Free probability, random matrices and the enumeration of maps

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Abstract

These are the lecture notes for the Colloquium lectures to be held in San Diego, january 2012. These lectures will discuss the relations between free probability, random matrices, and combinatorics. They will also show how these relations can be used to solve problems in each of these domains. Even if related, we will try to make each lecture independent of the others.

Introduction

Free probability is a theory initiated by D. Voiculescu in the eighties that studies non-commutative random variables. It is equipped with a notion of freeness, which is related with free products, and which plays the same role as independence in standard probability. These two ingredients allows to translate many problems from operator algebra into words more familiar to probabilists, and eventually import tools form probability theory to try to solve them.

It turns out that free probability is not only analogue to classical probability but that there is a natural bridge between both ; it is given by random matrices with size going to infinity. Indeed, matrices with size going to infinity provide a reach class of non-commutative random variables, in fact it is still an open question whether all non-commutative laws can be described in this way. Moreover, the notion of freeness of two variables is related with the notion of independence of the basis of the eigenvectors of the two underlying random matrices, in the sense that one is uniformly distributed with respect to the other. Thus random matrices, which in some sense pertain to the usual classical probability theory, are a source of inspiration in free probability, and therefore in operator algebra.

On the other hand, random matrices are well known since the seventies to be related with the enumeration of maps, that is connected graphs sorted by the genera of the surface in which they can be properly embedded. In fact 't Hooft and Brézin-Itzykson-Parisi-Zuber showed that matrix integrals can be seen as generating functions for the enumeration of maps, with the inverse of the dimension being the parameter governing the genus of the maps. Such matrix integrals, when they converge, also define non-commutative laws in free probability. Hence, it turns out that lots of non-commutative laws from free probability can be defined in terms of the enumeration of planar maps. Reciprocally, free probability can help to analyze combinatorial questions related with the enumeration of planar maps. Maps with higher genus appear as the correction to the limit and can be obtained by some surgery from the latter ; hence the full expansion is based on the first order which is related with free probability.

In these lectures, we will try to describe more precisely these relations between free probability, random matrices and the enumeration of maps.

During the first lecture, we shall give the basics of free probability. We will then see show some ideas from classical probability were imported to try to answer central questions in operator algebras related with isomorphisms between C^* or von Neumann algebras.

The second lecture will concentrate on the relation with combinatorial objects, namely the enumeration of maps and the so-called topological expansion. Topological expansion are expansions of integrals in terms of the number N of its parameters whose coefficients can be interpreted as a generating functions for the enumeration of maps. The relation with matrix integrals sometimes allows to study or even compute these generating functions and hence solve the underlying combinatorial question. We will see that this relation can be established with usual Gaussian calculus and Feynman diagrams. However, this relation is deeper and can be also deduced from the so-called loop equations which are consequences of analytic tools such as integration by parts. It is related with the non-commutative derivatives which have themselves a combinatorial description. We shall see that this point of view allows to show that many integrals, even not directly related with random matrices, have a topological expansion.

The last lecture will show how to put together the previous ones in order to solve a combinatorial problem motivated by physics, namely the Potts model on random planar maps. It is based on the relation between random matrices and loop models which elaborates on the relation between random matrices and planar maps. The construction of a matrix model related with loop models requires an additional idea coming from subfactors theory and more precisely the construction of the planar algebra of a bipartite graph given by V. Jones. Once this construction is done, it turns out that miraculously the matrix model can be computed at the large N limit, hence allowing to solve the initial combinatorial question.

1 Free probability

Citing D. Voiculescu, *"Around 1982, I realized that the right way to look at certain operator algebra problems was by imitating some basic probability theory. More precisely, in noncommutative probability theory a new kind of independence can be defined by replacing tensor products with free products and this can help understand the von Neumann algebras of free groups. The subject has evolved into a kind of parallel to basic probability theory, which should be called free probability theory."*

Thus, Voiculescu's first motivation to introduce free probability was the analysis of the von Neumann algebras of free groups. One of his central observations was that such groups can be equipped with tracial states (also called traces), which resemble expectations in classical probability, whereas the property of freeness, once properly stated, can be seen as a notion similar to independence in classical probability. This led him to the statement

free probability theory=noncommutative probability theory+ free independence.

These two components are the basis for a probability theory for noncommutative variables where many concepts taken from probability theory such as the notions of laws, convergence in law, independence, central limit theorem, Brownian motion, entropy, and more can be naturally defined. For instance, the law of one self-adjoint variable is simply given by the traces of its powers (which generalizes the definition through moments of compactly supported probability measures on the real line), and the joint law of several self-adjoint noncommutative variables is defined by the collection of traces of words in these variables. Similarly to the classical notion of independence, freeness is defined by certain relations between traces of words. Convergence in law just means that the trace of any word in the noncommutative variables converges towards the right limit.

