set M in which the "structural" concept of *limit point* has been established may constitute a *space* in Topology. Most of the early work in Topology was confined to euclidean spaces or their subsets, but it gradually became apparent that subsets of euclidean spaces were of themselves types of general spaces. The method of defining limit point in a given set M is quite arbitrary. One may proceed [1] by giving for each set $A \subset M$ the set A' (*derived set*) of all its limit points. Or one may prefer first to introduce *neighborhoods* of points and then define limit points in terms of neighborhoods. When one is studying the topology of an n-dimensional euclidean space E_n, the latter method is the more natural.

The simplest system or collection of neighborhoods for a euclidean space is that obtained by letting the set of points interior to a sphere be a neighborhood of any point in the set; more specifically, if x is a point and ϵ a positive number, the set $S(x, \epsilon)$ of all points y such that the distance $\rho(x, y)$ from x to y is less than ϵ is a neighborhood of any $y \in S(x, \epsilon)$. Thus, in E_1, the neighborhoods are intervals minus end points, and in E_2 they are interiors of circles. Finally, a point x is called a *limit point* of a set of points M, or M is said to *have x as a limit point*, provided that for every $\epsilon > 0$ the set $M \cdot S(x, \epsilon) - x \neq 0$; in words, every neighborhood of x contains at least one point of M distinct from x. It must be noted that M may have x as a limit point no matter whether $x \in M$ or not.

Now all we have just done is to set up a *neighborhood system* based on, or in agreement with, the euclidean *metric*. Suppose that we have an abstract set M on which there is defined a *single-valued*, *real* function $\rho(x, y)$ satisfying the following conditions: (1) $\rho(x, y) = 0$ if and only if $x = y$; (2) for any $x, y, z \in M$, distinct or not, $\rho(x, y) \leqq \rho(x, z) + \rho(y, z)$. Then if limit points are defined as above, M is called a *metric space*, the function $\rho(x, y)$ being its *metric*. (It follows readily from the definition that $\rho(x, y)$ is nonnegative and symmetric.) If a neighborhood system for a metric space is set up exactly as was done in the preceding paragraph for E_n, such a system is said to be in agreement with the metric.

If S is a given space, hence a set in which limit points have been defined either by a metric or otherwise, and $A \subset S$, then by $\overline{A}$ (*closure* of A) we denote the set $A + A'$, where A' is the set of all limit points of A. If $A = \overline{A}$, the set A is called *closed*. If $A \subset S$ is a closed set, then $S - A$ is called open; thus open sets are the *complements* of closed sets. Ordinarily, as for instance in a metric space, neighborhoods are special cases of open sets, and it is easy to see that a necessary and sufficient condition that a point x be a limit point of a set of points $M \subset S$ is that for every open set U containing x, the set $M \cdot U - x \neq 0$; in other words, that x be a limit point of M as given by the neighborhood system consisting of all open sets of S. Thus in a metric space we have defined two neighborhood systems, that consisting of its spherical neighborhoods $S(x, \epsilon)$ and that consisting of all its open sets. As a corollary of the statement just made above, if a point x is a limit point of a set A in a metric space M in terms of its spheres, then x is a limit point of A in terms of its open sets, and conversely.

This brings us to the notion of equivalent neighborhood systems of a set S: Two neighborhood systems of S are called *equivalent* if, given a neighborhood U of a point x in either system, there is a neighborhood V of x in the other system such that $V \subset U$. It follows immediately that if Σ_1 and Σ_2 are equivalent neighborhood systems of S, and a subset M of S has a limit point x in terms of neighborhoods of Σ_1, then x is a limit point of M in terms of Σ_2; and conversely. In a metric space the system Σ_1 of spheres $S(x, \epsilon)$ is equivalent to the system Σ_2 of open sets. It is important to note that by the imposition of nonequivalent neighborhood systems, the same set S may constitute different spaces in an essential sense (see §1.3 below). Hereafter when a space is defined in terms of a certain neighborhood system, we call the latter the *defining system* of neighborhoods.

In a metric space S, we may extend the symbol $\rho(x, y)$ to the form $\rho(X, Y)$, where $X \subset S$, $Y \subset S$; it will denote the greatest lower bound of the set of all numbers $\rho(x, y)$ such that $x \in X$, $y \in Y$. Also, if $M \subset S$, we shall find useful the notion of the *diameter* of M, to be denoted by $\delta(M)$, which is the least upper bound of the set of all numbers $\rho(x, y)$ where $x \in M$ and $y \in M$.

1.3. **Homeomorphisms, invariants.** If S_1 and S_2 are two spaces, then a *continuous mapping* of S_1 into S_2 is a single-valued function $y=f(x)$ such that (1) $x \in S_1$ and f is defined for all $x \in S_1$, (2) $y \in S_2$, and (3) f is continuous; that is, if $y=f(x)$ and $U \subset S_2$ is a neighborhood of y, then there is a neighborhood $V \subset S_1$ of x such that $f(V) \subset U$. In an expression of the sort $Y=f(X)$ as used here we interpret the symbol Y simply as the set of all $y \in S_2$ such that for some $x \in X$, $y=f(x)$, and as such we call it the *image set* of X (in S_2, rel. f).

There is another sense in which the symbol Y in the expression $Y=f(X)$ may be interpreted, and in which it signifies a duplicate of X in S_2 wherein for $y=f(x_1)=f(x_2)$, $(x_1 \neq x_2)$, we consider the values $f(x_1), f(x_2)$ as two points which coincide on y, and in this sense we call Y the *image* of X (in M, rel. f). Two mappings of X into M may have the same image sets but different images [68]. The set X will be called the counter image of Y. When Y is considered as the image set of X (in M, rel. f), then for $y \in Y$, the counter image of y is the set of all $x \in X$ such that $y=f(x)$, and this set is denoted by $f^{-1}(y)$.

If f is such that every point of S_2 is in the image set, then f is called a mapping *on* S_2. Finally, if f is a mapping of S_1 on S_2 and the inverse function $x=f^{-1}(y)$ is single valued and constitutes a continuous mapping of S_2

on S_1, then f (or f^{-1}) is called a *homeomorphism* (or topological mapping). The *topology* of a space S is the study of those properties that remain invariant under homeomorphisms of S. It is easy to show that a necessary and sufficient condition that a $(1-1)$-correspondence T between two spaces S_1, S_2 be a homeomorphism is that the neighborhood system of S_1 be equivalent to the system consisting of the image sets (in S_1, rel. T) of the elements of the neighborhood system of S_2.

In metric spaces, there is associated with the notion of continuous mapping that of *uniformly continuous mapping*, whose definition is like that of the special case of uniform continuity for real functions. And if, between two metric spaces M_1, M_2, there exists a homeomorphism f such that both f and f^{-1} are uniformly continuous, then f is called a *uniformly bicontinuous mapping*. The following theorem, whose proof is left to the reader, is of importance later on.

THEOREM. *If A_1 and A_2 are subsets of the compact* (definition in next paragraph) *metric spaces M_1 and M_2, respectively, such that $\overline{A}_i = M_i$, $(i=1, 2)$, and f is a uniformly bicontinuous mapping between A_1 and A_2, then f may be extended to a homeomorphism between M_1 and M_2.*

One of the most important topological invariants is compactness; a space is called *compact* if every infinite subset has a limit point. As in the case of most topological invariants, this property may be *localized*: A space S is called *locally compact* if for every $x \varepsilon S$ there exists a neighborhood U of x such that the closure of U is compact. Thus the space E_n is locally compact, but not compact. Another important invariant is *separability*: A space S is called *separable* if it has a denumerable subset A such that $\overline{A} = S$. A stronger invariant is that of perfect separability: A space S is called *perfectly separable* if it has a denumerable neighborhood system which is equivalent to the defining system. We introduce one more invariant at this stage: Let us call two subsets A and B of a space S *separated* if $A \neq 0 \neq B$ and $\overline{A} \cdot B = A \cdot \overline{B} = 0$; then a space S is called *connected* if it is not the sum of two separated sets. By way of example, the space E_n is connected and perfectly separable. As an example of a space which is separable but not perfectly separable, consider, in the coordinate plane, the set M_1 of all points (x, y) such that $x^2+y^2<1$ and the set M_2 such that $x^2+y^2=1$; let $M = M_1+M_2$. Thus far, M is only a *set* and not a *space*. Now for $p \varepsilon M_1$, let each neighborhood of p consist of all points of M_1 interior to a circle with center p; for $q \varepsilon M_2$, let each neighborhood of q consist of q together with all points of M_1 interior to a circle with center q. The space M so defined is separable, since if A consists of all points of M_1 both of whose coordinates (as points of the coordinate plane) are rational, $\overline{A} = M$; it is not, however, perfectly separable. The reader may also note that M is connected but not locally compact; and that the space M is not *metrizable* in that there does not exist a metric such that the system of spherical neighborhoods defined thereby is equivalent to the system of neighborhoods defining M above. (In regard to metrization problems, see [2].)

1.4. **Subspaces, imbedding.** Let A be a subset of a space S. Since limit points are defined in S, the set A may be considered as a space whose limit points are defined as in S (thus, if in S, $x \varepsilon A$ is a limit point of $M \subset A$, then x is a limit point of M in A). We then call A a *subspace* of S, or, which is equivalent, say that A is *imbedded in* S. The set M described in the preceding paragraph is a subset of the plane E_2, but is *not* a subspace of E_2 since in E_2 the point (1, 0) is a limit point of M_2, whereas in M this is not the case. A natural way to consider a subset A of a space S as a subspace of S is to let the neighborhoods in A be the "overlappings" of A with neighborhoods of S; more specifically, if U is a neighborhood in S of $x \varepsilon A$, let the set $U \cdot A$ be a neighborhood of x in A.

More generally, suppose S_1 and S_2 are spaces. If there exists a subspace A of S_2 such that A and S_1 are homeomorphic, then S_1 is said to be imbeddable in S_2. There exists, for example, a complete characterization of the compact Peano spaces (Chap. II) that are imbeddable in the 2-sphere [3], and in the Menger-Urysohn dimension theory [4] it is known, for example, that every n-dimensional compact metric space is imbeddable in E_{2n+1}. When a space S_1 is imbeddable in a space S_2, we shall often denote any homeomorph of S_1 in S_2 by the same symbol S_1.

The invariants defined in §1.3 may now be applied to subsets A of a space S; we speak of "connected subsets," "separable subsets," etc. It should be noted, however, that a majority of authors call a set $A \subset S$ compact if every infinite subset of A has a limit point *in* S, and introduce the term *self-compact* to denote compactness of A as a *subspace* of S; we use the term compact in this sense hereafter. However, for metric spaces, sets which are compact and closed are self-compact, and conversely, so that for closed sets in metric spaces the distinction is not significant. When a point set consisting of more than one point is both closed and connected, we call it a *continuum*. The reader will note that a connected space S_1 containing more than one point is a continuum, but that if it is embedded in a space S_2 it may fail to be a continuum in S_2, due to the non-invariance of the closure property (thus the segment $0<x<1$ of the space of real numbers is a continuum in itself as space, but not in the complete real number system). For this reason, when using either of the terms closed, continuum, the space relative to which these properties are used must be evident (in Chap. II below, for example, these terms will be used relative to C).

1.5. **Some important special spaces. A. The real number continuum E_1.** This section is basic in all that follows, since it contains the fundamental connecting link with ordinary analysis. Thus, the material in Chap. III is ultimately dependent on the structure of E_1, as also are the definitions of E_n and S_n below which are basic in the later discussions. Furthermore, particular pains have been taken in Chap. II to establish the homeomorphism between any set satisfying the arc definition used therein and the *bounded* real number continuum E_1^* consisting of all real numbers x such that $0 \leqq x \leqq 1$. Of course, the ordinary conception of the real number continuum is not topological; probably the Cantor-Huntington [5] axioms best express the usual intuitive conception, namely, a simply ordered set satisfying the Dedekind cut axiom, having a denumerable separating set, but no first and no last point (for E_1^* the last condition is replaced by the assumption of both a first and a last point). To consider E_1 as a space we may let, for any $x \varepsilon E_1$, a neighborhood of x consist of all points y such that $a < y < b$, where a and b are any points such that $a < x < b$. In terms of such neighborhoods one may show that E_1 is a locally compact, connected, perfectly separable space. In E_1^* neighborhoods would be similarly defined, except for 0 and 1; thus for $x = 0$ a neighborhood would consist of all y such that $0 \leqq y < b$.

The following example is instructive: In the cartesian plane let M be the set of points consisting of (0, 0) and all points (x, y) such that $y = \sin(1/x)$, $(0 < x \leqq 1)$. The set M may be linearly ordered in obvious fashion, and with neighborhoods defined as in the preceding paragraph is homeomorphic with E_1^*. However, as a subspace of the cartesian plane, the set M is not homeomorphic with E_1^*, inasmuch as the set K of points whose ordinates equal 1 has no limit point in M, whereas E_1^* is compact. What happens here is that the point (0, 0), which is a limit point of K when neighborhoods are defined in M as on the linear continuum, is no longer a limit point of K when the metric on the cartesian plane is used to define neighborhoods of M. This example will perhaps also make clear later the significance of Theorem 12 in Chap. II.

B. The euclidean n-dimensional space E_n. This is a metric space whose points are the ordered n-tuples $(x_1, x_2, \cdots, x_n)$, where x_i, $(i = 1, 2, \cdots, n)$, takes on all possible values in the real number continuum E_1, and whose metric is defined thus: if $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$, then $\rho(x, y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2}$. This is a special case of a *product space* (although as a product space the neighborhoods are parallelopipeds forming a system equivalent to the spherical neighborhood system given by the above metric). The fact that E_n is locally compact, connected, and perfectly separable follows from the same properties of E_1.

C. The n-sphere, S_n. This is the subspace of E_{n+1} consisting of all points $(x_1, x_2, \cdots, x_{n+1})$ such that $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$. This space is compact.

Since our point of view is to be topological, any space homeomorphic with one of the above spaces will receive the same name. For example, if $x \varepsilon S_n$, then the subspace $S_n - x$ is an E_n. When we speak of an S_n imbedded, say, in some E_k, the set of points constituting S_n is not in general a "sphere" in the sense of euclidean geometry, and, indeed, its relations to the complementary set $E_k - S_n$ may be greatly different from those of the euclidean sphere to its complement [6]. The 1-sphere S_1 is ordinarily called the *simple closed curve* or Jordan curve, and (Chap. III, Theorems J_1, J_2 [7]) an essential part of the proof of the Jordan curve theorem is to show that the complement of an S_1 imbedded in S_2 is exactly two components, just as in the analytically demonstrable case of the circle $x_1^2+x_2^2=1$, $x_3=0$, in the 2-sphere $x_1^2+x_2^2+x_3^2=1$. We remark here that in general, if A and B are metric spaces, and $B'=f(B)$, f a homeomorphism, $B'\subset A$, then the distance $\rho(x, y)$ for $x, y \varepsilon B'$ will be that given by the metric of A rather than that of B.

D. The Hilbert fundamental cube, Q_ω. This is the space whose points are all sequences $(x_1, x_2, \cdots, x_n, \cdots)$ of real numbers x_n, where $0 \leqq x_n \leqq 1/n$, and where $\rho(a,b)$, $a=(x_1, x_2, \cdots, x_n, \cdots)$, $b=(y_1, y_2, \cdots, y_n, \cdots)$ is the number $(\sum_{n=1}^{\infty}(x_n-y_n)^2)^{1/2}$.

E. Group-space. [8]. If a given set forms both a space and a group, then, if certain continuity conditions relating the limit point notion to the group operation are satisfied, the set is called a group-space (topological group, continuous group). In terms of neighborhoods, these continuity conditions usually imply that, given a neighborhood $U(c)$ of c and $a\cdot b=c$, then there exist neighborhoods $U(a)$, $U(b)$ such that $U(a)\cdot U(b)\subset U(c)$; and there exists $U(c^{-1})$ such that $(U(c^{-1}))^{-1}\subset U(c)$. In a metric space, where a limit point of a set is a limit point of a sequence of points of that set, these conditions may be stated in terms of limits of sequences.

II. Locally compact metric spaces and Peano spaces

We give in this section what may be considered an introduction to the material and methods exploited so effectively by the set-theoretic topologists. We list a set of five axioms. From two of these we derive (in §2.1) a number of fundamental theorems which include topological characterizations of E_1^* and S_1, the Cantor product theorem, the Borel theorem, and so on.

The complete set of five axioms serves to define a Peano space, and the theorems given in §2.2 are true for any such space. Historically, this material originates in the "continuous curves" of Jordan's *Cours d'Analyse* [9], defined for the coördinate plane by continuous functions of a real parameter. It is probably true that the investigation of these curves owes its liberation from the confines of analysis to the attention accorded them in connection with the so-called "space-filling curve" of Peano [10], and partly for this reason, as well as to avoid confusion with the term "Jordan curve" (S_1), the name Peano space has been applied in recent years to their topological analogues (although the terms "continuous curve," "Jordan continuum," are still used by some authors). However, the topological possibilities of these "curves" were first discovered by Schoenflies [11], whose set-theoretic researches culminated in their topological characterization among the subspaces of the euclidean plane. (See Chap. V.)

This type of work, concerning the relations between a continuous curve in the plane and its complement, was not, except for certain isolated papers, systematically continued and extended until taken up by R. L. Moore and his students and the Polish school of topologists. The material given here on Peano spaces actually relates, however, to the later development in point of view, which considers the curve as a space in itself, without reference to any larger space in which it may be imbedded, and particularly from the structural point of view introduced by G. T. Whyburn, which regards this space as being made up of certain subspaces called "cyclic elements." This material was selected not only for its importance as a powerful tool, but because it furnishes a convenient basis for the proof that the only compact Peano space which can satisfy the Jordan curve theorem non-vacuously is the 2-sphere.

