Let $\mathbb{F}_q$ be a finite field with $q$ elements and $\phi$ a rational function with coefficients in $\mathbb{F}_q$. For each $n \geq 1$, the orbit of every $x \in \mathbb{F}_{q^n}$ under $\phi$ is either periodic or strictly preperiodic. Little is known about how the proportion of such $x$ that are periodic changes as $n$ grows. Part of what makes the problem difficult is that, since $\phi$ is defined over a finite field, it is necessarily post-critically finite. We study the case where $\phi$ is quadratic, and show that in all but a few special cases the lim inf of this proportion is zero. The proof begins by using the Chebotarev density theorem over function fields and a result of Pink on lifting to characteristic zero. But the real guts of it are group-theoretic and complex dynamical: we make a careful study of the iterated monodromy groups of post-critically finite quadratic rational functions over $\mathbb{C}$, including several new results about the fixed-point proportion of the natural action of these groups on the infinite rooted binary tree. (Received September 15, 2020)