In this lecture, we will present the basics of free probability. Then we will highlight a few uses of this theory to tackle the question of the isomorphisms between C^* and von Neumann algebra.

1.1 Non-commutative laws

To extend the notion of law (or probability measure) to a noncommutative set up, one should consider classical laws say on $\mathbb{R}^d$ simply as linear functions on a set of test functions with values in $\mathbb{R}$, for instance the bounded continuous functions, the bounded measurable functions or if one restricts oneself to compactly supported functions, to polynomial functions. The additional requirements are that they are non-negative, that is that the value of this linear map at any nonnegative function is non-negative, and with mass one.

These concepts can all be extended to the non-commutative set-up. Let us restrict ourselves to d bounded variables and consider the set of polynomials

$$\mathbb{C}\langle X_1,\ldots,X_d\rangle = \{z_0 + \sum z_p X_{i_1^p}\cdots X_{i_{n_p}^p}, i_k^\ell \in [1,d], n_p \in \mathbb{N}^*, z_i \in \mathbb{C}\}$$

as the set of test functions. Then, a law of d non-commutative variables will be any linear form on $\mathbb{C}\langle X_1,\ldots,X_d\rangle$

$$\tau : P \in \mathbb{C}\langle X_1,\ldots,X_d\rangle \to \tau(P) \in \mathbb{C}\,. \tag{1}$$

If the variables are assumed to be self-adjoint, we may endow $\mathbb{C}\langle X_1,\ldots,X_d\rangle$ with the involution

$$\left(\sum z_p X_{i_1^p}\cdots X_{i_{n_p}^p}\right)^* = \sum \bar{z}_p X_{i_{n_p}^p} X_{i_{n_{p-1}}^p}\cdots X_{i_1^p}\,.$$

Non negative elements of $\mathbb{C}\langle X_1,\ldots,X_d\rangle$ can be written as PP^* for some $P \in \mathbb{C}\langle X_1,\ldots,X_d\rangle$ (see e.g. [Mu90, Theorem 2.2.4]). The positivity requirement is therefore

$$\tau(PP^*) \geq 0 \qquad \forall P \in \mathbb{C}\langle X_1,\ldots,X_d\rangle\,. \tag{2}$$

The mass condition is simply

$$\tau(1) = 1\,. \tag{3}$$

On the top of these conditions analogous to the classical setting, we shall assume additionnally that non-commutative laws satisfy the so-called tracial conditions

$$\tau(PQ) = \tau(QP) \qquad \forall P,Q \in \mathbb{C}\langle X_1,\ldots,X_d\rangle$$

The most obvious example of such a law is given by matrices. Indeed, if $X = (X_1,\ldots,X_d)$ are $N\times N$ Hermitian matrices then

$$\tau_X(P) := \frac{1}{N}\mathrm{Tr}(P(X_1,\ldots,X_d)) \qquad \forall P \in \mathbb{C}\langle X_1,\ldots,X_d\rangle$$

defines a non-commutative law. If $X = (X_1, \ldots, X_d)$ are random $N \times N$ Hermitian matrices then

$$\tau_X(P) := \mathbb{E}[\frac{1}{N}\mathrm{Tr}(P(X_1, \ldots, X_d)]) \qquad \forall P \in \mathbb{C}\langle X_1, \ldots, X_d\rangle \tag{4}$$

also defines a non-commutative law. We shall call this law the empirical distribution of $(X_1, \ldots, X_d)$. In fact, any limit of such laws will also define a non-commuative law, that is if $(X_1^N, \ldots, X_d^N)$ is a sequence of random $N \times N$ Hermitian matrices then if for all polynomial P the limit

$$\tau(P) = \lim_{N\to\infty} \tau_{X^N}(P)$$

exists, the limit τ is as well a non-commutative law. It is still an open question whether all non-commutative law can be constructed in this way. We shall see later in these lectures notes that random matrices at list provide a very useful bridge between classical probability and free probability.