Axiom 4 embodies the so-called "connectedness im kleinen" or local connectedness property of Peano spaces, and characterizes, among the compact, connected metric spaces, those which are image sets relative to continuous mappings of E_1^* (§1.5). Regarding the more detailed history of the property, the reader is referred to [12]; it will suffice to remark here that it embodies an internal characteristic of image-sets of E_1^*, dependent in no way on dimensionality or imbedding, whereas the Schoenflies characterization embodies an external characterizing property, relative only to sets imbedded in E_2. Our assumption in Axiom 1 of local compactness instead of compactness in no way complicates the proofs of theorems, and allows sufficient greater generality to achieve our purposes. We shall, in general, leave the details to the reader, but will, in the case of more difficult proofs, give references to the literature. The aim, as explained in the introduction, is to encourage the reader to supply his own proofs.

We consider a set C, whose elements we call points, satisfying:

AXIOM 0. *C contains at least two distinct points.*

AXIOM 1. *C is a metric space.*

AXIOM 2. *C is locally compact.*

AXIOM 3. *C is connected.*

AXIOM 4. *If $x \in C$ and ϵ is a positive number, then there exists an open connected subset U of C such that $x \in U$ and $U \subset S(x, \epsilon)$.*

A set C satisfying these axioms we call a Peano space.

2.1. **Some consequences of Axioms 1 and 2; topological characterizations of E_1^* and S_1.** In the following theorems all point sets mentioned are assumed to lie in a space C satisfying Axioms 1 and 2.

THEOREM 1. *If x is a limit point of a set of points M, then every open set which contains x also contains infinitely many points of M.*

COROLLARY. *No finite set of points has a limit point.*

THEOREM 2. *If x is a limit point of the sum of two point sets A and B, then x is a limit point of at least one of the sets A, B.*

THEOREM 3. *If A is a connected subset of a connected set M, and $M-A=B+D$ where B and D are separated, then both sets $B+A$, $D+A$ are connected.*

DEFINITION. *If a and b are distinct points of a connected set I such that no connected proper subset of I contains both a and b, then I is said to be irreducibly connected from a to b. In the theorems below, I will be a set irreducibly connected from a to b* [13].

THEOREM 4. *If N is a connected subset of I which contains a or b, then $I-N$ is connected.*

THEOREM 5. *If N is a connected subset of I which contains neither a nor b, then $I-N$ is the sum of two separated connected sets containing a and b respectively.*

THEOREM 6. *If M and N are connected subsets of I each of which contains a, then either $M \subset N$ or $N \subset M$.*

THEOREM 7. *Every $x \in I$ determines in unique fashion a decomposition of I into two sets $A(x)$ and $B(x)$ such that* (1) *$A(x)$ is irreducibly connected from a to x, and $B(x)$ is irreducibly connected from x to b;* (2) $A(x)\cdot B(x)=x =A(x)\cdot \overline{B}(x)+\overline{A}(x)\cdot B(x)$.

THEOREM 8. *If for distinct points $x, y \in I$ such that $A(x) \subset A(y)$ there is set up the binary relation $x<y$, then the set I is simply ordered in terms of the relation $<$.*

THEOREM 9. *For every $x \in I$, the set $A(x)$ consists of those points $y \in I$ such that $y \leqq x$.*

DEFINITION. *A subset M of I is called a portion of I if every element of I which is between (in terms of the relation $<$) two elements of M is itself an element of M.*

DEFINITION. *An $x \in I$ is called a lower bound of a set $M \subset I$ if $x \leqq M$ and there exists no $y \in M$ such that $x<y\leqq M$.*

(We write $x \leqq M$ if for every $y \in M$, $x \leqq y$.) Upper bound is defined in an obvious way.

THEOREM 10. *If a point x of a portion M of I is not an upper bound or a lower bound of M, then x is not a limit point of $I-M$.*

THEOREM 11. *In terms of the relation $<$, the set I satisfies the Dedekind cut axiom.*

THEOREM 12. *A closed I is compact.*

DEFINITION. *Let* Σ_1 *denote the set of all* $S(x, \epsilon)$, $x \in C$. *Then if* $M \subset C$, *a collection* Σ *of subsets of* C *will be said to be equivalent to* Σ_1 *relative to* M *if for every* $x \in M$ *and* $U \in \Sigma$ *such that* $x \in U$, *there exists* $\epsilon > 0$ *such that* $S(x, \epsilon) \subset U$; *and conversely, given an* $S(x, \epsilon)$ *there exists* $U \in \Sigma$ *such that* $x \in U \subset S(x, \epsilon)$.

THEOREM 13. *If* M *is a compact point set, there exists a denumerable collection* Σ *of neighborhoods* $S(x_n, \epsilon_n)$ *such that* Σ *is equivalent to* Σ_1 *relative to* M.

COROLLARY. *Every compact point set is separable.*

THEOREM 14. *A closed set* I *contains a sequence of points* $x_1, x_2, \cdots, x_n, \cdots$ *such that for any* $x, y \in I$ *and* $x < y$, *there exists an* n *such that* $x < x_n < y$.

THEOREM 15. *Every closed set* I *is homeomorphic with* E_1^*.

Theorem 15 embodies a topological characterization of those subsets of locally compact metric spaces that are homeomorphic with E_1^*, since it is obvious that every homeomorph of E_1^* is a set I. Hereafter, we shall call any homeomorph of E_1^* an *arc*.

THEOREM 16. *In order that a connected set* M *be irreducibly connected between two of its points* a *and* b, *it is necessary and sufficient that if* $x \in M$ *and* $a \neq x \neq b$, $M - x$ *is the sum of two separated sets neither of which contains both* a *and* b.

COROLLARY. *Among the continua, the arc is characterized by the condition of Theorem* 16.

DEFINITION. *If a set* M *is the sum of two* I's, *say* I_1, I_2, *such that* $I_1 \cdot I_2 = a + b = \bar{I}_1 \cdot I_2 = I_1 \cdot \bar{I}_2$, *then* M *is called a quasi-closed curve.*

THEOREM 17. *In order that a connected set* M *be a quasi-closed curve, it is necessary and sufficient that* (1) *for every* $x \in M$, $M - x$ *be connected, and* (2) *for* $x, y \in M$, $x \neq y$, *the set* $M - (x + y)$ *be not connected.*

COROLLARY, *In order that a continuum* M *be an* S_1, *it is necessary and sufficient that conditions* (1), (2) *of Theorem* 17 *hold.* [14].

THEOREM 18 (Cantor product theorem). *If* $M_1, M_2, M_3, \cdots$ *form an infinite sequence of compact sets such that* $M_n \supset \overline{M}_{n+1}$, (§1.2), $n = 1, 2, 3, \cdots$, *then the product* $\Pi_{n=1}^{\infty} M_n$ *is nonvacuous and closed.* (See definition of compact, §1.3, §1.4.)

DEFINITION. *If* M *is a point set and* G *is a set of point sets such that for every* $x \in M$ *there exists at least one* $g \in G$ *such that* $x \in g$, *then* G *is said to cover* M.

THEOREM 19 (Denumerable to finite Borel theorem). *If M is a closed and compact point set and G is a denumerable set of open sets covering M, then there exists a finite subset of G which covers M.*

(Consider the sets $M - M \cdot \sum_{i=1}^{n} g_i$, where $g_i \varepsilon G$.)

Now from Theorem 13 above it follows easily that if G is a collection of open sets covering a compact and closed set M, then a denumerable subset of G covers M. From this and Theorem 19 we have the following theorem.

THEOREM 20 (Borel). *If G is a collection of open sets covering a compact and closed point set M, then a finite subset of G covers M.*

THEOREM 21. *Under the hypothesis of Theorem 18, with the additional assumption that each set M_n is connected, the product $\prod_{n=1}^{\infty} M_n$ is connected.*

DEFINITION. *If $x, y \varepsilon C$, then a finite sequence of sets $M_1, M_2, \cdots, M_n$ will be said to form a simple chain of sets from x to y if (1) $x \varepsilon M_1$ if and only if $i=1$, (2) $y \varepsilon M_i$ if and only if $i=n$; and (3) $M_i \cdot M_j \neq 0$, $(i<j)$, if and only if $j=i+1$.*

THEOREM 22. *If M is a connected point set and G is a collection of open sets covering M, then for any two points $x, y \varepsilon M$ there exists a simple chain of elements of G from x to y.*

2.2. **Some properties of Peano spaces; topological characterization of S_2.**

THEOREM 23. *If each connected open subset of C is called a neighborhood of each of its points, then the system Σ_2 of all such neighborhoods is equivalent to Σ_1.*

Hereafter we call each element of Σ_2 a *domain*. By Theorem 1 and Axiom 3, every domain contains infinitely many points.

If a set M is an arc, and a, b are the points of M corresponding to 0, 1 of E_1^*, then we shall often denote M by ab. If M is a point set such that for every $a, b \varepsilon M$ there exists an arc ab in M, then M will be called *arcwise connected.*

Related to the following so-called "arc theorem," there exists a not inconsiderable literature. For a proof that can be adapted to the present framework, the reader is referred to [15].

THEOREM 24. *The space C is arcwise connected.*

COROLLARY. *Every domain is arcwise connected.*

(For the Axioms $0-4$ all hold for every domain; that is, every domain is itself a Peano Space.)

Remark. For reference in Chap. VI, we note here that it now follows readily that in order that a locally compact, connected metric space S containing more than one point be a Peano space, it is necessary and sufficient that for every $x \in S$ and $\epsilon>0$ there exist a $\delta>0$ such that for every pair $a, b \in S(x, \delta)$ there is an arc ab in $S(x, \epsilon)$.

DEFINITION. *If M is a point set and $p \in M$, then the component of M determined by p is the set of all points $x \in M$ such that both p and x lie in a connected subset of M. It may be shown directly from the definition of connectedness that if L is the component of M determined by $p \in M$, then for every $q \in L$, the component of M determined by q is again L, so that every point set possesses a unique decomposition into its components.*

THEOREM 25. *A necessary and sufficient condition that a subset R of C be a domain is that it be a component of an open subset of C.*

DEFINITION. *If R is a domain, then the boundary of R is the set of points $\overline{R}-R$.*

THEOREM 26. *If R is a domain not identical with C, then the boundary B of R is a nonvacuous, closed point set. Furthermore, if $C-\overline{R}\neq 0$, then $C-B$ is the sum of the separated sets R and $C-\overline{R}$.*

DEFINITION. *A point p is called an end point of C if for every $\epsilon>0$ there exists a point x such that $C-x=C_1+C_2$, where C_1 and C_2 are separated and $p \in C_1 \subset S(p, \epsilon)$* [16].

DEFINITION. *If M is a connected set and $x \in M$ is such that $M-x$ is not connected, then x is called a cut point of M. If an $x \in M$ is not a cut point of M, it is called a non-cut point of M* [17].

DEFINITION. *If M is a metric space and $x \in M$, $\epsilon>0$, then by $F(x, \epsilon)$ we denote the set of all $y \in M$ such that $\rho(x, y)=\epsilon$.*

THEOREM 27. *If x is a non-cut point of C and ϵ a positive number such that $F(x, \epsilon)$ is compact, then there exists a $\delta>0$ such that for any $a, b \in F(x, \epsilon)$ there is an arc ab in $C-S(x, \delta)$.*

COROLLARY. *If x is a non-cut point of C and ϵ a positive number, then there exists a $\delta>0$ such that all points of $C-S(x, \epsilon)$ lie in one component of $C-S(x, \delta)$.*

DEFINITION. *With every non-cut point p of C, we associate a set C_p which consists of all points x of C such that for no $y \in C$ is $C-y=C_1+C_2$ where C_1 and C_2 are separated sets containing p and x, respectively. By a cyclic element of C will be meant either a cut point of C or a set C_p.*

By way of examples, the end points of E_1^* are sets C_p and the other points of E_1^* are cut points, so that each point of E_1^* is a cyclic element of E_1^*. The 1-sphere S_1 is a cyclic element of S_1. The set M consisting of two tangent circles in E_2 has just three cyclic elements consisting respectively of the point of tangency and the two sets of points lying on the respective circles; this example shows, incidentally, that cyclic elements are not in general mutually exclusive point sets [18].

THEOREM 28. *A* $C_p = p$ *only if* p *is an end point.*

THEOREM 29. *Every* C_p *is a closed point set.*

THEOREM 30. *A* C_p *has the property that each component of* $C - C_p$ *has just one limit point in* C_p.

THEOREM 31. *If* M *is a connected set, then for any* C_p *the set* $M \cdot C_p$ *is connected.*

DEFINITION. *A set which has more than one element will hereafter be called nondegenerate.*

THEOREM 32. *Every nondegenerate* C_p *is a Peano space having no cut point.*

THEOREM 33. *If* M *is a set* C_p *and* A_1, A_2 *are closed, mutually exclusive, nondegenerate subsets of* M, *then there exist in* M *two arcs* $a_{1i}a_{2i}$, $(i = 1, 2)$, *such that* $a_{1i} \varepsilon A_1$, $a_{2i} \varepsilon A_2$, *and* $(a_{11}a_{21}) \cdot (a_{12}a_{22}) = 0$.

LEMMA. *If* M *is a nondegenerate* C_p *(and consequently a Peano space),* D *is a domain of* M, *and* x *is a non-cut point of* D, *then the set* C_x *of the Peano space* D *is nondegenerate.*

THEOREM 34. *If* M *is a* C_p *and* $a, b, c \varepsilon M$, *where* a, b, *and* c *are distinct, then there exists in* M *an arc* ac *which contains* b.

DEFINITION. *A Peano space* C *is called cyclicly connected if each pair of points of* C *lie together on some* 1*-sphere of* C.

THEOREM 35. *Every* C_p *is cyclicly connected.*

DEFINITION. *If* M *is a connected set,* $A \subset M$ *and* $B \subset M - A$, *and* $M - A$ *is the sum of two separated sets, each of which contains points of* B, *then we say that* A *separates* B *in* M.

THEOREM 36. *If* K *is a nondegenerate subset of* C *such that no point of* C *separates* K *in* C, *then* K *lies in a single* C_p *of* C.

THEOREM 37. *If* p *is a non-cut point of* C, *then the set* C_p *may be defined as the maximal, cyclicly connected subset of* C *that contains* p.

DEFINITION. *A Peano space* C *will be said to satisfy the Jordan curve theorem nonvacuously if there exists at least one* 1*-sphere in* C, *and if for every* 1*-sphere* S_1 *of* C, *it is true that* $C - S_1$ *consists of two mutually exclusive domains each of whose boundaries is* S_1.

THEOREM 38. *If C is compact and satisfies the Jordan curve theorem non-vacuously, then C is a 2-sphere.*

THEOREM 39. *If C is compact and contains at least one 1-sphere, and every 1-sphere of C separates C but no subarc of a 1-sphere separates C, then C is a 2-sphere.*

For history and proof of Theorems 38 and 39 (originally due to Zippin), the reader is referred to [19].

III. ELEMENTARY COMBINATORIAL TOPOLOGY; THE JORDAN CURVE THEOREM

In the Introduction it was stated that we would try to introduce notions in as simple a form as feasible, proceeding gradually to the more refined concepts embodied in generalizations and extensions, with the double purpose (1) of building up that intuitive background so necessary to an understanding and appreciation of the abstract formulations and (2) of observing the historical side of the development of topology. It is in accordance with this purpose that the present section is included.

The mode of development here is obviously inspired by [20], which seemed best adapted to our scheme. Except for what seems a restriction to topology on the 2-sphere (which is apparent rather than real, since concepts such as chain, homology, and so on, are phrased in terms and symbols which extend immediately to higher dimensional cases), the material is supposed to present a sample of the activity of the combinatorial school of thought, as represented particularly by Veblen and Alexander, during the decade approximately from 1915 to 1925. The simple modulo 2 topology is used throughout this section. Where orientation is not essential, this apparatus is most convenient. Of course, without orientation, certain topological invariants are missed, the most obvious one of which is perhaps the notion of orientability of manifolds (in terms of which and the Betti numbers, for instance, the 2-dimensional manifolds may be entirely classified).

As an application of the restricted theory developed in this section, we give a proof of the Jordan curve theorem, again adapted from [20]. It is accurate, we believe, to say that the earliest topological activity in this country was devoted to proofs of this theorem, and today the literature on the subject is immense. The first statement and proof were given by Jordan in [9]. In the opening words of a paper [21] which many consider as presenting the first completely rigorous and satisfactory proof of the theorem, Veblen states: "Jordan's explicit formulation of the fundamental theorem that a simple closed curve lying in a plane decomposes this plane into an inside and an outside region is justly regarded as a most important step in the direction of a perfectly rigorous mathematics."

As we shall see later, the Jordan curve theorem is now recognized to be, in its essential parts, only an instance of a general topological duality which in turn is an instance of a general group-theoretic theorem. It is hoped, by the way, that the reader will observe that whereas much emphasis is laid on the fundamental analytic basis of the real number system in these early sections, we gradually become more immersed in geometric considerations, and from there we fall inevitably into the algebraic (see Chap. IV).

