Uniqueness in labelings of tree-depth-critical graphs.

A $k$-ranking of a graph $G$ is a labeling of the vertices of $G$ with values from $\{1, \ldots, k\}$ such that any path joining two vertices with the same label contains a vertex having a higher label. The tree-depth of $G$ is the smallest value of $k$ for which a $k$-ranking of $G$ exists. The graph $G$ is critical if every proper minor of $G$ has smaller tree-depth than $G$ has.

Focusing on $k$-rankings of critical graphs, we define a graph $G$ to be 1-unique if for every vertex $v$ in $G$, there exists an optimal ranking in which $v$ is the unique vertex with label 1. We explore the seemingly close relationships between 1-uniqueness and criticality, showing that (contrary to an earlier conjecture) not all critical graphs are 1-unique, though many are. We show that for graphs that are 1-unique, we have useful tools to more easily conclude criticality, construct larger critical graphs, and prove conjectured properties of critical graphs. (Received September 16, 2016)