The notion of convergence of non-commutative laws will always refer to weak convergence, that is a sequence $\tau_n, n \geq 0$ of non-commutative laws of d variables converges towards a law τ iff for any polynomial $P \in \mathbb{C}\langle X_1, \ldots, X_d\rangle$

$$\lim_{n\to\infty} \tau_n(P) = \tau(P)\,.$$

The notion of random variables also makes sense in this setting as by the so-called Gelfand-Naimark-Segal (or GNS) construction, being given a non-commutative law τ, one can always construct "random variables" with law τ in the sense of constructing a Hilbert space H equipped with a scalar product $< .,. >_H$, a vector ξ in H and bounded linear operators $\tilde{X}_1, \ldots, \tilde{X}_d$ on H so that for any polynomial P

$$< \xi, P(\tilde{X}_1, \ldots, \tilde{X}_d)\xi >_H = \tau(P(X_1, \ldots, X_d))\,. \tag{5}$$

In this construction, one takes the set of polynomials and separate it by quotienting by the left ideal $\tau(PP^*) = 0$. H is then obtained by completing for the L^2 norm under τ. The letters $X_1, \ldots, X_d$ are seen as left multiplication operators, and therefore as operators on H.

1.2 C^* algebras and W^*-algebras

In classical probability theory, test functions can be taken to be bounded continuous or bounded measurable, resulting with different weak* topologies on the space of laws. The same holds in the non-commutative case with the notions of C^* algebras and W^*-algebras respectively.

We will restrict our discussion throughout to unital C^*-algebras without further mentioning it. Thus, in the following, a C^*-algebra $\mathcal{A}$ is a unital algebra equipped with a norm $\|\cdot\|$ and an involution $*$ so that

$$\|xy\| \leq \|x\|\|y\|, \quad \|a^*a\| = \|a\|^2.$$

Recall that $\mathcal{A}$ is complete under its norm.

If one is given a non-commutative law τ of d bounded non-commutative variables, one can associate a C^*-algebra by the Gelfand-Naimark-Segal construction mentionned above. The variables are seen as bounded operators on a Hilbert space (namely left multiplication operators on the set $L^2(\tau)$), the C^* algebra is the set of polynomials in these variables completed under the norm given by

$$\|P\|_\tau = \lim_{n\to\infty} \tau\left((PP^*)^n\right)^{1/2n} .$$

A C^* -algebra $\mathcal{A} \subset B(H)$ for some Hilbert space H is a *von Neumann algebra* (or W^* -algebra) if it is closed with respect to the weak operator topology. In the case of the von Neumann algebra associated with a non-commutative law τ, one takes the C^*-algebra and complete it for the weak topology in the associated Hibert space, see (5).

1.3 Freeness

Freeness is the non-commutative analogue of independence in probability. In some sense, probability theory distinguishes itself from integration theory by the notion of independence and of random variables which are the basis to treat problems from a different perspective. Similarly, free probability differentiates from noncommutative probability by this very notion of freeness which makes it a noncommutative analogue of classical probability.

Independence of two families $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_d)$ can be defined by saying that for any test functions f, g,

$$\mathbb{E}[f(x_1, \ldots, x_p)g(y_1, \ldots, y_d)] = \mathbb{E}[f(x_1, \ldots, x_p)]\mathbb{E}[g(y_1, \ldots, y_d)]$$

or in other words that the expectation of the product of two test functions in the x's and the y's respectively vanishes as soon as the expectation of each test function vanishes.

Freeness is a natural extension of the notion of independence to noncommutative variables ; we say that random variables $X = (X_1, \ldots, X_p)$ and $Y = (Y_1, \ldots, Y_d)$ with joint law τ are free iff for any polynomials $P_1, \ldots, P_k \in \mathbb{C}\langle X_1, \ldots, X_p\rangle$ and $Q_1, \ldots, Q_k \in \mathbb{C}\langle X_1, \ldots, X_d\rangle$ so that $\tau(P_i(X)) = 0$ and $\tau(Q_i(Y)) = 0$ we have

$$\tau(P_1(X)Q_1(Y)P_2(X) \cdots P_k(X)Q_k(Y)) = 0\,. \tag{6}$$

It is easy to check by induction over the degree that this relation defines uniquely joint moments if the marginals are known.

The name of freeness is indeed related with the standard notion of freeness for groups. Take ϕ to be the linear functional on $\mathcal{A}$ so that for all $g \in G$, $\phi(g) = 1_{g=e}$. Then, if g_1 and g_2 satisfy the freeness relation (6), they are free in the usual sense that you can not build non trivial words in g_1 and g_2 that equals the neutral element. More precisely, let G be generated by two free generateors g_1, g_2, that is elements of G are of the form $g_1^{n_1} g_2^{m_1} \cdots g_1^{n_p} g_2^{m_p}$ with $n_i \neq 0$ for $i \geq 2$ and $m_i \neq 0$ for $i \leq p-1$. By freeness, $g_1^{n_1} g_2^{m_1} \cdots g_1^{n_p} g_2^{m_p} \neq e$ unless $p = 1$ and $n_1 = m_1 = 0$.