It is also noteworthy that, with the close of the present chapter we shall have, as by-products of Chaps. II and III, a complete proof of this theorem:

THEOREM. *A necessary and sufficient condition that a Peano space be a 2-sphere is that it satisfy the Jordan curve theorem nonvacuously.*

3.1. **Geometric basis.** Choose an S_2 represented in E_3 by the equation $x^2+y^2+z^2=1$. The intersection of this S_2 with the plane $z=0$ is a 1-sphere S_1, which separates S_2 into two open hemispheres E_1^2, E_2^2. Similarly the plane $y=0$ intersects S_1 in two points E_1^0, E_2^0 forming a 0-sphere S_0 which separates S_1 into two open arcs E_1^1 and E_2^1. The geometric configuration Q_1 consisting of the elements E_i^k, $(k=0, 1, 2; i=1, 2)$, we call an *elementary subdivision* of S_2. Pairwise, these elements have no common points, and we call E_i^k a k-dimensional cell, or simply a *k-cell* (the 0-cells may also be called *vertices*). Write

$$E_i^k \to E_1^{k-1} + E_2^{k-1}, \qquad k = 1, 2, \tag{3.1}$$

a so-called "boundary relation," where the $\to$ may be read "is bounded by."

From the elementary subdivision Q_1 we pass to the *derived subdivisions* of S_2. These are obtained by further intersections with planes through (0, 0, 0). For example, we may first consider the plane $x=0$, which intersects E_1^1 in a point E_3^0, *subdividing* E_1^1 into open segments one of which we continue to denote by E_1^1, the other by E_3^1; the configuration consisting of the new cells E_i^k, $(k=0, 1, 2)$, we call the *first derived subdivision* Q_2. Then Q_3 may consist of the new configuration in S_2 resulting from subdivisions of E_2^1 by the same plane $x=0$; Q_4 of the configuration on S_2 resulting from subdivision of E_1^2 into two new 2-cells and a new 1-cell; and so on. In an obvious manner we may define, inductively, a sequence of *subdivisions* $Q_1, Q_2, \cdots, Q_n, \cdots$ such that (1) for each natural number n, Q_{n+1} is derived from Q_n by the subdivision of a single cell as above, and (2) for arbitrary $\epsilon>0$, there exists an integer m such that for $n>m$, all cells of Q_n are of diameter less than ϵ (where diameter is the set-theoretic diameter obtained from the metric on E_3). For each Q_n we may write a set of bounding relations such as (3.1), except that the right-hand member usually contains symbols of more than two $(k-1)$-cells.

A collection K of cells E_i^k of Q_n will be called a *complex* if, for $E_i^k \varepsilon K$, and $E_i^k \to \cdots + E_h^{k-1} + \cdots$, each cell E_h^{k-1} is in K. If we denote the set of all points in cells of K by (K), then the latter condition is equivalent to requiring that (K) be a closed point set. If, moreover, there is a fixed integer m such that for the cells E_i^k of K, (1) $k \leqq m$, and (2) there is at least one $E_i^m \varepsilon K$, then K is called an m-dimensional complex, or simply an *m-complex*. Thus Q_n is itself a 2-complex, and any complex whose cells are elements of Q_n is a *subcomplex* of Q_n. If K_1 and K_2 are subcomplexes of Q_n and Q_{n+h}, respectively, and $(K_1) \equiv (K_2)$, then we call K_2 a subdivision of K_1.

3.2. **Algebraic basis; homology groups.** Let K be a complex, and denote the cells of K by E_i^k, $(k=0, 1, \cdots, m; i=1, 2, \cdots, \alpha_k)$. With K we associate certain abelian groups $C^k(K)$ whose elements are polynomials of the type

$$\sum_{i=1}^{\alpha_k} c^i E_i^k, \qquad c^i = 0 \text{ or } 1,\ E_i^k \epsilon K, \tag{3.2}$$

and whose operation is ordinary addition modulo 2. The polynomials (3.2) are called *k-chains*. If C^k is a k-chain, denote by $|C^k|$ the smallest subcomplex of K containing all cells E_i^k for which $c^i=1$, and we call $|C^k|$ the complex associated with C^k. In particular, each symbol E_i^k is a k-chain, a *cell-chain*; however, in this case we shall continue to let E_i^k denote both a k-chain and a k-cell (except where confusion may result).

As in (3.1), let us denote the boundary relations for K thus:

$$E_i^k \rightarrow \sum_{h=1}^{\alpha_{k-1}} e_{ik}^h E_h^{k-1}, \qquad e_{ik}^h = 0 \text{ or } 1,\ k > 0. \tag{3.1$'$}$$

Then we may interpret (3.1)′ as an association of a unique $(k-1)$-chain with each cell-chain E_i^k. This may be extended as follows:

$$\sum_{i=1}^{\alpha_k} c^i E_i^k \rightarrow \sum_{i=1}^{\alpha_k} \sum_{h=1}^{\alpha_{k-1}} c^i e_{ik}^h E_h^{k-1}. \tag{3.3}$$

Then by (3.3) there is associated with each k-chain, a unique $(k-1)$-chain; the latter we call the *boundary-chain* of the former. The reader may prove the following theorem.

THEOREM 1. *The boundary-chain of the sum of two chains is identical with the sum of their boundary-chains.*

Hence relations (3.3) may be added termwise, and we have this corollary:

COROLLARY 1. *The relations* (3.3) *establish a homomorphism of the group* $C^k(K)$ *into a subgroup* $H^{k-1}(K)$ *of* $C^{k-1}(K)$.

If $C^k \rightarrow 0$, where 0 is the identity of $C^{k-1}(K)$, then the chain C^k is called a *cycle*, or when we wish to indicate dimension, a *k-cycle*. We also have another corollary:

COROLLARY 2. *The set of all k-cycles, $k>0$, consists of the elements of that subgroup $Z^k(K)$ of $C^k(K)$ that is mapped by* (3.3) *into the identity of* $C^{k-1}(K)$.

In Theorem 1 and its corollaries, we have assumed $k>0$, since relation (3.1)′ is not defined otherwise. We now make the convention that any 0-chain which has an even number of non-zero coefficients is a 0-cycle. Then the set of all such 0-cycles is an abelian group $Z^0(K)$.

Since, as is easily shown, the right-hand member of (3.1)′ is a cycle, we have another corollary of Theorem 1.

COROLLARY 3. *Every boundary chain is a cycle, hence $H^{k-1}(K)$ is a subgroup of $Z^{k-1}(K)$.*

DEFINITION. *For any $k\geqq 0$, the factor group of $Z^k(K)$ modulo $H^k(K)$ is called the kth homology group of K modulo 2, or the kth Betti group of K modulo 2, and will be denoted by $B^k(K)$. The number of linearly independent generators of this group is called the kth Betti number of K modulo 2, and is denoted by $p^k(K)$.*

The relation $\gamma^k \in H^k(K)$ is usually expressed by a relation

$$\gamma^k \sim 0, \tag{3.4}$$

to be read "γ^k is homologous to zero." Thus two cycles γ_1^k, γ_2^k lie in the same coset of $B^k(K)$ if and only if $\gamma_1^k+\gamma_2^k\sim 0$. In general, if cycles $\gamma_1^k, \gamma_2^k, \cdots, \gamma_n^k$ satisfy a relationship $\gamma_1^k+\gamma_2^k+\cdots+\gamma_n^k\sim 0$, we call them linearly dependent with respect to homologies on K. Evidently the number $p^k(K)$ is the maximal number of k-cycles linearly independent with respect to homologies on K.

Remark. Since it may be shown that any two 0-cells of a connected complex are the end points of an arc of that complex made up of 0-cells and 1-cells, this theorem follows readily:

THEOREM 2. *The number $p^0(K)+1$ is the number of components in* (K).

3.3. **Invariance under subdivision.** Suppose K_2 is a subdivision of K_1, as defined in the final paragraph of §3.1. Then the passage from K_1 to K_2 consists of a series of steps, each step consisting of a subdivision of a single cell. It is easily shown that this operation leaves $p^k(K)$ unchanged.

THEOREM 3. *The numbers $p^k(K)$ are invariant under subdivision.*

Since $p^k(Q_1)=0$ for $k<2$, and equals 1 for $k=2$, we have the following corollary:

COROLLARY 4. *The Betti numbers $p^k(Q_n)$ are all 0 if $k<2$, and 1 if $k=2$.*

3.4. **Open subsets of S_2.** Let U be an open subset of S_2, and let U_n denote the set of all cells E_i^k of Q_n such that the closure of $(E_i^k)\subset U$. Then

U_n is a complex, and furthermore $(U_n) \subset (U_{n+1})$. We define chain-groups $C^k(U)$ of U, whose elements are the chains in $C^k(U_n)$ for $n=1, 2, 3, \cdots$, making the convention that if the complex $|C_2^k|$, where $C_2^k \varepsilon C^k(U_{n+h})$, is a subdivision of $|C_1^k|$, where $C_1^k \varepsilon C^k(U_n)$, then the chains C_1^k, C_2^k are the same element of $C^k(U)$. To add two elements of $C^k(U)$, it is necessary only to find n great enough so that the two chains are elements of $C^k(U_n)$; their sum in the latter group determines their sum in $C^k(U)$.

Groups $Z^k(U)$, $H^k(U)$, $B^k(U)$, and numbers $p^k(U)$ are defined analogously as for the case of a complex K; the essential difference here is that we are actually dealing with U as an infinite complex.

THEOREM 4. *The number $p^0(U)+1$ is the number of domains in U.*

(Theorems of Chap. II are applicable in the proof of this theorem.)

3.5. **Miscellaneous theorems.** Hereafter, if C is a chain, (C) denotes the set of points $(|C|)$.

THEOREM 5. *A 1-cycle γ^1 of Q_n is the boundary-chain of exactly two different chains K_1^2, K_2^2 of Q_n, and $(K_1^2)\cdot(K_2^2)=(\gamma^1)$.*

(For proof, use Corollary 4 and Theorem 1.)

COROLLARY 5. *If $x \varepsilon S_2$ and $\gamma^1 \varepsilon Z^1(S_2-x)$, then $\gamma^1 \varepsilon H^1(S_2-x)$.*

COROLLARY 6. *If a subset M of S_2 consists of exactly two points, then $p^1(S_2-M)=1$.*

The following theorem is an example of a so-called "addition theorem":

THEOREM 6. *Let A, B be closed subsets of S_2, and suppose that for a given $\gamma^0 \varepsilon Z^0(S_2-A-B)$ there exist chains $K_A^1 \varepsilon C^1(S_2-A)$, $K_B^1 \varepsilon C^1(S_2-B)$, $K^2 \varepsilon C^2(S_2-A-B)$ such that $K_A^1\to\gamma^0$, $K_B^1\to\gamma^0$ and $K^2\to K_A^1+K_B^1$. Then $\gamma^0\sim 0$ in S_2-A-B.*

(For proof see [20, Corollary W^i].)

THEOREM 7. *If t is an arc of S_2, then $p^k(S_2-t)=0$ for $k=0, 1$.*

As t is an E_1^*, it is the sum of arcs t_1, t_2 where t_1 corresponds to the interval $0\leqq x\leqq\frac{1}{2}$, and $t_1\cdot t_2$ is a single point a. By Corollary 5 and Theorem 6 a $\gamma^0 \varepsilon Z^0(S_2-t)$ which is nonbounding in S_2-t is also nonbounding in at least one of the sets S_2-t_1, S_2-t_2, say the former; subdivide t_1, etc. Continuing in this manner one shows the existence of a sequence T of arcs such that (1) each arc of T contains its successor, and the product of all arcs of T is one point x; (2) γ^0 is nonbounding in S_2-t' for all $t' \varepsilon T$. But (Theorem 4) $p^0(S_2-x)=0$; hence every γ^0 bounds a chain K' of S_2-x. The remainder of the proof is left to the reader.

3.6. **The Jordan curve theorem.**

THEOREM J_1. *If J is an S_1 imbedded in S_2, then $p^0(S_2-J)\geqq 1$.*

Proof. Let $a, b \in J$; then $J = A + B$, where A and B are arcs having only a and b in common. By Corollary 6, there exists a 1-cycle γ^1 of $S_2 - a - b$ which is nonbounding in the latter set, and by application of Theorem 7, $A \cdot (\gamma^1) \neq 0$ and $B \cdot (\gamma^1) \neq 0$.

Now $\gamma^1 = K_A^1 + K_B^1$, where $(K_A^1) \cdot A = 0 = (K_B^1) \cdot B$. Let $K_A^1 \rightarrow \gamma^0$. Then γ^0 is a cycle in $S_2 - J$ and is nonbounding in the latter set. For suppose we have an $L^1 \rightarrow \gamma^0$ in $S_2 - J$. Then not both $K_A^1 + L^1 \sim 0$, $K_B^1 + L^1 \sim 0$, hold in $S_2 - a - b$, else the sum would. Hence $K_A^1 + L^1$, say, is nonbounding in $S_2 - a - b$. But then by Theorem 7, $(K_A^1 + L^1) \cdot A \neq 0$.

THEOREM J_2. *If J is an S_1 imbedded in S_2, then $p^0(S_2 - J) \leqq 1$.*

Proof. Suppose $p^0(S_2 - J) \geqq 2$. Then for n large enough, there exist in Q_n cycles γ_1^0, γ_2^0 that are independent with respect to homologies in $S_2 - J$. Using the notation A, B of the above proof, and applying Theorem 7, there exist chains K_A^1, K_B^1 in $S_2 - A$, $S_2 - B$ respectively, such that

$$K_A^1 \rightarrow \gamma_1^0, \quad K_B^1 \rightarrow \gamma_1^0, \tag{3.5}$$

and by Theorem 6, the cycle $K_A^1 + K_B^1$ is nonbounding in $S_2 - a - b$. Similarly, there exist chains L_A^1 and L_B^1 in $S_2 - A$ and $S_2 - B$ respectively, such that

$$L_A^1 \rightarrow \gamma_2^0, \quad L_B^1 \rightarrow \gamma_2^0, \tag{3.6}$$

and $L_A^1 + L_B^1$ is nonbounding in $S_2 - a - b$.

By Corollary 6, the cycles $K_A^1 + K_B^1$, $L_A^1 + L_B^1$ cannot be independent with respect to homologies in $S_2 - a - b$, hence their sum bounds in $S_2 - a - b$. But from (3.5) and (3.6) we have $K_A^1 + L_A^1 \rightarrow \gamma_1^0 + \gamma_2^0$ in $S_2 - A$ and $K_B^1 + L_B^1 \rightarrow \gamma_1^0 + \gamma_2^0$ in $S_2 - B$, and then by Theorem 6, the cycle $\gamma_1^0 + \gamma_2^0$ bounds in $S_2 - J$.

THEOREM J_3. *If J is an S_1 imbedded in S_2, then $S_2 - J$ is the sum of two mutually exclusive domains whose common boundary is J.* (For definitions see Chap. II.)

Proof. Since by Theorems J_1, J_2, $p^0(S_2 - J) = 1$, there are just two components in $S_2 - J$. Select Q_n so that there exists a cycle γ^0 of Q_n associated with two 0-cells E_1^0, E_2^0 and nonbounding in $S_2 - J$, and let $x \in J$. Consider any $S(x, \epsilon)$ (the metric in E_3 may be used here). By continuity of the homeomorphism defining J, and using the notation introduced above, A and B exist such that $x \in A \subset S(x, \epsilon)$. By Theorem 7, there exists in $S_2 - B$ a chain $K^1 \rightarrow \gamma^0$. But since γ^0 is nonbounding in $S_2 - J$, $(K^1) \cdot A \neq 0$.

Now the point set (K^1) contains an arc $E_1^0 E_2^0$ (see Remark preceding Theorem 2), and the common part of this arc and A is a closed point set M. From the "I-theorems" of Chap. II, there exist portions of the arc $E_1^0 E_2^0$ in each domain of $S_2 - J$ that lie in $S(x, \epsilon)$. Hence J is the common boundary of these domains.

IV. Generalizations; duality theorems

From the elementary formulation of the combinatorial method in Chap. III, the reader has probably guessed the great possibilities of generalization. That the geometric set-up of §3.1 may be paralleled on an S_n in E_{n+1} is obvious, and the modulo 2 algebraic machinery of §3.2 has been so phrased for general k that it applies immediately to the higher-dimensional case. It was with such a basis as this that Alexander established his celebrated duality theorem [20]:

Theorem. *If K is a complex imbedded in S_n, then* $p^k(K) = p^{n-k-1}(S_n - K)$.

Applied to the $(n-1)$-sphere and $(n-1)$-cell so as to parallel the results at the end of Chap. III, this duality yields immediately the extension to S_n of the Jordan theorem (first established by Brouwer [23]).

Historically, the modulo 2 coefficients of §3.2 were preceded by the integer coefficients used in Poincaré's fundamental work; a systematic exposition has been given [24] by Veblen who in the same connection develops the modulo 2 theory for complexes introduced earlier [25]. One of Poincaré's most notable results was his duality theorem for orientable manifolds:

Theorem. *If M is an orientable n-manifold, then* $p^k(M,G_0) = p^{n-k}(M, G_0)$.

(These symbols will be explained below.) Cell-orientation was essential in Poincaré's theory, as well as in the general theory described below. General modulo m coefficients were introduced by Alexander [26] and rational coefficients together with the useful notion of relative chains by Lefschetz. The latter made a most notable advance in the enunciation of a general duality which generalized both the Poincaré and Alexander dualities, as well as establishing the intimate relationship between these two theorems [27]. More recently, Pontrjagin demonstrated that these dualities were consequences of a group-theoretic theorem [28]. Our exposition in §4.4 and §4.5 below is based on the later more refined and conclusive papers [29].