Consider an orthonormal basis $\{v_g\}_{g \in G}$ of $\ell^2(G)$, the set of sums $\sum_{g \in G} c_g v_g$ with $c_g \in \mathbb{C}$ and $\sum |c_g|^2 < \infty$. $\ell^2(G)$ is equipped with a scalar product

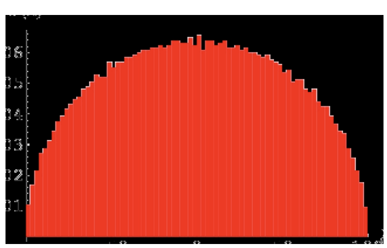
$$\langle \sum_{g \in G} c_g v_g, \sum_{g \in G} c'_g v_g \rangle = \sum_{g \in G} c_g \bar{c}'_g\,,$$

which turns it into a Hilbert space. The action of each $g' \in G$ on $\ell^2(G)$ becomes $\lambda(g')(\sum_g c_g v_g) = \sum_g c_g v_{g'g}$, yielding the left regular representation determined by G, which defines a family of unitary operators on $\ell^2(G)$. These operators are determined by $\lambda(g)v_h = v_{gh}$. By definition $\phi(\lambda(g)) = 1_{g=e}$ defines by linearity a tracial state. Moreover by freeness $\phi(g_1^{n_1} g_2^{m_1} \cdots g_1^{n_p} g_2^{m_p}) = 0$ as soon as $n_i \neq 0$ for $i \geq 2$ and $m_i \neq 0$ for $i \leq p-1$, that is as soon as $\phi(g_1^{n_i}) = 0, i \geq 2$ and $\phi(g_2^{m_i}) = 0$ for $i \leq p$. Since this relation extends by linearity we conclude that $\lambda(g_1)$ and $\lambda(g_2)$ are free in the free probability sense.

1.4 The semi-circle law

In classical probability, the normal (or Gaussian) law plays a very particular role as it describes the fluctuations of many models. Indeed, the sum of independent variables with finite second moment converges when the number of variables goes to infinity, once properly centered and renormalized, towards the Gaussian law. Therefore, the Gaussian law describes many models in probability.

Similarly, if one sums bounded free variables, recenters and renormalizes, one will see a universal law appearing, but this will not be the normal law but the semi-circle law.



More precisely, we have (see [Voi91] and [AGZ10][Section 5.3.4]) :

Theorem 1.1 *Let* $\{a_i\}_{i\in\mathbb{N}}$ *be a family of free self-adjoint random variables in a noncommutative probability space with a tracial state* ϕ. *Assume that for all* $k \in \mathbb{N}$,

$$\sup_j |\phi(a_j^k)| < \infty. \tag{7}$$

Assume $\phi(a_i) = 0$, $\phi(a_i^2) = 1$. *Then,*

$$X_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} a_i$$

converges in law as N *goes to infinity to a standard semicircle distribution.*

The semicircle law σ is given by

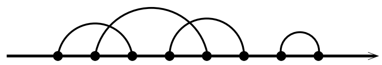
$$\sigma(dx) = \sqrt{4 - x^2} dx/\pi \,.$$

Eventhough this does not look like the Gaussian law it is deeply connected to it by the combinatorial characterization of its moments. Indeed, the moment of a centered Gaussian vector $(G_1, \ldots, G_{2n})$ is characterized by Wick formula

$$\mathbb{E}[G_1 G_2 \cdots G_{2n}] = \sum_{\substack{1\le s_1<s_2..<s_n\le 2n \\ r_i>s_i}} \prod_{j=1}^{n} \mathbb{E}[G_{s_j} G_{r_j}].$$

Wick formula shows that Gaussian expectation are described by pair partitions, that is matching. This description was used by Feynman to represent Gaussian integration by diagrams, hence allowing a combinatorial point of view on many integrals. In particular, if $G_i = G$ follows the standard Gaussian distribution

$$E[G^p] = \#\{\text{ pair partitions of p ordered points}\}$$



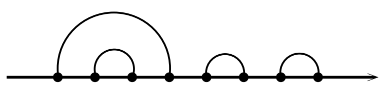
The semicircle law has a similar description but in terms of *non-crossing* partition. Indeed,

$$\int x^p d\sigma(x) = \#\{\text{ non-crossing pair partitions of p ordered points}\}.$$

A non-crossing pair partition is a pair partition of ordered points so that if (a,b) and (c,d) are two blocks of the partition so that $a < c$,

$$a < b < c < d \qquad \text{or} \qquad a < c < d < b\,.$$

In other words, if we represent the blocks of the partitions by arcs, it can be done so that the arcs do not intersect.



The proof of Theorem 1.1 is straightforward once one is aware of the combinatorial description of σ. Indeed, it is enough to show that for all $p \in \mathbb{N}$

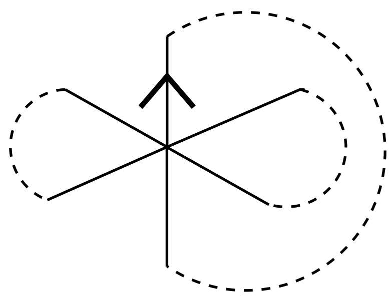
$$\lim_{N\to\infty} \phi\left((\frac{1}{\sqrt{N}} \sum_{i=1}^{N} a_i)^p \right) = \int x^p d\sigma(x)\,.$$

But when expanding the left hand side

$$\phi\left((\frac{1}{\sqrt{N}} \sum_{i=1}^{N} a_i)^p \right) = \frac{1}{N^{\frac{k}{2}}} \sum_{i_1,\ldots,i_p=1}^{N} \phi(a_{i_1} \cdots a_{i_p})$$

we see that the terms in the right hand side vanishes by freeness except if each index i_k appears at list twice as the a_i are centered. But this means at most $N^{k/2}$ indices contribute. As we divide by this quantity we see that sets of indices where an index is repeated more than twice will be neglectable. Hence, we may concentrate on the case where indices are repeated exactly twice. But then, by freeness, the contribution will also vanish except if the indices are paired by a non crossing partition. Indeed for a term to contribute, there must be a couple of neighbors indices $i_p i_{p+1}$ which are equal. Since all other indices have to be different by the previous point, we can remove their expectation by freeness. We then can continue inductively to see that the partition must be constructed inductively by pairing neighbors ; this is exactly the construction of non-crossing partitions. Hence, the right hand side converges towards the number of non-crossing pair partitions of p points.

For later uses, we may represent the p points of the partition as the end point of half-edges of a vertex with valence p, drawn on the sphere with one marked half-edge. Non crossing pair partitions then become matchings of the end points of these half-edges so that the graph is properly embedded into the sphere (that is drawn into the sphere so that no edges intersect).



We next see how non-crossing partitions arise in random matrix theory.

1.5 Random matrices and freenesss

Random matrices played a central role in free probability since Voiculescu's seminal observation that independent Gaussian Wigner matrices converge in distribution as their size goes to infinity to free semicircular variables. This result can be extended to approximate any law of free variables by taking diagonal matrices and conjugating them by independent unitary matrices (see [AGZ10][section 5.4]). To be more precise, let X^N be a matrix following the Gaussian Unitary Ensemble, that is a $N \times N$ Hermitian matrix with i.i.d centered Gaussian entries with covariance N^{-1}, that is

$$\begin{aligned} X^N(k\ell) &= \bar{X}^N(\ell k) = \frac{1}{\sqrt{2}}(x_{k\ell} + iy_{k\ell}) \quad 1 \le k \le \ell \le N \\ X^N(kk) &= x_{kk} \quad 1 \le k \le N \end{aligned}$$

with

$$\begin{aligned} d\mathbb{P}_N(X^N) &= \frac{1}{Z^N} \exp\{-\frac{N}{2}\sum_{k\le\ell}(x_{k\ell}^2 + y_{k\ell}^2)\} \prod dx_{k\ell}dy_{k\ell} \\ &= \frac{1}{Z^N} \exp\{-\frac{N}{2}\mathrm{Tr}((X^N)^2)\} dX_N \end{aligned}$$

Then Wigner [Wig55] showed the following.

Theorem 1.2 (Wigner) *For any $p \ge 0$*

$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}((X^N)^p)] = \int x^p d\sigma(x)\,.$$

However, the relation of random matrices and freeness was discovered much later by Voiculescu [Voi91]. A central result is the following.

Theorem 1.3 (Voiculescu) *For any polynomial $P \in \mathbb{C}\langle X_1, \cdots, X_d\rangle$*

$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}(P(X_1^N, \cdots, X_d^N))] = \sigma^d(P)$$

exists. σ^d is the law of d free semi-circular law.