4.1. **General homology theory of a complex.** If K is a subcomplex of some subdivision of an S_r, we may associate with K the abelian chain-groups $C^k(K,G)$ whose elements are again polynomials (3.2), but where now the coefficients c_i are elements of some given abelian group G. In Chap. III we dealt exclusively with the case where G is the modulo 2 group of integers. The most important special cases of G with which we shall be concerned later on are (1) the finite additive groups of integers modulo m, to be denoted by G_m, $m>1$; (2) the additive group of integers G_0, often called the integer group modulo 0; (3) the group of rotations of the 1-sphere, or additive group of real numbers modulo 1, which we denote by R; (4) the additive group of rational real numbers G_r. It will be noted that if the usual multiplication is introduced, the groups G_m(m prime) and G_r are algebraic fields; all G_m are rings, but R is not even a ring. Except for considerations of a special nature, however, these facts are largely irrelevant. It is important chiefly that it be possible to associate with K certain abelian groups which have geometric significance in the sense of yielding topological invariants.

Simplicial ("barycentric") subdivision. Denote the cells of K by E_i^k, ($i=1, 2, \cdots, \alpha_k$; $k=0, 1, \cdots, n$), denote E_i^0 temporarily by P_i^0, and let P_i^k denote a point in the cell E_i^k. Hereafter, if in any complex a cell E_j^h is in the complex $|E_i^k|$, ($h<k$), we shall call the cells E_j^h, E_i^k *incident* to one another, and indicate this incidence by the relation $E_j^h<E_i^k$. Now there can be set up a new complex K', the *simplicial subdivision* of K, each of whose k-cells has $k+1$ vertices $P_{i_0}^a, P_{i_1}^b, \cdots, P_{i_{k-1}}^l, P_{i_k}^h$, where, $E_{i_0}^a<E_{i_1}^b<\cdots<E_{i_k}^h$ and where the points of the k-cell are the points (excepting $P_{i_k}^h$ itself, and, in case $k=1$, excepting $P_{i_0}^a$) of S_r on geodesics joining $P_{i_k}^h$ to the points of the cell whose vertices are $P_{i_0}^a, P_{i_1}^b, \cdots, P_{i_{k-1}}^l$ (such a "cell," no longer obtained as were the original cells, is nevertheless like the latter homeomorphic with the set of points $x_1^2+x_2^2+\cdots+x_k^2<1$ in E_k, hence with the space E_k itself; *hereafter a cell is characterized by this property*). It will be noted that in the point set-theoretic sense, each cell E_i^k is the sum of all cells of K' in E_i^k having P_i^k as a vertex. And a simple argument shows that in each cell E_i^k, the $(k-1)$-cells of K' are incident with exactly two k-cells of K'. Furthermore, the set of points in $|E_i^k|-E_i^k$ is an S_{k-1}.

Orientation. If, in a complex, a k-cell has exactly $k+1$ vertices, we call it a *simplex*. A simplex may be assigned positive and negative *orientations* as follows: A certain simple ordering of its vertices, $v_0, v_1, \cdots, v_k$ is selected, and the symbol $v_0v_1\cdots v_k$, or any even permutation of the v's therein, regarded as an algebraic symbol associated with the simplex and its positive orientation; the symbol derived from $v_0v_1v_2\cdots v_k$ by any odd permutation of the v's, or the symbol for a positive orientation preceded by a minus sign, is associated with a negative orientation. We usually denote a positive orientation, $v_0v_1v_2\cdots v_k$, by a single symbol such as σ, and let $-\sigma$ denote the negative orientation; of course $-(-\sigma)=\sigma$.

We define a function (or linear operator) F as follows:

$$F(v_{i_0}v_{i_1}\cdots v_{i_k}) = \sum_{s=0}^{s=k}(-1)^s v_{i_0}v_{i_1}\cdots v_{i_{s-1}}v_{i_{s+1}}\cdots v_{i_k} \tag{4.1}$$

and, more generally,

$$F\left(\sum_j \eta^j\sigma_j^k\right) = \sum_j \eta^j F(\sigma_j), \tag{4.2}$$

the algebraic operations being commutative, associative and distributive.

Returning to K', all of whose cells are simplexes, we assign orientations to all its cells a_j^k, denoting their positive orientations by τ_j^k. Each cell will be oriented independently of all others of the same or different dimension, except as follows: If $E_i^k \varepsilon K$, denote the k-cells of K' in $|E_i^k|$ by a_{ij}^k, and to a_{i1}^k assign a positive orientation τ_{i1}^k. Then to the other cells a_{ij}^k of E_i^k we assign orientations τ_{ij}^k such that the terms with non-zero coefficients in $F(\Sigma_j \tau_{ij}^k)$ are associated with the cells of K' on $|E_i^k| - E_i^k$; of course, the latter fact holds also for $F(\Sigma_j(-\tau_{ij}^k))$. We call τ_{i1}^k the *indicatrix of* E_i^k, and associate with E_i^k and τ_{i1}^k, a positive orientation σ_i^k.

If we formally set $\sigma_i^k = \Sigma_j \tau_{ij}^k$, then we note that $F(\sigma_i^k) = \sum \eta^m \sigma_m^{k-1}$. In this manner we can extend F to the σ's, and write a relation in the σ's analogous to (4.2). That F^2 operating on the τ's gives 0, and hence also on the σ's gives 0, is readily shown.

Finally, if G is any abelian group, we let $C^k(K, G)$ denote the abelian group whose elements are polynomials of type $\sum_{i=1}^{\alpha_k} c^i \sigma_i^k$. By defining

$$F\left(\sum_i c^i \sigma_i^k \right) = \sum_i c^i F(\sigma_i^k), \tag{4.3}$$

algebraic operations being as before commutative, associative, and distributive, and with $c^i \cdot (-1) = -c^i$, and so on, we obtain a homomorphism of $C^k(K, G)$ onto a subgroup $H^{k-1}(K, G)$ of $C^{k-1}(K, G)$. Denoting the identity of $C^k(K, G)$ by 0, it turns out that the group $Z^k(K, G)$ of $C^k(K, G)$ which maps into 0 by (4.3) contains $H^k(K, G)$ as subgroup.

The elements of $C^k(K, G)$, $Z^k(K, G)$, $H^k(K, G)$ we call respectively *k-chains*, *k-cycles*, *bounding k-cycles*, all *over G*. The factor group of $Z^k(K, G)$ modulo $H^k(K, G)$ we call the *kth Betti group of K over G*. For the case $k=0$, one has to adopt, as was done in Chap. III, a convention regarding what is to constitute a 0-cycle. The reader will recognize the connection of all this with the material in §3.2, and perhaps supply the notion of homology and Betti number over G (number of linearly independent generators of the corresponding Betti group); the use of the $\rightarrow$ of Chap. III instead of the operator F may, of course, be introduced if desired.

4.2. **Dual complexes.** Suppose that the structure of the n-complex K is such that there exists a complex D, the *dual* of K, defined as follows: The 0-cells B_i^0 of D are the points P_i^n of K'; in general, the $(n-k)$-cells B_i^{n-k} of D are, in a point set sense, the sum of all cells of K' with vertices $P_i^k, P_j^l, \cdots, P_s^h$ such that $E_i^k < E_j^l < \cdots < E_s^h$; in particular, each n-cell B_i^n is the sum of all cells of K^1 having P_i^0 as a vertex. Then if (K) is connected, we call K a closed n-manifold. Cells such as E_i^k, B_i^{n-k} are called *dual*; note that $(E_i^k) \cdot (B_i^{n-k}) = P_i^k$. Also, if $E_i^k < E_j^{k+1}$, then $B_j^{n-k-1} < B_i^{n-k}$.

Suppose, now, that the structure of the n-manifold K is such that, starting with an indicatrix for a particular n-cell of K, say the indicatrix τ_{11}^n of E_1^n, it is possible so to orient the other cells E_i^n, $(i=2, \cdots, \alpha_n)$, that $F(\sum_{i=1}^{\alpha_n} \sigma_i^n) = 0$. There being, from the definition, two and only two

n-cells E_p^n, E_q^n, incident with a given cell E_i^{n-1}, this means that σ_p^n and σ_q^n are so defined that σ_i^{n-1} occurs in $F(\sigma_p^n)$ and $F(\sigma_q^n)$ with opposite signs. We then call K a closed *orientable n-manifold*, and when oriented so that $F(\sum_{i=1}^{\alpha_n}\sigma_i^n)=0$, we call it *coherently* oriented relative to the indicatrix τ_{11}^n.

Let K be an orientable n-manifold coherently oriented relative to some indicatrix. We may orient its dual, D, as follows: B_i^{n-k} being a cell of D, we choose an indicatrix b^{n-k} in B_i^{n-k} with vertices $P_i^k, P_j^{k+1}, \cdots, P_s^n$, and let $\eta_1 P^0P^1\cdots P_i^k$ (where η_1 is $+1$ or -1) be a positive orientation of a cell of K' in E_i^k. Let $\sigma^n=\eta_2 P^0P^1\cdots P_i^kP_j^{k+1}\cdots P_s^n$, the η_2 having been determined in K' during the orientation of K. Then we let $\beta_i^{n-k}=\eta_1\eta_2 P_i^kP_j^{k+1}\cdots P_s^h$ be the positive orientation of b^{n-k}. As K' is a simplicial subdivision of D, this indicatrix determines the positive orientation of B_i^{n-k}, which we denote by δ_i^{n-k}.

4.3. **Intersection numbers in an orientable n-manifold.** We define a number $\chi(\sigma_i^k, \delta_j^{n-k})$ as follows: If $i\neq j$, its value is zero; if $i=j$, let $\tau_{i1}^k=\eta_1P^0P^1\cdots P_i^k$, and let $\eta_3P_i^kP^{k+1}\cdots P^n$ be the indicatrix of δ_j^{n-k}. If we choose η_2 as above so that $\eta_2P^0P^1\cdots P_i^kP^{k+1}\cdots P^n$ is a positive orientation of the n-cell indicated, then $\chi(\sigma_i^k, \delta_i^{n-k})=\eta_1\eta_2\eta_3$. Obviously $\chi(\sigma_i^k, \delta_i^{n-k})=1$. It may be shown that any of the τ's or β's in E_i^k, B_i^{n-k} would yield the same result. Furthermore, $\chi(-\sigma_i^k, \delta_j^{n-k})$, may be defined in an obvious manner. We call $\chi(\sigma_i^k, \delta_j^{n-k})$ the *intersection number* or Kronecker index of σ_i^k, δ_j^{n-k}.

Since K is the dual of D, the numbers $\chi(\delta_i^{n-k}, \sigma_i^k)$ are defined, and it may be shown by counting transpositions that

$$\chi(\delta_i^{n-k}, \sigma_i^k) = (-1)^{k(n-k)}\chi(\sigma_i^k, \delta_i^{n-k}).$$

Consider $\sum_{i=1}^{\alpha_k}c^i\sigma_i^k=C^k\varepsilon C^k(K,G')$ and $\sum_{i=1}^{\alpha_k}d^i\delta_i^{n-k}=D^{n-k}\varepsilon C^{n-k}(D,G'')$. If we can attach a suitable meaning to the product $c^i\cdot d^i$, we may define

$$\chi(C^k, D^{n-k}) = \sum_{i=1}^{\alpha k} c^id^i\chi(\sigma_i^k, \delta_i^{n-k}). \tag{4.4}$$

For example, if $G'=G''=G_m$ (see §4.1), the number χ is defined and may be called the *intersection number* of the chains C^k, D^{n-k}; in Chap V, where we use G_2 exclusively as coefficient group, the intersection numbers will all be 0 or 1. For the general case, we introduce the notions below.

4.4. **Character groups.** Consider three additive abelian groups X, G, and R, whose identities we denote by 0, and suppose that to each ordered pair x, g, $x\varepsilon X$, $g\varepsilon G$, there corresponds a unique $r\varepsilon R$; we write $xg=r$. If this operation satisfies the distributive laws $(x_1+x_2)g=x_1g+x_2g$, $x(g_1+g_2)=xg_1+xg_2$, then we say that X and G form a *group pair*. If H is a subgroup of G, denote by (X, H) the set of all $x\varepsilon X$ such that for all $h\varepsilon H$, $xh=0$; we call (X, H) the *nullifier* of H in X. This nullifier is a subgroup of X.

Similarly, we define the nullifier (G, Y) of Y in G. Finally, if $(X, G)=0 =(G, X)$, we call G and X *orthogonal*.

Suppose now that R is the group R of §4.1 and G is any, at most denumerable, discrete group. The set of all homomorphisms h of G into R forms a group X, whose operation is defined by the rule $h=h_1+h_2$ if for all $g\varepsilon G$, $h(g)=h_1(g)+h_2(g)$, and which is continuous in the sense that $\lim h_n=h$ if for all $g\varepsilon G$, $\lim h_n(g)=h(g)$. We call X the *group of characters* of G; it is a compact metric space [29(a)]. In case G is finite it is isomorphic to G[29(a)]. The groups X and G form an orthogonal group pair rel. R, where $hg=h(g)$ for all $h\varepsilon X$, $g\varepsilon G$.

Reversing the above procedure and starting with a continuous group X there is obtained a discrete group G of continuous homomorphisms (homomorphisms that preserve limits) of X into R, and in similar fashion G, is called the group of characters of X. In general, it may be shown [29 (a)] that if a compact metric topological group X and a discrete group G are orthogonal, then each is the group of characters of the other.

4.5. **Linking numbers; dualities.** If the groups G', G'' determining the chains C^k, D^{n-k} of (4.4) form an orthogonal group pair, then the number $\chi(C^k, D^{n-k})$ is defined by (4.4) in an obvious manner. Suppose, finally, that $\Gamma^k\varepsilon H^k(K, G)$, $\Gamma^{n-k-1}\varepsilon H^{n-k-1}(D, X)$, where G, X form an orthogonal group pair, and consider any $C^{n-k}\varepsilon C^{n-k}(D, X)$ such that $F(C^{n-k})=\Gamma^{n-k-1}$. Then the number $\chi(\Gamma^k, C^{n-k})$ is called the *linking number* of Γ^k and Γ^{n-k-1} (in K) and is denoted by $\nu(\Gamma^k, \Gamma^{n-k-1})$. It is dependent only on the Γ's, being the same for all chains C^{n-k} as well as for successive (concordantly oriented) subdivisions of K.

It was shown by Pontrjagin in [29(b)], to which we refer the reader for proofs, that if K is an S_n, and M a complex imbedded [22] in S_n, then the Betti groups $B^k(M, X)$, $B^{n-k-1}(S_n-M, G)$ form an orthogonal group pair, if for Γ^k, Γ^{n-k-1} as cycles in elements of these respective groups we define the product as yielding the linking number $\nu(\Gamma^k, \Gamma^{n-k-1})$ and let this be the product of those elements. (In order to apply the above definitions, chains in M are approximated by chains in the basic subdivisions Q_m of S_n, and chains of S_n-M are obtained relative to complexes dual to the basic subdivisions of S_n.) The duality thus obtained, establishing as a by-product the invariance of the Betti groups of the complement of a complex in S_n (under homeomorphisms of that complex) is of course a generalization of the Alexander duality. It holds, moreover, under suitable restrictions as to cycles employed [27] for the case of a general orientable closed manifold instead of S_n, as well as for the case where M is any closed point set. (See Chap. VI.)

With the geometric and algebraic background which we hope has been furnished in this section, the reader who is interested in these particular aspects of topology is urged to pursue the matter further, particularly with reference to the generalizations of the notions of cell and manifold [30] as well as the abstract approach in terms of cell-spaces [31] and the formulation in terms of cohomology groups [32] (also see Chap. VI).

V. The S_{n-1} imbedded in S_n, $(n=2, 3)$

In his fundamental work on point sets imbedded in S_2, Schoenflies showed that not only is the complement of a point set M, which is an S_1 imbedded in S_2, the sum of exactly two domains of which M is common boundary, but that if D is either one of these domains, then every point of M is *accessible* from D (that is, given $x \varepsilon M$, $y \varepsilon D$, there exists an arc xy such that $xy-x \subset D$). He then proceeded to show that the Jordan curve theorem, extended in this manner, allows a converse. Using the characterization of S_1 in the corollary to Theorem 17 (§2.1), we give below a simple proof of this converse, using an hypothesis weaker than that of Schoenflies [11, Chap. V].

As we have pointed out elsewhere [33], it was Schoenflies' feeling that when a suitable theory of connectivity for polyhedrals became available, such results as this could be extended to higher dimensions. Using the homology and duality theory developed above, as well as the material on Peano spaces in Chap. II, we give a sample in Theorem 2 below of such a result (see also §7.1). We include the proofs, adapted to present circumstances, as examples of the use to which the preceding material may be put. Since orientation has no advantages in these problems, we use modulo 2 chains exclusively.

5.1. **The S_1 in S_2.** We shall first prove the following theorem:

THEOREM 1. *In order that a point set M in S_2 should be an S_1, it is necessary and sufficient that it be a common boundary of two domains D_1 and D_2 from each of which every point of M is accessible.*

Proof. As for the necessity, besides the obvious application of the Jordan curve theorem, we may proceed in a manner which is quite uniform for all cases of S_{n-1} in S_n, and which we prefer to exemplify in the proof below of the u.l.0-c. property in Theorem 2; since it is easily shown that the u.l.0-c. property as defined below is stronger than the accessibility property.