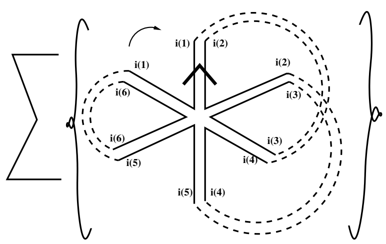
The proof of these two results relies on Wick calculus which in fact provides not only the asymptotic of traces of words in random matrices but also the whole N expansion. Let us first consider the $d = 1$ case. Then, one just expands the trace in terms of the matrix entries

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}((X^N)^p)] = \frac{1}{N}\sum_{i(1),\ldots,i(p)=1}^{N} \mathbb{E}[X^N_{i(1)i(2)}X^N_{i(2)i(3)}\cdots X^N_{i(p)i(1)}]\,.$$

Using Wick formula together with $\mathbb{E}[X^N_{ij} X^N_{k\ell}] = N^{-1} 1_{ij=\ell k}$, one gets

$$\mathbb{E}[X^N_{i(1)i(2)} \cdots X^N_{i(p)i(1)}] = \frac{1}{N^{p/2}} \sum_{\text{pair partition}} \prod_{(k,\ell) block} 1_{i(p)i(p+1)=i(\ell+1)i(\ell)}$$

The later matching can be more conveniently represented by seeing the Gaussian entries as the end points of half-edges of a vertex with valence p with one marked vertex



A face is obtained by cutting the graph along the edges. As $\mathbb{E}[X^N_{ij} X^N_{k\ell}] = N^{-1} 1_{ij=\ell k}$, only matchings so that indices are constant along the boundary of the faces contribute.Hence, since indices take any value between 1 and N,

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}((X^N)^p)] = \sum_{\substack{\text{graph 1 vertex} \\ \text{degree p}}} N^{\#\text{faces} - p/2 - 1} \quad (8)$$

But, by Euler formula, any connected graph satisfies that its genus is given by

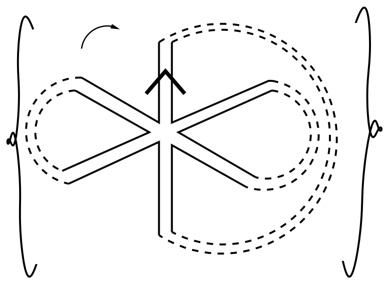
$$2 - 2g = \#\{\text{vertices}\}\#\{\text{faces}\} - \#\{\text{edges}\} = 1 + \#\{\text{faces}\} - p/2$$

so that

$$\#\text{faces} - p/2 - 1 = -2g \leq 0$$

with equality only if the graph is planar. Hence

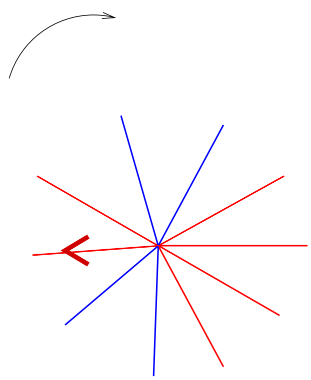
$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}((X^N)^p)] \;\; = \;\; \#\{\text{planar graph with 1 vertex with degree } p\}$$



A similar strategy can be used when one deals with several GUE matrices. To extend this graphical view point, one needs to associate (bijectively) to any word in d non-commutative variables either ordered colored points or a "vertex with colored half-edges" in order to make the difference between the different matrices in the word. Namely, associate to

$$q(X_1, \ldots, X_d) = X_{i_1} X_{i_2} \cdots X_{i_p}$$

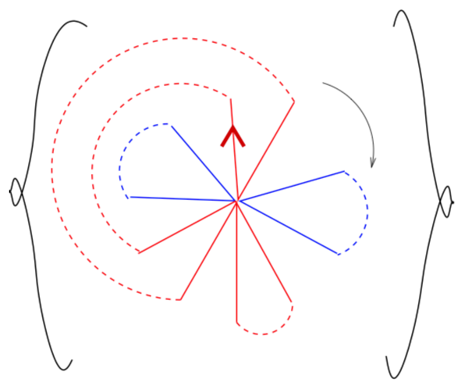
a "star of type q" given by the vertex with p colored half-edges drawn on the sphere so that the first branch has color i_1, the second of color i_2 etc until the last which has color i_p. For instance, if $q(X_1, X_2) = X_1^2 X_2^2 X_1^4 X_2^2$ and 1 is associated to red whereas 2 is associated with blue, the star of type q is a vertex with first two half-edges which are red, then two blue, four red and finally two blue.



Then

$$\sigma^d(q) = \lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}(q(X_1^N, X_2^N, \cdots, X_d^N))]$$

is the number of planar graphs build on a star of type q by matching only branches of the same color.



Indeed, the very same arguments hold, except that since the covariance between entries of two different matrices vanishes, pairing between half-edges of different colors do not contribute to the sum.

In fact, it is not hard to see that σ^d is the law of free variables. Indeed, it is enough to verify the property for a monomial q whose decomposition into free components has been reentered, that is

$$q = (q_1(X_{i_1}) - \sigma_d(q_1(X_{i_1})))(q_2(X_{i_2}) - \sigma_d(q_2(X_{i_2}))) \cdots (q_\ell(X_{i_\ell}) - \sigma_d(q_1(X_{i_\ell})))$$

with $i_p \neq i_{p+1}$. If we call $S_1, \ldots, S_\ell$ the successive sets of half-edges of the same color (but so that S_k and S_{k+1} are sets of different color for all k) in the star of type q, then any planar map is such that there exists a $k \in \{1, \ldots, \ell\}$ so that the half-edges of S_k only match among themselves (and not with half-edges of $S_j, j \neq k$). But then this contribution vanishes due to the centering. Hence $\sigma_d(q) = 0$.

1.6 More general laws

Let V be a polynomial in d non-commutative variables which is self-adjoint in the sense that $V(X_1, \ldots, X_d) = V(X_1, \ldots, X_d)^*$ for any self-adjoint variables and set for $M > 0$ large enough

$$\mathbb{P}_V^N(dX_1^N, \ldots, dX_d^N) = \frac{1_{\|X_i^N\| \le M}}{Z_{V,M}^N} e^{-N\mathrm{Tr}(V(X_1^N, \ldots, X_d^N))} dX_1^N \cdots dX_d^N$$

where $\|X\|$ denotes the spectral radius of the matrix X. As V is self-adjoint, the above measure has a real density. Then, it was proved in [GMS06] that if $M > 2$ is fixed and $V - \frac{1}{2}\sum X_i^2$ is small, the empirical distribution τ_{X^N} of $(X_1^N, \ldots, X_d^N)$ under $\mathbb{P}_V^N$ (see (4)) converges. This result was generalized in [GS09] to "locally convex potential". We say that V is (c, M) convex if for any N, $N\mathrm{Tr}V(X_1, \ldots, X_d)$ has Hessian (as a function of the entries of the matrix) bounded below by Nc for some $c > 0$, when evaluated at any self-adjoint matrices so that $\|X_i\| \le M$. Then it was proved that

Theorem 1.4 *Let $c > 0$ be fixed. Then, there exists $M(c) < \infty$ so that if $M \ge M(c)$ and V is (c, M) convex, there exists a non-commutative law τ_V so that for any polynomial P*

$$\tau_V(P) = \lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(P(X_1^N, \ldots, X_d^N)) d\mathbb{P}_V^N(X_1^N, \ldots, X_d^N)$$

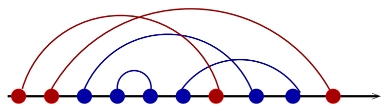
Note that small perturbations of the quadratic potential satisfy our local convexity property. The cutoff M is only needed to make the integral converges; in fact if V is strictly convex, that is is (c, ∞), then the cutoff can be removed. The convergence of the empirical distribution of matrices corresponding to laws $\mathbb{P}_V^N$ with potential V which does not satisfy such a convexity property is still a wide open problem.

However, there exists natural non-commutative laws which are described as limits as in the above theorem. This is indeed the case of q-Gaussian laws according to Dabrowski [Dab10].

A d-tuple of q-Gaussian variables is such that

$$\tau_{q,d}(X_{i_1} \cdots X_{i_p}) = \sum_{\pi} q^{i(\pi)} \qquad \forall i_k \in \{1, \cdots, d\}$$

where the sum runs over pair partitions of colored dots whose block contains dots of the same color and $i(\pi)$ is the number of crossings. For instance the figure below has $i(\pi) = 4$.



Here, the ordered colored dots are equivalent to colored stars, that is are in bijection with monomials.