Suppose that M is a common boundary of domains D_1 and D_2. That M is closed follows from Theorem 26 (§2.2). Suppose M not connected. Then in some subdivision Q of S_2, the closed 2-cells of Q that contain points of M form a complex K such that $p^0(K) \geqq 1$. By the duality of Chap. IV there exist cycles γ^1 of $S_2-(K)$ and γ^0 of K such that $\nu(\gamma^0, \gamma^1)=1$. The cycle γ^1 is the sum of cycles γ_i^1, $(i=1, 2, \cdots, k)$, such that for each i, (γ_i^1) is a component of (γ^1). Since, from the definition of ν,

$$\nu(\gamma^0, \gamma^1) = \sum_{i=1}^{k} \nu(\gamma^0, \gamma_i^1), \qquad \text{(mod 2)}$$

at least one linking number $\nu(\gamma^0, \gamma_i^1)$, say $\nu(\gamma^0, \gamma_1^1)$, is 1. The set (γ_1^1), being connected, lies wholly in one component of $S_2 - M$. And since (γ_1^1) must separate (K) in S_2, it also separates M. But every point of M is a limit point of both A and B, so that each of these domains contains points of (γ_1^1), contradicting the fact that (γ_1^1) lies in one component of $S_2 - M$.

No point of M separates M. For if $x \varepsilon M$ were such that $M - x = M_1 + M_2$, where M_1, M_2 are separated, and γ^0 a cycle based on a single pair of 0-cells one from each of the domains D_1, D_2, then γ^0 would have to be nonbounding in $S_2 - (M_1 + x)$ (see first part of proof of Theorem 7 (§3.5)).

Finally, every two points of M separate M. For, using the accessibility property, if $x, y \varepsilon M$ there exists an S_1, say J, consisting of two arcs t_i such that $t_i - (x+y) \subset D_i$ and x, y are the end points of t_i. By Theorem J_3 (§3.6), $S_2 - J$ is the sum of two domains A, B having J as common boundary. Each of the sets A, B contains points of M. For consider the point set $A + s_1 + s_2$, where $s_i \varepsilon t_i - (x+y)$; A is connected and both s_1, s_2 are limit points of A, and hence $A + s_1 + s_2$ is connected. But a connected set which contains points of both D_1 and D_2 is easily shown to contain points of M. Thus $M - (x+y) = M_1 + M_2$ where $M_1 \subset A$, $M_2 \subset B$ and hence M_1, M_2 are separated sets.

That M is an S_1 now follows from the corollary to Theorem 17 (§2.1).

Further properties of the complement of an S_1 in S_2 will be given when the general case is discussed in Chap. VII.

5.2. **The S_2 in S_3.** In one of those papers of Brouwer [34] that seem to have been inspired by Schoenflies' work, he pointed out that accessibility is not a sufficient tool to characterize simple surfaces in S_3. In the same connection he proved that the domains complementary to surfaces, among which in S_3 the S_2 is a particular case, have a stronger property (*Unbewalltheit*) which we call *uniform local* 0-*connectedness* (u.l.0-c.), and this property, as we shall see, is enough to "smooth out" the boundary of a domain in a manner not effected by accessibility.

A domain D is called u.l.0-c. if for every $\epsilon > 0$ there exists a $\delta > 0$ such that any 0-cycle γ^0 of D such that $\delta(\gamma^0) < \delta$ bounds a chain K^1 of D such that $\delta(K^1) < \epsilon$.

THEOREM 2. *A necessary and sufficient condition that a point set M in S_3 be an S_2 is that it be a common boundary of two u.l.*0-*c. domains A and B such that $p^1(A) = 0$.* [35].

Proof of necessity. Let M be an S_2 imbedded in S_3. Then, by the Alexander duality theorem (Chap. IV), $p^0(S_3 - M) = p^2(M) = 1$ (Corollary 4, §3.3), and hence $S_3 - M$ consists of just two domains A, B. That M is the common boundary of these domains follows from the duality theorem and an argument similar to that used in Theorem J_3 (§3.6).

The domain A is u.l.0-*c*. Consider an $x \varepsilon M$ and $\epsilon > 0$. From the continuity of the homeomorphism defining M, there exists in M an S_1, say J, which lies in $S(x, \epsilon)$ and such that if C and R are the domains in M whose common boundary is J, then $x \varepsilon C \subset S(x, \epsilon)$, and C and R are 2-cells. Let $\delta > 0$ be such that $M \cdot S(x, \delta) \subset C$, and consider any cycle γ^0 of $A \cdot S(x, \delta)$. There exists a chain K_1^1 of $S(x, \delta)$ such that $K_1^1 \rightarrow \gamma^0$; hence we may write

$$(5.1) \qquad K_1^1 \rightarrow \gamma^0 \quad \text{in} \quad S_3 - [\overline{R} + F(x, \epsilon)].$$

Also, since (γ^0) lies in the component A of $S_3 - M$, there is a K_2^1 such that

$$(5.2) \qquad K_2^1 \rightarrow \gamma^0 \quad \text{in} \quad S_3 - [\overline{C} + F(x, \epsilon)].$$

The cycle $\gamma^1 = K_1^1 + K_2^1$ bounds a chain of $S_3 - J$. For γ^1 is also a cycle of $S_3 - \overline{R}$, and $\overline{R}$ being the closure of a 2-cell, we have by the Alexander duality that $p^1(S_3 - \overline{R}) = p^1(\overline{R}) = 0$. Then by Theorem 6 (§3.5), which extends immediately to S_3, γ^0 bounds a chain of $S_3 - [\overline{C} + \overline{R} + F(x, \epsilon)]$, and hence in $A \cdot S(x, \epsilon)$.

The property just established holds for all points of $\overline{A}$, its proof being trivial for $x \varepsilon A$, and employing the Borel theorem 20 (§2.1) the u.l.0-c. property of A follows at once.

Since $p^1(M) = 0$ (Corollary 4, §3.3), it follows from the Alexander duality that $p^1(A) = 0$.

Proof of sufficiency. Let M be a common boundary of two u.l.0-c. domains A and B in S_3, such that $p^1(A) = 0$. We shall show that M satisfies the hypothesis of Theorem 39 (§2.2). That M is a continuum follows as in the proof of Theorem 1.

The set M satisfies Axiom 4 of Chap. II. Consider any $x \varepsilon M$ and $\epsilon > 0$, and let $C(x, \epsilon)$ denote the component of $M \cdot S(x, \epsilon)$ determined by x. If $C(x, \epsilon)$ is an open subset of M, Axiom 4 is satisfied. Suppose this is not the case. Then there exists $y \varepsilon C(x, \epsilon)$ such that y is a limit point of $M - C(x, \epsilon)$. Then if η_1 is such that $0 < \eta_1 < \epsilon - \rho(x, y)$, there exists in $S(y, \eta_1)$ an $x_1 \varepsilon M - C(x, \epsilon)$. The component $C(x_1, \epsilon)$ has no limit point in $C(x, \epsilon)$, hence there exists $\eta_2 < \eta_1$ such that $0 < \eta_2 < \rho(y, C(x_1, \epsilon))$. Suppose $x_2 \varepsilon [M - C(x, \epsilon)] \cdot S(y, \eta_2)$. Continuing in this manner, with $\lim \eta_i = 0$, it is shown that there exist mutually exclusive components $C(x_i, \epsilon)$, $(i = 1, 2, \cdots, n, \cdots)$, such that y is a limit point of the set of all points x_i.

Let δ be such that $\overline{S}(y, \delta) \subset S(x, \epsilon)$. Then no two points x_i are in the same component of $M \cdot S(y, \delta)$. As A is u.l.0-c., there exists a $\delta_1 > 0$ such that any 0-cycle of $A \cdot \overline{S}(y, \delta_1)$ bounds a chain of $A \cdot S(y, \delta)$; and such that a similar statement holds for B. Let x_k, $x_j \varepsilon S(y, \delta_1)$. By adapting the procedure used above to the present case, there can be shown to exist a cycle γ^2 of $S_3 - M \cdot S(x, \delta)$ such that (γ^2) is connected and separates x_k and x_j in S_3. Let T_1 be the sum of those components (finite in number) of $(\gamma^2) \cdot S(y, \delta)$ that have points in $S(x, \delta_1)$. Then $(\gamma^2) \cdot S(y, \delta) = T_1 + T_2$, where $T_2 = (\gamma^2) \cdot S(y, \delta) - T_1$. The set $N = F(y, \delta) + T_1 + T_2$ separates x_k and x_j in S_3. It follows readily from the extension to S_3 of Theorem 6 (§3.5) that

$N-T_2$ separates x_k and x_j in S_3, and finally that $C_1+F(y, \delta)$, where C_1 is the symbol for a single component of T_1, separates x_k and x_j.

But x_k and x_j are boundary points of both A and B, hence in some subdivision Q of S_3 can be found cycles γ_1^0, γ_2^0 of A and B, respectively, which lie in $S(y, \delta_1)$ and fail to bound a chain of $S_3-[C_1+F(y, \delta)]$. But there exist chains $L_i^1 \to \gamma_i^0$, $(i=1,2)$, in $A \cdot S(y, \delta)$ and $B \cdot S(y, \delta)$, respectively. Both sets $(L_i^1) \cdot C_1$ are therefore nonvacuous, and, consequently, C_1 contains points of both A and B, and $C_1 \cdot M \neq 0$. But this is impossible since $C_1 \subset (\gamma^2) \cdot S(y, \delta)$ and $(\gamma^2) \cdot S(y, \delta) \cdot M = 0$. We must therefore conclude that M is a Peano space.

By hypothesis, $p^1(A)=0$; we now prove that $p^1(B)=0$. Assume γ^1 a cycle nonbounding in B. In each complex Q_i of a sequence of subdivisions $\{Q_i\}$ of S_3, let P_i denote the complex composed of all closed 3-cells of Q_i that contain points of M. Then for i large enough, γ^1 is a nonbounding cycle of S_3-P_i and by the duality theorem is linked with a cycle z^1 of P_i. Now every vertex v of the complex associated with z^1 is a vertex of a closed 3-cell of P_i that contains at least one point of M; hence if every cell of Q_i is of diameter $<\epsilon_i$, where $\lim_{i\to\infty} \epsilon_i=0$, there is a point v' of A whose distance from v is $<\epsilon_i$. Then for i large enough, there exists, by virtue of the fact that A is u.l.0-c., a cycle Γ^1 of A composed of a set of chains of A bounded by 0-cycles associated with the v''s, and such that $\Gamma^1 \sim z^1$ in $S_3-(\gamma^1)$. Thus there exists a chain $K_1^2 \to \Gamma^1+z^1$ in $S_3-(\gamma^1)$. Now by hypothesis $p^1(A)=0$, hence there exists a chain $K_2^2 \to \Gamma^1$ in A. Let $K^2=K_1^2+K_2^2$; then $K^2 \to z^1$ in $S_3-(\gamma^1)$, $\chi(\gamma^1, K^2)=0$, and $\nu(\gamma^1, z^1)=0$. This contradiction establishes that $p^1(B)=0$.

No arc of M separates M. The proof of this is like that used in the third paragraph of the proof of Theorem 1.

Finally, every S_1 in M separates M. For suppose J an S_1 in M such that $M-J$ is connected. By the duality theorem, there exists a cycle γ^1 of S_3-J which is nonbounding in S_3-J, and such that for i large enough, there exist cycles z_i^1 in the dual of Q_i approximating J such that $\nu(\gamma^1, z_i^1)=1$. The set $(\gamma^1) \cdot M$ is a closed subset F of S_3-J. Letting $\rho((\gamma^1), J)=4\epsilon$, every $x \in F$ is in a domain of M of diameter $<\epsilon$, since M is peanian. By the Borel theorem, a finite number of these domains, say $U_1, U_2, \cdots, U_k$ contain all points of F. For each i, let $a_i \in U_i$. Since $M-J$ is a domain, there exists in $M-J$ an arc a_1a_i for each $i>1$, by the corollary to Theorem 24 (§2.2). The point set H formed by the closure of the set

$$\sum_{i=1}^{k} U_i + \sum_{i=2}^{k} a_1a_i$$

is a subcontinuum of $M-J$.

Letting $\rho(H, J)=\epsilon_1$, suppose η is a positive number less than both ϵ and ϵ_1. Using the u.l.0-c. of A and B, there exist, for i great enough, cycles Z_A^1 and Z_B^1 of A and B respectively, approximating z_i^1 and such that there exists a $K^2 \to Z_A^1 + Z_B^1$, every point of whose intersection $(K^2) \cdot M$ with M lies at a distance $< \eta$ from J, and for which $\nu(\gamma^1, Z_A^1) = \nu(\gamma^1, Z_B^1) = 1$. Since $p^1(A) = p^1(B) = 0$, there exist chains $K_A^2 \to Z_A^1$, $K_B^2 \to Z_B^1$ lying in A and B respectively. We have the relations

$$K_A^2 \to Z_A^1 \quad \text{in} \quad S_3 - [(\gamma^1) - (\gamma^1) \cdot A],$$

$$K^2 + K_B^2 \to Z_A^1 \quad \text{in} \quad S_3 - (\gamma^1) \cdot \overline{A}.$$

By the extension to S_3 of Theorem 6 (§3.5), the 2-cycle $M^2 = K_A^2 + K^2 + K_B^2$ must be nonbounding in $S_3 - (\gamma^1) \cdot M$, else Z_A^1 would bound in $S_3 - (\gamma^1)$. But this implies that the point set (M^2) separates points of H in S_3. However, H is a continuum and $H \cdot (M^2) = 0$, so that this is impossible.

Thus M is a Peano space satisfying the hypothesis of Theorem 39 (§2.2), and, consequently, M is a 2-sphere.

VI. Continuous mappings

Mapping spaces. In Chap. I we have defined the notion of continuous mapping of one space into another. Hereafter we drop the word "continuous"—it will be understood that all mappings are continuous. If A and M are spaces, there exist in general many different mappings of A into M; for example, the trivial "constant" mappings of type $f(A) = x$ where $x \in M$. The set of all mappings of A into M we denote by the customary symbol M^A. If M is compact [36] metric, the set M^A becomes a metric space (which we continue to denote by M^A) if, given f, $g \in M^A$, we define $\rho(f, g) = \text{LUB}_{x \in A} \rho[f(x), g(x)]$ (LUB = least upper bound).

6.1. **Homotopy and mapping classes.** Consider f, $g \in M^A$. We call f *homotopic to* g if there exists a continuous function $\phi(x, t)$, $x \in A$, $(0 \leqq t \leqq 1)$, whose "values" are points of M and such that $\phi(x, 0) = f(x)$, $\phi(x, 1) = g(x)$ for all $x \in A$. The fact that f is homotopic to g may also be expressed by saying simply that f and g are *homotopic*, since the relationship is plainly symmetric. Furthermore, being also reflexive and transitive, the relationship constitutes an equivalence relation in the set M^A, and the corresponding classes [37] are called the *mapping classes* of M^A. Where it will cause no confusion and f, $g \in M^A$ are homotopic, we shall sometimes speak of the images themselves as being homotopic; for example, if $K = f(S_1)$ is the image of a circle S_1 in M, we may say that K is homotopic to a point x in M meaning thereby that f is homotopic to the mapping $g(S_1) = x$.

6.2. **Special cases of mapping spaces.** We shall often be concerned below with particular, but important, choices of M or A. The case where M or A is the n-sphere S_n or the euclidean n-space E_n will be of special interest. One of the earliest examples of the latter type is due to Brouwer [38] (to whom, indeed, the notion of mapping classes is due), who took $M = E_1$ and A as any compact closed subset of E_n and showed the existence of an $f \in E_1^A$ where each $f^{-1}(x)$ is vacuous or a component of A. From R. L.

Moore's study of upper-semicontinuous collections of continua in the plane [39], it follows that if $f \in S_2^{S_2}$ such that each counter image $f^{-1}(x)$ is nonvacuous and connected and does not separate S_2, then the image set is S_2 itself, and as a matter of fact, using terminology to be explained below, f is a mapping of degree one.

The spaces $S_n^{S_m}$ for general values of m and n have been given considerable attention, particularly by Hopf. When $m<n$, the space $S_n^{S_m}$ is connected—indeed, arcwise connected, since every two mappings are homotopic. Likewise, $S_1^{S_m}$ for $m>1$ is connected. For $m=n$, $S_n^{S_m}$ has infinitely many components as shown by Hopf, who has also shown that the space $S_2^{S_3}$ has infinitely many components [40]. For $m>n>2$ very little is known concerning the structure of the space $S_n^{S_m}$. We return to these matters later on.

The more general case where either M or A is assumed merely to be peanian has also received considerable attention, especially by American topologists. In 1921 Hahn showed that a decomposition of a continuum C that is irreducibly connected between two points into its "prime parts" generates an $f \in E_1^{*C}$. Following out this idea, R. L. Moore in 1925 showed that a decomposition of any compact metric continuum K into its prime parts generates a mapping of K onto a compact Peano space C such that for every $x \in C, f^{-1}(x)$ is a prime part of K (unless of course K has only one prime part). This procedure has been made abstract and considerably extended both as to breadth of application and results by G. T. Whyburn and the present author [41]. The case A and M both Peano spaces has been studied extensively of late by topologists in this country, usually, however, with attention confined to special subspaces of M^A, as for instance the subspaces of arc-preserving mappings, interior mappings, etc. [42].

But to return to our main interest—the case where M or A is an S_n—it will simplify our exposition if we treat separately the cases where (1) M is an S_n and (2) A is an S_n. In accomplishing this, it will be necessary for us to broaden our basis of discussion, particularly with reference to the extension of the notion of Peano space.

6.3. **Some extensions of the notion of Peano space.** We remarked in §2.2 that in order for a locally compact, metric, connected space M to be peanian, it is necessary and sufficient that for any $x \in M$ and $\epsilon>0$ there exist a $\delta>0$ such that any two points of $S(x, \delta)$ are the end points of an arc in $S(x, \epsilon)$. In terms of homotopy, this is equivalent to requiring that there exists $\delta>0$ such that any image of S_0 in $S(x, \delta)$ be homotopic in $S(x, \epsilon)$ to a point.

Following this line of thought, we may define [43] a metric space M as *n-dimensionally locally connected* (n-LC) if, given any $x \in M$ and $\epsilon>0$, there exists $\delta>0$ such that any image of S_n in $S(x, \delta)$ is homotopic in $S(x, \epsilon)$ to a point. If M is n-LC for $n=0, 1, \cdots, p$, we call M an LC^p, and if M is n-LC for all non-negative integers n, we call M an LC^∞. In this terminology a Peano space is an LC^0. But we may go further. Let us call a subcomplex L of a complex K dense in K if all vertices of K are in L. A partial mapping of K into M will be any $f \varepsilon M^L$ where L is any complex dense in K. The mesh of such a mapping will be the maximum $\delta(f(E))$ for all cells E of L. Then we call M an LC if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every complex K and partial mapping of K into M of mesh less than δ, the mapping can be extended to a mapping of K of mesh less than ϵ. As an example (Borsuk) of a set which is LC^∞ but not LC, consider in Q_ω the closed segment L joining the origin to the point $(1, 0, 0, \cdots)$. Let $x_p = (1/p, 0, 0, \cdots)$, $(p = 2, 3, 4, \cdots)$, and let S_p be a p-sphere of center x_p and diameter less than $[\rho(x_p, x_{p+1})]/2$. The set of all such spheres together with the points of L not on diameters of any of these spheres is an LC^∞ but not an LC. Finally, M is an LC^* if it is an LC and for all n every image of an S_n in M is homotopic in M to a point.

Below we shall point out other types of local connectedness; but first we shall define certain spaces, which, while the fact is not at all apparent at first, are intimately related to the various types of spaces LC.

6.4. **Retracts.** Following a line of thought suggested by the notion of homotopy, suppose $M \subset A$ and that there exists $f \varepsilon M^A$ such that $M = f(A)$ and $f[f(x)] = f(x)$ for all $x \varepsilon A$; then we call M a *retract* of A [44]. Those sets which are retracts of Q_ω turn out to be identical with those sets that are retracts of every metric space in which they are imbedded, and are called, by Borsuk, *absolute retracts* (AR-set). The AR-sets are peanian, satisfy a fixed point theorem (see §6.9), and do not separate any E_n containing them; when imbedded in E_2, they are identical with those Peano continua that do not separate E_2.

Related to the AR-sets are the *absolute neighborhood retracts* (ANR-sets) [45]: If $M \subset A$, and M is a retract of some open subset of A containing it, then M is called a neighborhood retract of A; if, moreover, M is a neighborhood retract of every metric space in which it is imbedded, then it is an ANR-set; such sets are identical with the neighborhood retracts of Q_ω. Apart from their many interesting intrinsic properties, it is interesting to note that if $M \subset E_n$ is an ANR-set, then $E_n - M$ is the sum of a finite number of domains each of whose boundaries is peanian and accessible from the corresponding domain.

We come directly, below, to the relations of the sets just defined to local connectedness. Before this, however, we define another related notion, that of *local contractibility*. A metric space M is called *locally contractible* if, given $x \varepsilon M$ and $\epsilon > 0$, there exists $\delta > 0$ such that the set $S(x, \delta)$ is homotopic to a point in $S(x, \epsilon)$. It is immediately obvious that a locally contractible space is n-LC for all n. *Among the finite dimensional spaces, the ANR-sets are identical with the compact metric spaces that are locally contractible*

[45]; *and identical with the compact metric spaces that are n-LC for all n.* Furthermore, as shown by Lefschetz [43], *the properties ANR and LC are equivalent; and the properties AR and LC* are equivalent.*

Aside from these relationships, these sets have many interesting intrinsic properties—for instance, if M is an ANR-set there exists a complex K such that all Betti groups (§6.5) of M as well as the fundamental group (§6.7) are homomorphic mappings of the corresponding groups of K [46]—but it is necessary to refer the reader to the works cited for these, and return to the main line of thought.

6.5. **Betti groups of abstract spaces.** In Chapter IV we defined Betti groups $B^k(M, G)$ where M is a complex and G an abelian group. We consider now the possibility of setting up like groups for the case of an abstract space M, the nature of these groups being such that in case K is a complex, $B^k(K, G)$, $B^k((K), G)$ are isomorphic. This problem was first attacked from a standpoint closely allied to the approach outlined in Chaps. III and IV above. It should be moderately clear, for example, how a definition for Betti groups of LC-sets might be set up on a purely combinatorial basis, using cell-images. For spaces with no local connectedness properties, the first difficulty, apparently, to be overcome is the inability to realize cells. But if one recalls the manner of orientation of a simplex, as in setting up an indicatrix, for example (§6.1), it is apparent that the notion of k-cell as the homeomorph of the set of points $(x_1, x_2, \cdots, x_k)$ in E_k such that $x_1^2+x_2^2+\cdots+x_k^2<1$ is for the moment of less importance than the set of $k+1$ vertices related thereto; for the orientation of the simplex at least, only the latter are of importance.

Taking this into account, a theory of homology has been developed by Čech for very general spaces based on systems of coverings by open sets [47]. For compact metric spaces, in which we are especially interested below, this theory is equivalent to that based on V-cycles [48] which we now describe.

Let M be a compact metric space and G an abelian group. A specified set E^n of $n+1$ points of M, such that $\delta(E^n)<\epsilon$, we call an ϵ-n-cell of M; the points of E^n we call its "vertices." To E^n corresponds a positively oriented cell σ^n as in §4.1—merely an assignment of a certain ordering (and its even permutations) to the vertices. Each proper subset of $k+1$ points of E^n constitutes the vertices of a cell E^k, which is again oriented, and the boundary function F may be defined as before. An ϵ-n-chain of M over G is simply an n-chain in terms of the oriented ϵ-n-cells and G. A sequence $C^n=\{C_i^n\}$ of ϵ_i-n-chains over G is called a *V-n-chain over G* of M if $\lim_{i\to\infty}\epsilon_i=0$. If the elements of a V-n-chain γ^n are cycles such that for each i there exists a $\delta_i-(n+1)$-chain C_i^{n+1} of M such that $F(C_i^{n+1})=C_i^n-C_{i+1}^n$, where $\lim_{i\to\infty}\delta_i=0$, then we call the sequence γ^n a *V-n-cycle.* Thus, if we define the sum of V-n-chains $C_1^n=\{C_{i1}^n\}, C_2^n=\{C_{i2}^n\}$ as $C_1+C_2=\{C_{i1}+C_{i2}^n\}$, we have additive groups of chains and cycles for M. If we call a V-n-cycle

$\gamma^n = \{Z_i^n\}$ *bounding* provided there exists a V-n-chain $\{C_i^n\}$ such that $F(C_i^n) = Z_i^n$ for all i, then we obtain a group of bounding cycles; and hence the Betti group $B^n(M, G)$. These groups are obviously topological invariants, and by showing that for a complex they agree with the groups defined in Chap. IV, the topological invariance of the latter may be established. And using these groups on M, the Pontrjagin duality (§4.5) extends to the case where M is any closed subset of S_n [29].

Many variations of the definitions given in the preceding paragraph have proved useful. For example, if G is a subgroup of a group G', a V-n-cycle γ^n over G is also a V-n-cycle over G', and can be called *bounding over* G' provided there exists a V-n-chain over G', say C^n, such that $F(C^n) = \gamma^n$, obtaining a Betti group $B^n(M, G, G')$. The most common case of this sort is that where G is G_0 and G' is G_r. Or one may, for example, use only ϵ-n cycles over G as elements of a sequence $\{C_i^n\}$, and call the latter a cycle if the differences $C_i^n - C_{i+1}^n$ bound $\delta_i-(n+1)$-chains over G', etc. [49].

In this connection, it is interesting to note a result of Steenrod [50] to the effect that for any compact metric space M (or for a general topological space in case the Čech homology theory is employed), the group R (§4.1) is a universal coefficient group for the homology theory in the sense that if the groups $B^k(M, R)$ are known and G is a given abelian group, then the groups $B^k(M, G)$ are determined.

For a locally compact metric space, one may obtain a homology theory by restricting the V-n-chains used to those which are at the same time V-n-chains of compact subspaces of the given space.

In terms of V-cycles, we may state another extension of the notion of Peano space. We first note that in order that a space C satisfying Axioms $0-3$ of Chapter II should be a Peano space, it is necessary and sufficient that for $x \in C$ and $\epsilon > 0$ there exist $\delta > 0$ such that any 0-cycle (mod 2) of $S(x, \delta)$ bound in $S(x, \epsilon)$ (the proof of this is left to the reader). The group G_2 may of course be replaced by other groups, and having selected a coefficient group G, one may call a locally compact metric space *locally i-connected* (l.i-c.) in the sense of V-chains over G if for $x \in M$ and $\epsilon > 0$ there exists $\delta > 0$ such that every V-cycle over G of $S(x, \delta)$ bounds in $S(x, \epsilon)$. Similar to the notions LC^p, LC^∞, we have the notions lc^p, lc^∞ defined in terms of the V-cycles.

Regarding the relations of this new type of local connectedness with the types previously described, one can show by simple examples that the properties n-LC and l.n-c. are independent. But it can be shown that LC^n is in general a stronger property than lc^n. The most striking result in this connection, however, is that of Hurewicz [51] to the effect that if one uses chains over G_0 in defining the l.n-c. property, and confines himself to those compact spaces that are 1-LC, then the properties lc^n, LC^n are equivalent. Thus, *in terms of homotopies relating to mapping of S_1 alone*, one is able to determine a class of spaces in which these two important types of local connectedness are equivalent. For a discussion of the implications of the property lc^n, one is referred to [52]. We noted in §6.4 that if a continuum M in S_n is an ANR-set, hence an LC-set, the domains of $S_n - M$ are finite in number and have accessible peanian boundaries. In terms of the lc^n-property, this result may be generalized as follows: If a continuum M in S_n is lc^{n-2}, then the domains D_i, $(i=1, 2, \cdots)$, of $S_n - M$, if infinite in number, are such that $\lim_{i\to\infty} \delta(D_i)=0$, their boundaries are all peanian, and are accessible from their respective domains. For $n=2$ this result (with the exception of the peanian property of the domain boundaries, which was obtained later by M. Torhorst) was established by Schoenflies [11]. If one generalizes the notion of accessibility as has been done by Alexandroff [53], all points of an lc^{n-2} compact set in S_n are i-accessible for $i=1, 2, \cdots, n-2$. [54].

6.6. **Spaces S_n^A.** We are now equipped to discuss more fully the spaces S_n^A, where A is compact and metric, and their applications. When A is an S_m, we have already described certain known results (§6.2). The suggestions for the more general results which we presently recount are contained in certain results obtained by Alexandroff, Borsuk, Čech, and Hopf. Alexandroff [49] showed that if A is n-dimensional, then A contains a certain type of nonbounding cycle ("power-cycle") if and only if S_n^A is not connected, and as a corollary, the latter is a necessary and sufficient condition that, in case $A \subset S_{n+1}$, A separate S_{n+1}; the latter result was also obtained independently by Borsuk [55]. In [56] Hopf found that the number of components in S_n^A, if A is an n-dimensional complex, is infinite if $p^n(A, G_0) > 0$, otherwise the number is equal to the order of the $(n-1)$th torsion group (the group generated by the generators of finite order of the $(n-1)$th Betti group). Earlier Hopf [40] established that if A is any complex, the condition $p^1(A, G_0)=0$ is equivalent to the connectedness of S_1^A; a result which was extended by Borsuk [57] to the case where A is compact metric. The latter results were carried by Bruschlinsky [58] to the determination of the "reduced" 1-dimensional Betti group $B^1(A, G_0, G_r)$, by the device of turning S_1^A into a group.

Suppose T is a topological group, and consider $f, g \in T^A$. For any $x \in A$, let $h(x)=f(x)\cdot g(x)$, the operation on the right being that in T. Since T is a topological group, it follows that $h \in T^A$, and we may write $h=fg$, where now the operation on the right is the one induced in T^A, and relative to this operation the elements of T^A form a group whose identity is the mapping of A into the identity. The component determined by the identity in T^A is a normal divisor D of T^A, and the factor group T^A/D has as its elements the components of T^A. Let us call this factor group the *mapping class group of A rel. T*, as distinguished from the mapping group T^A. If we consider the group operation in S_1 to be that of the rotations of the circle, then the mapping class group of A rel. S_1 is isomorphic to the 1-dimensional reduced Betti group of A. In particular, then, the Betti number

$p^1(A, G_0)$ is determined solely from a consideration of the mapping group S_1^A.

Similarly, by setting up a suitable group operation in S_3, Bruschlinsky showed that if A is 3-dimensional, then $B^3(A, G_0, G_r)$ is isomorphic to the factor group of the mapping class group of A rel. S_3 modulo the subgroup formed of elements of finite order; and if A is in addition a complex, then the "full" Betti group $B^3(A, G_0)$ is isomorphic to this mapping class group of A rel. S_3. The difficulty encountered in extending these results to other dimensions is due to the lack of a group operation in the corresponding S_n's. However, Freudenthal [59] overcomes this obstacle by setting up a "partial" operation in S_n, sufficient to define a group, analogous to the mapping class group, whose elements are the components of S_n^A. This group he calls the "Hopf group" and shows that for an n-dimensional A, it completely determines the Betti group $B^n(A, G_m)$, $m=0$, or $m>1$.

It will be noted that, with the single exception $n=1$, the results quoted concerning the determination of Betti groups through mappings of A into S_n all hold only for dimension $A=n$. To attack the problem for the general dimension $A>n$, one obviously needs to modify or refine the above procedures in some way, in view of known examples; for instance, as shown by Hopf [40], the space $S_2^{S_3}$ is not connected, and yet $p^2(S_3, G)=0$. That this is possible has been shown by Lefschetz and Hopf [60], who, for any complex K, have been able to determine the Betti groups of all dimensions purely from a consideration of certain types of mappings into the various S_n's. We omit further details here, while observing that it is quite apparent, from a consideration of results already obtained, that from mappings into S_n, considerable information concerning the topology of a space, hitherto derivable only by purely combinatorial methods, may be secured. Our citations above do not, as a matter of fact, give a complete picture by any means; for example, Borsuk and Eilenberg [61] have made a special study of S_1^A and employed mappings into S_1 in a most effective manner in the investigation of plane topology, establishing with this tool many theorems ordinarily obtained only by combinatorial methods.

6.7. **Spaces A^{S_n}.** We saw in §6.3 how an important type (n-LC) of local connectedness may be introduced by imposition of homotopy conditions on certain elements of A^{S_n}, namely those elements whose related images in A are "small." Without the latter restriction, but with $n=1$, we encounter a notion basic in the definition of the so-called *Poincaré group*, or *fundamental group*. (For more detailed exposition than is possible here, see especially [62].) Poincaré discovered certain 3-dimensional manifolds ("Poincaré spaces") whose Betti groups were the same as those of S_3, but whereas $S_3^{S_1}$ is connected, the Poincaré spaces contain images of S_1 not homotopic to a point.

Suppose M is an arcwise connected space and $x \in M$. Let $\sigma_1=P^0P^1$ be any oriented 1-cell (on E_1^* for instance), and consider an $f \in M^{\sigma_1}$ such that

$f(P^0)=f(P^1)=x$. Then the image (§1.3) $u=f(\sigma_1)$ is called an *oriented* (closed) *path* of M; the image $f(-\sigma_1)$ is also an oriented path of M, and is denoted by u^{-1}. If $u=f(\sigma_1)$, $v=g(\sigma_2)$ are two oriented paths of M, we may define a path $s=u\cdot v$, *product* of u and v, by considering the "terminal" and "initial" points (and only these) of u and v, respectively, as coincident, and letting s be the image $\phi(\sigma)$ where σ is the geometric sum of σ_1 and σ_2, and identical with f, g on σ_1, σ_2 respectively. The fundamental group of M is a group $\pi_1(M)$ whose elements are sets of oriented paths of M (end points fixed at x), two paths u and v being in the same set if and only if $u\cdot v^{-1}$ is homotopic to x. The identity of $\pi_1(M)$ is the set of paths homotopic to x; the product $S=U\cdot V$ of U, $V \varepsilon \pi_1(M)$ is that set S which contains $s=u\cdot v$ for any $u \varepsilon U$, $v \varepsilon V$; and the inverse U^{-1} of $U \varepsilon \pi_1(M)$ is the set containing u^{-1} for any $u \varepsilon U$. In general, $\pi_1(M)$ is not abelian, in contrast to the abelian character of the groups $B^1(M, G)$. (For M a complex or, more generally, a locally contractible space, the factor group of $\pi_1(M)$ modulo its commutator group is isomorphic to $B^1(M, G_0)$.) It is not dependent on the particular choice of x, and its topological invariance is obvious.

It was conjectured by Poincaré that among the closed n-dimensional manifolds M_n, the sphere S_n is characterized by its Betti groups $B^k(S_n, G_0)$ and fundamental group $\pi_1(S_n)$. In particular, is S_3, then, the only orientable closed 3-manifold whose fundamental group "vanishes"? To date, no answer to this question has been published, although it is known [63] that the Betti groups and fundamental group of two 3-manifolds may agree without their being homeomorphic.

The fundamental group has furnished a new invariant making possible the differentiation between spaces which agree in other known invariants, as well as a tool fundamental in investigations such as those concerning knots and "covering manifolds" [64]. Rather than discuss these matters, we pass on to the problem of generalizing the group $\pi_1(M)$ and the rôle of the n-sphere therein.

A generalization of the fundamental group. The most successful generalization of $\pi_1(M)$ is due to Hurewicz [65]. With him, we consider M as not only connected, but as a locally contractible, metric, separable space. If A is a finite dimensional, compact metric space, then M^A is locally contractible, a fortiori 0-LC, and has an at most denumerable set of components; and each of the latter is open and arcwise connected. Hence if $f \varepsilon M^A$, we may consider the group $\pi_1(C_f)$, where C_f is the component of M^A containing f. Since C_f is locally contractible, oriented paths which approximate closely enough a given path u will be in the same element of $\pi_1(C_f)$ as u. Consequently the order of $\pi_1(C_f)$ will be at most denumerable.

The case which most interests us is that where A is S_{n-1}. We may then simplify the above procedure by choosing constant points $x \varepsilon S_{n-1}$, $y \varepsilon M$, and limiting ourselves to that subspace $M_{xy}^{S_{n-1}}$ of $M^{S_{n-1}}$ which consists of all $f \varepsilon M^{S_{n-1}}$ such that $f(x)=y$. It develops that a group $\pi_1(C_f)$ for the space $M_{xy}^{S_{n-1}}$ is independent, not only of the choice of C_f, but also of x and y, and *consequently depends only on M and n*. This group Hurewicz calls the *nth homotopy group*, $\pi_n(M)$, *of M*. For $n=1$, it obviously agrees with the fundamental group $\pi_1(M)$. However, whereas the latter is not abelian, the homotopy groups $\pi_n(M)$ for $n>1$ are all abelian. Also, whereas for complexes, the factor group of $\pi_1(M)$ modulo its commutator group agrees with $B^1(M, G_0)$, for $n>1$, $\pi_n(M)$ appears in general to be independent of the Betti group. For example, in the case of closed 2-manifolds with positive Betti numbers, $\pi_2(M)$ reduces to the identity. However, if the first $n-1$ homotopy groups $(n>1)$ of M reduce to the identity, then $B^n(M, G_0)$ and $\pi_n(M)$ are isomorphic; in particular, then, the first n homotopy groups reduce to the identity if and only if the like holds for the first n Betti groups and the group $\pi_1(M)$. Also the reduction of $\pi_n(M)$ to the identity is equivalent to the connectedness of the space M^{S_n}.

An interesting by-product of these investigations [65, II] is the establishing of an equivalence between the Poincaré conjecture cited above and each of the following statements (M_n denoting a closed n-manifold): (1) If every closed proper subset is contractible to a point in M_n, then $M_n=S_n$; (2) if S_n can be mapped on M_n with degree 1 (see below), then $M_n=S_n$.

6.8. **Degree of a mapping; cohomology groups.** Of central importance in the theory of mappings as developed by Brouwer is the *degree* of a mapping (*Abbildungsgrad*). Let M and N be two orientable closed n-manifolds, oriented as in Chap. IV, and consider any $f \in N^M$. On the basis of a well known deformation theorem of Alexander and Veblen [66], it may be shown that for some simplicial subdivisions of M and N there will exist $g \in N^M$ such that f and g are in the same mapping class and $g(M)$ constitutes a *simplicial mapping*—that is, each cell of M maps onto a cell of N, vertices of M going into vertices of N. If $g(\sigma^n)$ is the image of an oriented cell of M, and $g(\sigma^n)$ covers an oriented cell ξ^n of N, then, since vertices map onto vertices, one defines in an obvious way agreement of orientation of $g(\sigma^n)$ with ξ^n; if the orientation of $g(\sigma^n)$ agrees with that of ξ^n, we say the degree $d_g(\xi^n, \sigma^n)$ of the covering of ξ^n by σ^n is $+1$; otherwise it is -1. If $g(\sigma^n)$ does not cover ξ^n, then by definition $d_g(\xi^n, \sigma^n)=0$. The number

$$d_g(\xi^n, M) = \sum_{\sigma^n} d_g(\xi^n, \sigma^n)$$

turns out to be the same for all ξ^n, and hence may be denoted by $d_g(N, M)$; it is the degree of the mapping g of M onto N. As shown by Brouwer [67], it is the same for all choices of g, hence by defining $d_f(N, M)=d_g(N, M)$ an invariant of the mapping class is obtained [68].

One naturally asks, if two mappings are of the same degree, do they necessarily belong to the same mapping class? If $N=S_2$ and M is any closed 2-manifold, an affirmative answer was given by Brouwer, a result extended later by Hopf to the case $N=S_n$ and M a closed n-manifold.

Later Hopf went further [56], getting a corresponding result for the case of M any n-complex (a very simple proof is given by Whitney in [69]), by suitably defining the notion of degree modulo m, ($m=0, 1, 2, 3, \cdots$), of the mapping of a cycle modulo m into S_n, and classifying the mapping classes of S_n^M by the medium of the degrees modulo m. Following this line of thought, Hurewicz [65, III], considering the case where M and N are compact spaces, defined $f, g \in N^M$ to be of the same *homology type in dimension n* if for every abelian group G the homomorphisms induced by f, g of $B^n(M, G)$ into $B^n(N, G)$ agree. For the case where M is n-dimensional, and N is a connected LC^n whose groups $\pi_i(N)$, ($i=1, 2, \cdots, n-1$), reduce to the identity, he showed that if $f, g \in N^M$ are of the same homology type in dimension n, they belong to the same mapping class [70].

It should be emphasized here that most of the work of this type (as well as the treatment of dualities mentioned above) is considerably simplified by the use of R as coefficient group or the employment of the cohomology groups. Regarding the latter, in Chap. IV we introduced a boundary operator such that $F(\sigma_i^k)=\sum_h e_i^h\sigma_h^{k-1}$, or more generally, $F(\sum_i c\sigma_i^k)=\sum_i\sum_h c^i e_i^h\sigma_h^{k-1}$. The numbers e_i^h were determined by the incidence of the oriented cells, and the first of these relations may be paralleled by introducing a "coboundary" operator F_c such that $F_c(\sigma_i^{k-1}) =\sum_h u_i^h\sigma_h^k$, where $e_i^h=u_h^i$, the second relation having the analogue $F_c(\sum_i c^i\sigma_i^{k-1})=\sum_i\sum_h c^i u_i^h\sigma_h^k$. If we call any polynomial $\sum_i c^i\sigma_i^k$ a *co-chain*, we have the corresponding *co-cycles*, *coboundaries*, and finally the *cohomology groups* $B_c^k(K, G)$. If A, B are character groups of one another, then $B^k(K, A)$ and $B_c^k(K, B)$ are character groups of one another [71]. As formulated by Lefschetz and Whitney [72], the Hopf result on mapping classes may be stated as follows: *The mapping classes of* S_n^M *(M an n-complex) are in* $(1-1)$*-correspondence with the elements of the cohomology group* $B_c^n(M, G_0)$. For the Hurewicz generalization, a like formulation holds, except that G_0 is replaced by $\pi_n(N)$.

6.9. **Further remarks concerning** $S_n^{S_n}$. In §6.2 we have already reviewed briefly the case $S_n^{S_m}$. Here we mention some points regarding the case $m=n$. We remarked above that the components of $S_n^{S_n}$ are infinite in number. The early work of Brouwer on this case centered about the presence of *invariant* or *fixed points* of a mapping. If we call $f \in S_n^{S_n}$ *sense-preserving* in case $d_f(S_n, S_n)>0$, and *sense-reversing* in case $d_f(S_n, S_n)<0$, then (regarding the original S_n's as identical) for n even and sense-preserving, there is at least one fixed point; that is, there exists $x \in S_n$ such that $f(x)=x$; and similarly for n odd and sense-reversing. This work has been greatly extended by Lefschetz [73]; a corollary of his results, of special interest in regard to the material in §6.3 above, is that if M is a compact metric continuum which is an LC^∞ and whose nth Betti numbers ($n>0$) vanish, then any $f \in M^M$ has at least one fixed point. (Also see §6.4.)

The restriction to the case of an f which is periodic has also received attention. If f is a homeomorphism of such a type that f^n is the identity mapping (and not for $m<n$), then f is called a mapping of period n. It was shown by Brouwer and also by Kérékjarto that every periodic mapping of an S_2 is equivalent to a rotation, or to the product of a rotation by a reflection. This work has been greatly extended by P. A. Smith [74], who considered periodic mappings of certain general spaces having the same Betti groups as an n-sphere, and has succeeded in showing that for the period prime, the set of invariant points also has the same homology properties as a sphere. More particularly, he has shown [75] that the invariant points of a periodic mapping of an S_3 are either a vacuous set, or an S_i where $i=0$, 1 or 2; the last case being possible only when the period is two and the mapping is sense-reversing.

6.10. **General remarks.** The contents of the present section were planned as an indication of the tendencies of some of the most recent investigations in topology. The determination of homology groups by mappings into S_n, the introduction of general homotopy groups as generalizations of the Poincaré group and the extension to higher dimensions of the peanian idea by mappings of S_n in the space not only show the fundamental rôle still played by the euclidean n-sphere, but emphasize the complete unification of the abstract set-theoretic and combinatorial methods in topology. It seems hardly necessary to point out the problems still to be settled, as the incompleteness of the various topics outlined above is suggestive of itself. We have not, of course, touched on all the important investigations involving the n-sphere—we have omitted, for instance, the *sphere-spaces* recently introduced and investigated by Whitney and reported on in [76]; also the relations of the S_1 to its complement in S_3, among which are the *knot* properties which have received so much attention [64]. But we hope, despite these limitations, that we have achieved the general purpose of this section and that the reader who desires more complete information will find sufficient sources in the citations.

VII. General case of S_{n-1} in S_n; a general "Jordan curve theorem"

The contents of §7.1 below belong properly with the material of Chap. V. It seemed best, however, to defer the discussion until the more advanced notions of Chap. VI were available, particularly those of homotopy and general chain and cycle groups. Whereas the cases $n=2$, 3 of Chap. V were quite readily handled by the machinery introduced previously , the more powerful tools made available in Chap. VI are indispensable in giving a satisfactory picture of the higher dimensional situations. For detailed proofs of the theorems quoted we must refer to the bibliography. In §7.3 we show how the abstract chains and cycles may be applied to obtain a very general theorem of which the Jordan curve theorem is a special case, and which, moreover, applies to a much wider variety of spaces than the S_n's.

7.1. **The S_{n-1} in S_n.** In Chap. V we discussed the case of an S_{n-1} in S_n for $n=2, 3$. The unrestricted case offers as chief difficulty the lack of a topological characterization of S_k, for $k>2$, comparable to those provided for S_1 and S_2 in Chap. II. However, suppose M is an S_{n-1} in S_n; what can be said regarding the topological character of S_n-M? From the duality theorem we know that S_n-M is the sum of two domains A and B of which M is the common boundary (the latter fact being established by an argument like that of Theorem J_3, §3.6); also that the Betti numbers of A and B are all zero.

Regarding the homotopy properties of A and B, we have the well-known example [6] for the case $n=3$ for which the group $\pi_1(A)$ has an infinite set of generators. So far as we know, the question as to whether $\pi_1(A)$ can have a finite set of generators has not been investigated. At any rate, the example referred to shows that neither A nor B need be an n-cell, if $n>2$. Here we have one of the essential differences between the cases $n=2$ and $n>2$, so common in the case of more general point sets, since for $n=2$, both A and B are 2-cells [77]. For $n>2$, the question concerning whether, when $\pi_1(A)$ reduces to the identity, the point set A is an n-cell, has not been answered. From the work of Hurewicz on homotopy groups (§6.7) we know that if $\pi_1(A)=0$, then $\pi_k(A)=0$ for all $k>0$; or, to state the matter in different terms, no matter how great n may be, if every S_1 in A is homotopic to a point in A, then every S_k in A for $k>0$ is homotopic to a point in A.

As for the "smoothness" of the general S_{n-1} in S_n, we may show that not only are A and B u.l.0-c., but also u.l.i-c. for all $i>0$: A set A is called *uniformly locally i-connected* (u.l.i-c.) if for arbitrary $\epsilon>0$ there exists $\delta>0$ such that every i-cycle of A of diameter $<\delta$ bounds a chain of A of diameter $<\epsilon$. The proof, except for the different dimensions involved, is the same as for the $i=0$ case in Chap. V.

Since A and B are u.l.0-c., all points of M are accessible from both A and B. Moreover, for any non-negative i, the points of M are i-accessible in the senses of Alexandroff and Čech [53, 54]. These are weaker properties, however, than the u.l.i-c. properties.

What can be said by way of converse theorems? Thus, what is the nature of a common boundary in S_n of two u.l.i-c., $(i=0, 1, \cdots, n-2)$, domains whose Betti numbers are zero? As a matter of fact, it turns out [78] that *if all we know of M is that it is the boundary of a single u.l.i-c., $(i=0, 1, \cdots, n-2)$, domain A whose $(n-1)$th Betti number is zero, then we may conclude that in S_n-M there is one and only one other domain B whose boundary is again M and which is u.l.i-c. for $i=0, 1, \cdots, n-2$; and furthermore that for each i, the ith Betti number of A is equal to the $(n-i-1)$th Betti number of B.* It follows that M satisfies the Poincaré duality theorem.

If the Betti numbers of A are all zero, is M an S_{n-1}? We saw in Chap. V that the answer is affirmative for $n=2, 3$. For $n>3$ we encounter the difficulty mentioned at the beginning of this section. In this situation we may do the next best thing, which appears to be to show that M has many of the topological properties of S_{n-1}.

7.2. **Generalized manifolds.** The condition that the Betti numbers of A be zero is not of much moment in the investigation of such a question, since for any $(n-1)$-manifold in S_n the local properties, internally and externally, are the same as for an S_{n-1} in S_n, and the essential properties in the large are embodied in the dualities mentioned in the preceding paragraph; and that $p^{n-1}(M)=1$ and $p^{n-1}(F)=0$ for any closed proper subset F of M are, of course, a result of the common boundary property of M. In short, just as the notion of sphere leads to that of manifold, so are we led in generalizing the notion of sphere to generalizing the manifold notion. This has been accomplished in a variety of ways; in the theory of complexes by replacing "cellular" by "cell-like in homology properties," for instance [30]. In passing to general spaces, a like approach may be made. The so-called *generalized manifolds*, recently introduced by a number of authors [78, 79], not only give natural generalizations of the manifold notion, but furnish directions for attack on such a problem as that proposed above.

So far as the various definitions of generalized manifolds that have been given to date are concerned, it turns out that, whenever a continuum M is the boundary of a u.l.*i*-c., $(i=0, 1, \cdots, n-2)$, domain A in S_n, then M is a generalized $(n-1)$-manifold in all these senses—indeed, for M to satisfy the definition of any one of them is to satisfy them all; and conversely, if M is an $(n-1)$-generalized manifold in S_n it is the boundary of such a domain. We do not review here the various definitions of generalized manifold, nor the proof of the statements just made; the reader may refer to the above citations for more detailed information. In passing, however, we note that instead of limiting oneself to the boundary of a *domain* A, one can obtain the following general result [78]: *A necessary and sufficient condition that an open subset* D *of* S_n *be u.l.i-c. for* $i=0, 1, \cdots, n-2$ *is that* (1) D *consist of a finite number of domains* $D_1, D_2, \cdots, D_m$ *such that* $D_j \cdot D_k=0$ *if* $j\neq k$; (2) *each component of* B_k, *the boundary of* D_k, $(1\leqq k\leqq m)$, *be either a point or a generalized closed* $(n-1)$-*manifold, and* $B_k=B_{k0}+B_{k1}+\cdots+B_{kh}+B_{k(h+1)}+\cdots$, *where* B_{k0} *is the set of point components of* B_k *and the manifolds* B_{kj}, *when* $j>h$, *satisfy the conditions* $p^i(B_{kj})=0$ *for* $1\leqq i\leqq n-2$; (3) $\lim_{j\to\infty}\delta(B_{kj})=0$. *Furthermore, the Poincaré duality holds for* B, *the boundary of* D, *and the duality* $p^i(D)=p^{n-i-1}(S_n-B-D)$. *For* $n=3$, *the sets* B_{kj}, $(j>k)$, *are all* S_2's.

A recent attempt to define abstractly a space which may possess most of the topological properties of S_n is of peculiar interest [80]: A space S such that for each $x \varepsilon S$ there exists a neighborhood U of x such that $S-U$ is an absolute retract (§6.4). For 1- or 2-dimensional S, such a space is the corresponding S_n, and for higher dimensions has all the homology and homotopy properties of S_n. A more general type of space, defined by the same author, is obtained by requiring in the above definition only that $S-U$ have the homology properties of a closed n-cell—the homotopy properties of S_n no longer hold here, although for the 1- and 2-dimensional cases the S_n is the only space satisfying these conditions.

7.3. **A general "Jordan curve theorem."** Throughout Chaps. II–V the Jordan curve theorem played a basic rôle; it furnished a motive for the first two of these sections as well as for the duality theorem of Chap. IV and the theorems of Chap. V. In Chap. II, we found that among Peano spaces the theorem characterizes S_2. We conclude with a theorem of which the Jordan curve theorem is a very special case [81]. In the case of S_n, note that we deal with a Peano space such that $p^n(S^n)=1$ and $p^n(F)=0$ for every closed proper subset F of S_n; and that the set M which decomposes S_n into just two domains of which it is common boundary satisfies a like condition with "n" replaced by "$n-1$." Now consider the S_0 in S_1—the former is the common boundary of two domains in S_1. A more or less plausible reason for this is that S_1 has no cut point (§2.2). Let us generalize the notion of cut point [82] as follows:

DEFINITION: *A point $x \in M$ is called a k-dimensional cut point of M if there exists a cycle γ^k of $M-x$ which bounds on M, but does not bound on any closed subset of $M-x$.*

Intimately related to the cut point notion, but having reference to a specified cycle is that of a barrier: If γ^k is a cycle of M, let us call a *carrier* of γ^k any closed subset F of M such that γ^k is also a cycle of F. Then, given a γ^k which bounds on M, and a carrier F of γ^k, a point $x \in M-F$ will be called a *barrier* to γ^k (in M) if γ^k fails to bound on every closed subset of $M-x$.

In terms of these definitions, no point of an S_n is an $(n-1)$-dimensional cut point of S_n and a fortiori none is a barrier to an $(n-1)$-cycle. We state and prove the following theorem, using (as in Chap. III) G_2 as coefficient group and the V-cycles of §6.5. For brevity, let us call a cycle *essential* if it has a carrier on which it fails to bound.

THEOREM J. *Let M be a compact Peano space such that $p^n(F)=0$ for every closed proper subset F of M; and let J be a closed subset of M such that $p^{n-1}(J)=1$ and $p^{n-1}(F)=0$ for every closed proper subset F of J. Denoting by γ^{n-1} the essential $(n-1)$-cycle of J, let γ^{n-1} bound on M and no point of $M-J$ be a barrier to γ^{n-1}. Then the set $M-J$ is the sum of two mutually exclusive domains of which J is the common boundary.*

The proof we give is chiefly an application of Lemmas 2 and 3 below. If a cycle γ bounds on a closed set F but bounds on no proper closed subset of F, then F will be called an *irreducible membrane relative to* γ. If a cycle γ

bounds on a set M, then there exists in M at least one irreducible membrane relative to γ; and every cycle γ has an *irreducible* carrier [83].

LEMMA 1. *In a metric space, let γ^n be an essential n-cycle and let M_1, M_2 be irreducible membranes relative to γ^n such that $M_1 \neq M_2$. Then M_1+M_2 is a carrier of an essential $(n+1)$-cycle.*

Proof. If K_1, K_2 are chains bounded by γ^n on M_1, M_2, respectively, and J is the irreducible carrier of γ^n, then some subsequence of the sequence of $\delta_i-(n+1)$-chains forming K_1+K_2 is a cycle γ^{n+1} whose irreducible carrier is M_1+M_2 with possibly certain points of J deleted [84]. We continue to denote the chains derived from harmonizing [84] K_1, K_2, γ^n with γ^{n+1} by the same symbols, and the new J, M_1, M_2 by the same symbols; we still have $M_1 \neq M_2$. Suppose $x \varepsilon M_1-M_2$.

Let $\epsilon>0$ be such that $M_2 \cdot \overline{S}(x, \epsilon)=0$. The portion of K_1 in $S(x, \epsilon)$ is a chain $F_1 \rightarrow \Gamma^n$, where Γ^n is a cycle of $F(x, \epsilon)$ [85]. The chain $E_1=K_1+F_1 \rightarrow \gamma^n+\Gamma^n$ is a chain of $M-S(x, \epsilon)$, and the cycle K_1+K_2 is identical with $E_1+F_1+K_2$.

Suppose there exists $K_3 \rightarrow K_1+K_2$ on M_1+M_2. The portion of K_3 in $S(x, \epsilon)$ is a chain $K_3' \rightarrow F_1+F_2$, where F_2 is a chain of $F(x, \epsilon)$. Then, since $F_1 \rightarrow \Gamma^n$, and $F_1+F_2 \rightarrow 0$, we have $F_2 \rightarrow \Gamma^n$. But F_2 is a chain of M_1, and hence we have $E_1+F_2 \rightarrow \gamma^n$ on $M_1-S(x, \epsilon)$, contradicting the fact that M_1 is an irreducible membrane relative to γ^n. Hence K_1+K_2 must be an essential cycle.

LEMMA 2. *Let M be a compact metric space such that $p^n(M)>0$ and $p^n(F)=0$ for every closed proper subset of M, and J a closed subset of M such that $p^{n-1}(J)=k$. Then $M-J$ has at most $k+1$ components* [86].

Proof. Suppose $M-J$ has at least $k+2$ components, so that $M-J$ is the sum of $k+2$ mutually separated subsets M_i. Let γ^n be an essential cycle of M. Then $\gamma^n=\Gamma^n+\sum_i \gamma_i^n$ where Γ^n is a chain of J, γ_i^n a chain of M_i, and $\Gamma^n \rightarrow \gamma^{n-1}$, $\gamma_i^n \rightarrow \gamma_i^{n-1}$, the boundary cycles being cycles of J.

Since $p^{n-1}(J)=k$, some linear combination of the cycles γ_i^{n-1}, $(1 \leqq i \leqq k+1)$, bounds on J; say $K^n \rightarrow \sum_{i=1}^{k+1} c_i \gamma_i^{n-1}=\gamma^{n-1}+\sum_{i=1}^{k+2} d_i \gamma_i^{n-1}$ where $d_i=c_i+1$ (mod 2) and in particular $d_{k+2}=1$. Then $H^n=\Gamma^n+K^n \rightarrow \sum_i d_i \gamma_i^{n-1}$, H^n being a chain of J. Also, $\sum_i d_i \gamma_i^n \rightarrow \sum_i d_i \gamma_i^{n-1}$, hence $\gamma^n+\sum_i d_i \gamma_i^n \rightarrow \sum_i d_i \gamma_i^{n-1}$. The cycle γ^n is the sum (mod 2) of the cycles $\gamma^n+\sum_i d_i \gamma_i^n+H^n$ and $\sum_i d_i \gamma_i^n+H^n$, each of which bounds on M since $p^n(F)=0$ for every proper closed subset of M. But then γ^n bounds on M, contradicting the hypothesis.

LEMMA 3. *Under the hypothesis of Theorem J* [87], *$p^n(M)>0$ and the set J separates M.*

Proof. Let M_1 be an irreducible membrane of M relative to γ^{n-1}. Let $x \varepsilon M_1-J$. As x is not a barrier to γ^{n-1}, there exists in $M-x$ another irreducible membrane relative to γ^{n-1}, say M_2. By Lemma 1, M_1+M_2 carries an essential n-cycle, and since no proper closed subset of M carries an essential n-cycle, $M=M_1+M_2$. Consequently $p^n(M)>0$.

The set $M-M_1\neq 0$, else $M_2\subset M_1-x$ and M_1 would not be an irreducible membrane relative to γ^{n-1}. Let H be a component of $M-M_1$; then $H\subset M_2$. If $\overline{H}-H\subset J$, then J separates M by Theorem 26 (§2.2). If $J\not\supset\overline{H}-H$, then H has a limit point $y\in M_1-J$. Now $M-y$ contains an irreducible membrane M_3 relative to γ^{n-1}, and arguing as above, we see that $M=M_1+M_3$. But this is impossible, since if $0<\epsilon<\rho(y, M_3)$, $H\cdot S(y, \epsilon)\neq 0$ and $H\cdot S(y, \epsilon)\cdot(M_1+M_3)=0$. Consequently $\overline{H}-H\subset J$ and J separates M.

Proof of Theorem J. By Lemmas 2 and 3, $M-J$ has just two components A and B. The boundary of each of these is J. For if the boundary of A, say, were a proper subset J' of J, then J' would separate M. But $p^{n-1}(J')=0$ and by Lemma 2, such a set as J' cannot separate M.

An interesting feature of Theorem J is that it imposes no conditions on the dimensions of the point sets involved, at least not explicitly. Whether the conditions given imply, for instance, that the space M must itself be n-dimensional we leave unsettled. It would be interesting to know, for example, if in an lc^n, an n-cycle with an at most n-dimensional carrier bounds on an at most $(n+1)$-dimensional set (for $n=0$, Theorem 24 (§2.2) yields a positive answer).

References and notes

Due to space limitations, a complete list of citations is out of the question. It is expected, therefore, that references to be found in works cited will be consulted, especially since in the case of a sequence of papers on a special topic we sometimes cite only the last. We do not in general go into such historical matters as the origin of various terms used. For detailed bibliographies the reader is referred to S. Lefschetz, *Topology*, and R. L. Moore, *Foundations of Point Set Theory*, respectively, Volumes 12 and 13 of the Colloquium Publications of the American Mathematical Society; and to Seifert-Threlfall, *Topologie*, Leipzig, 1934. For a chronologically arranged list of books relating to topology, see the comprehensive Alexandroff-Hopf, *Topologie*, Berlin, 1935.

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2. E. W. Chittenden. *On the metrization problem and related problems in the theory of abstract sets.* Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 13–34.

3. S. Claytor. *Peanian continua not imbeddable in a spherical surface.* Annals of Mathematics, (2), vol. 38 (1937), pp. 631–646.

4. K. Menger. *Dimensionstheorie.* Leipzig and Berlin, 1928. G. Nöbeling. *Die neuesten Ergebnisse der Dimensionstheorie.* Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 41 (1931), pp. 1–17.

5. E. V. Huntington. *The Continuum.* 2d edition. Cambridge, Massachusetts, 1917.

6. As for instance in the case of a knotted S_1 in E_3, or the Alexander example of the S_2 in S_3 such that S_3-S_2 contains S_1's that are not deformable to a point in S_3-S_2. J. W. Alexander. *An example of a simply connected surface bounding a region which is not simply connected.* Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 8–10.

7. In citing a theorem given elsewhere than in the chapter under discussion, we shall state the chapter or section that contains it.

8. F. LEJA. *Sur la notion du groupe abstrait topologique.* Fundamenta Mathematicae, vol. 9 (1927), pp. 37–44. D. VAN DANTZIG. *Zur topologischen Algebra.* Mathematische Annalen, vol. 107 (1932–33), pp. 587–626. D. MONTGOMERY. *Continuity in topological groups.* Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 879–882.

9. C. JORDAN. *Cours d'Analyse.* Paris, 1893. 2d edition, p. 90.

10. For references, see: R. L. MOORE. *Report on continuous curves from the viewpoint of analysis situs.* Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 289–302, second footnote, p. 290.

11. A. SCHOENFLIES. *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten.* Jahresbericht der Deutschen Mathematiker-Vereinigung, supplementary vol. 2. Leipzig, 1908.

12. R. L. MOORE, Cit. [12], second footnote, p. 292. G. T. WHYBURN. *Concerning continuous images of the interval.* American Journal of Mathematics, vol. 53 (1931), pp. 670–674.

13. B. KNASTER and C. KURATOWSKI. *Sur les ensembles connexes.* Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255, §2. G. H. HALLETT, JR. *Concerning the definition of a simple continuous arc.* Bulletin of the American Mathematical Society, vol. 25 (1918–1919), pp. 325–326.

14. R. L. WILDER. *Concerning simple continuous curves and related point sets.* American Journal of Mathematics, vol. 53 (1931), pp. 39–55, §2.

15. R. L. MOORE. *Foundations of Point Set Theory.* Loc. cit., p. 86. See also: G. T. WHYBURN. *On the construction of simple arcs.* American Journal of Mathematics, vol. 54 (1932), pp. 518–524.

16. H. M. GEHMAN. *Concerning end points of continuous curves and other continua.* Transactions of the American Mathematical Society, vol. 30 (1928), pp. 63–84.

17. G. T. WHYBURN. *On the structure of continua.* Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 49–73.

18. See cit. [17], especially §3, as well as references therein, and the latter part of the same paper for extensions of the notion. For proofs of the next few theorems and brief historical comments, see: G. T. WHYBURN. *On the cyclic connectivity theorem.* Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 429–433.

19. E. R. VAN KAMPEN. *On some characterizations of 2-dimensional manifolds.* Duke Mathematical Journal, vol. 1 (1935), pp. 74–93.

20. J. W. ALEXANDER. *A proof and extension of the Jordan-Brouwer separation theorem.* Transactions of the American Mathematical Society, vol. 23 (1922), pp. 333–349.

21. O. VEBLEN. *Theory of plane curves in non-metrical analysis situs.* Transactions of the American Mathematical Society, vol. 6 (1905), pp. 83–98.

22. We recall from §1.4 that to say "S_k imbedded in S_n" means merely that the S_k is a point set in S_n that is homeomorphic with an S_k, and not in general a subcomplex of any subdivision of S_n; for example, an S_1 imbedded in S_2 may have a positive area in terms of the metric of S_2.

23. See citation, p. 333 of [20].

24. O. VEBLEN. *Analysis Situs.* American Mathematical Society Colloquium Publications, vol. V, Part 2. 1922.

25. O. VEBLEN and J. W. ALEXANDER. *Manifolds of N dimensions.* Annals of Mathematics, (2), vol. 14 (1913), pp. 163–178 (note footnote bottom p. 164).

26. J. W. ALEXANDER. *Combinatorial analysis situs.* Transactions of the American Mathematical Society, vol. 28 (1926), pp. 301–329.

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28. L. PONTRJAGIN. *Über den algebraischen Inhalt topologischer Dualitätssätze.* Mathematische Annalen, vol. 105 (1931), pp. 165–205.

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30. S. LEFSCHETZ. *Topology.* American Mathematical Society Colloquium Publications, vol. 12, p. 105. 1930.

31. A. KOLMOGOROFF. *Über die Dualität im Aufbau der kombinatorischen Topologie.* Recueil Mathématique (Moscou), vol. 1 (43) (1936), pp. 97–102.

32. H. WHITNEY. *On products in a complex.* Annals of Mathematics, (2), vol. 39 (1938), pp. 397–432 (contains both historical comments and bibliography).

33. R. L. WILDER. *Point sets in three and higher dimensions and their investigation by means of a unified analysis situs.* Bulletin of the American Mathematical Society, vol. 38 (1932), pp. 649–692.

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36. Obviously "bounded" is sufficient, where we would call a set M bounded if $\delta(M)$ is finite.

37. These classes are what we might call "arc-components" of M^A.

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41. G. T. WHYBURN. *A decomposition theorem for closed sets.* Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 95–96. R. L. WILDER. *Decompositions of compact metric spaces.* Ibid., vol. 43 (1937), p. 334, abstract no. 270.

42. For instance, see: G. T. WHYBURN. *Interior transformations on surfaces.* American Journal of Mathematics, vol. 60 (1938), pp. 477–490.

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58. N. BRUSCHLINSKY. *Stetige Abbildungen und Bettische Gruppen der Dimensionszahlen* 1 *und* 3. Mathematische Annalen, vol. 109 (1934), pp. 525–537.

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66. See LEFSCHFTZ. *Topology.* Chap. II, §3.

67. L. E. J. BROUWER. *Über Abbildung von Mannigfaltigkeiten.* Mathematische Annalen, vol. 71 (1911), pp. 97–115.

68. Consider such mappings as those defined by $w=e^{i\theta}$, $w=e^{2i\theta}$, $0\leqq\theta<2\pi$ of $|z|=1$ into $|w|=1$.

69. H. WHITNEY. *The maps of an n-complex into an n-sphere.* Duke Mathematical Journal, vol. 3 (1937), pp. 51–55.

70. As announced in the third paragraph of [69], conditions that f and g be homotopic have also been found by WHITNEY for $N=S_2$ and M any 3-complex (Bulletin of the American Mathematical Society, vol. 42 (1936), p. 338, abstract no. 220) and for N=projective n-space, M any n-complex.

71. See, for instance, the paper cited in [31].

72. See the paper of Lefschetz cited in [60]; also Whitney in [69].

73. See LEFSCHETZ, *Topology,* and citations therein to Brouwer.

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79. Besides the papers of Čech and Lefschetz mentioned in footnote 5 of [78], see [53]. E. ČECH. *On general manifolds.* Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 110–111. P. ALEXANDROFF and L. PONTRJAGIN. *Les variétés à n dimensions généralisées.* Comptes Rendus de l'Académie des Sciences, vol. 202 (1936), pp. 1327–1329.

80. K. BORSUK. *Über sphäroidale und H-sphäroidale Räume.* Recueil Mathématique (Moscou), vol. 1 (43) (1936), pp. 643–660.

81. For some very general decomposition theorems, applying to so-called "pseudomanifolds," see: E. ČECH. *On Pseudomanifolds.* Mimeographed lecture notes, Princeton, 1935.

82. Compare: R. L. WILDER. *Sets which satisfy certain avoidability conditions.* Časopis pro Pěstovani Matematiky a Fysiky, 1938, pp. 185–198.

83. P. ALEXANDROFF. *"Gestalt und Lage."* Annals of Mathematics, (2), vol. 30 (1928–1929), pp. 101–187, especially 168–169.

84. See first paper cited in [52].

85. That is, Γ^n may be shifted to $F(x, \epsilon)$ by infinitesimal alterations in the chain F_1; see for instance Anhang I of [83].

86. See [35], Theorem 3; also E. Čech, [81], and *Sur la décomposition d'une pseudovariété par un sousensemble fermé.* Comptes Rendus de l'Academie des Sciences, vol. 198 (1934), pp. 1342–1345.

87. As a matter of fact the only condition that J need satisfy here is that it carry at least one $(n-1)$-cycle nonbounding on J but bounding on M and to which no point of $M-J$ is a barrier.

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