Theorem 1.5 (Dabrowski (2010)) *If dq is small enough, there exists* $V_{q,d} = 1/2 \sum X_i^2 + W_{q,d}$ *with* $W_{q,d}$ *self-adjoint and small so that*

$$\tau_{q,d}(P) = \lim_{N\to\infty} \int \frac{1}{N} \mathrm{Tr}(P(X_1^N, \ldots, X_d^N)) d\mathbb{P}^N_{V_{q,d}}(X_1^N, \ldots, X_d^N)$$

1.7 The conjugate variables

One of the central difference between classical probability and free probability is the absence of notion of density in the latter. It turns out that a way to replace this notion is by integration by parts. Indeed, if P is a probability measure on $\mathbb{R}$, one way to characterize that it is absolutely continuous with respect to Lebesgue measure and with density $e^{-V(x)}$ with respect to Lebesgue measure for a smooth functionn V is to show that for any smooth test function f, we have

$$\int f(x)V'(x)dP(x) = \int f'(x)dP(x)\,.$$

This notion can be extended to the free probability setting. To get some intuition about what such an integration by parts should be, let us consider the non-commutative laws discussed in the previous section. Define the cyclic gradient as the linear derivative on the set of polynomial whose restriction to monomials is given by

$$\mathcal{D}_i(X_{i_1} \cdots X_{i_p}) = \sum_{i_\ell = i} X_{i_{\ell+1}} \cdots X_{i_p} X_{i_1} \cdots X_{i_{\ell-1}}$$

and observe that for any $(sr) \in \{1, \ldots, N\}^2$, since in the complex setting $\partial_{X_i(sr)} X_j(s'r') = 1_{i=j} 1_{sr=r's'}$,

$$\partial_{X_i(sr)} \mathrm{Tr}(P(X_1, \ldots, X_d)) = (\mathcal{D}_i P)(X_1, \ldots, X_d)(sr)\,.$$

Moreover, define the non-commutative derivative ∂_i by

$$\partial_i(X_{i_1}\cdots X_{i_p}) = \sum_{i_\ell = i} X_{i_1}\cdots X_{i_{\ell-1}} \otimes X_{i_{\ell+1}}\cdots X_{i_p}$$

and observe similarly that

$$\partial_{X_i(sr)}(P(X_1,\ldots,X_d))(s'r') = (\partial_i P)(X_1,\ldots,X_d)(s'r, sr') \,.$$

Therefore the classical integration by parts yields, at least when no cutoff is present, the following formula for any polynomial function P and any $sr, s'r'$

$$\begin{aligned} N\int P(s'r')(\mathcal{D}_i V)(sr) d\mathbb{P}_V^N &= \int P(s'r')\partial_{X_i(sr)} N \mathrm{Tr}(V) d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N) \\ &= \int \partial_{X_i(sr)} P(s'r') d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N) \\ &= \int (\partial_i P)(s'r, sr') d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N) \qquad (9) \end{aligned}$$

Taking $r' = s$ and $s' = r$ and summing over the indices yields the so-called Schwinger-Dyson (or loop) equation

$$\int \frac{1}{N}\mathrm{Tr}(P\mathcal{D}_i V) d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N) \qquad (10)$$

$$= \int \frac{1}{N}\mathrm{Tr}\otimes\frac{1}{N}\mathrm{Tr}(\partial_i P) d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N)\,.$$

Since we restrict ourselves to the case where the measure $d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N)$ as a strictly log-concave density and that for any polynomials P, $\frac{1}{N}\mathrm{Tr}(P(X_1,\ldots,X_d))$ is a Lipschitz function of the entries (at least when their spectrum radius is bounded) one finds by concentration of measure (see [GZ06, AGZ10, Gu09]) that for any polynomial P there exists a finite constant $C(P)$ so that

$$\int \left|\frac{1}{N}\mathrm{Tr}(P) - \int \frac{1}{N}\mathrm{Tr}(P) d\mathbb{P}_V^N\right|^2 d\mathbb{P}_V^N \le \frac{C(P)}{N^2}\,. \qquad (11)$$

Therefore, (9) yields, as $\int \frac{1}{N}\mathrm{Tr}(P)\mathbb{P}_V^N(dX_1^N,\dots,dX_d^N)$ converges towards $\tau_V(P)$ for all N, that for all polynomial P and all $i \in \{1,\dots,d\}$

$$\tau_V(P\mathcal{D}_iV) = \tau_V \otimes \tau_V(\partial_i P) \tag{12}$$

This is the natural analogue of classical integration by parts and in fact, as we have just seen, it can be seen as a limit of classical integration by parts at least for the laws described in the previous part as limits of random matrices. When V is locally strictly convex, Equation (12) as a unique solution [GMS06, GS09] and characterizes therefore τ_V. Moreover, the convergence of the empirical distribution of the random matrices following $\mathbb{P}_V^N$ towards τ_V can be proved by using this uniqueness and the fact that any limit point has to satisfy this equation.

Conjuguate variables are just defined so that such a formula holds, namely given a non-commutative law τ, its conjugate variables $\xi_i, 1 \le i \le d$ are simply the elements of the W^* algebra so that for all polynomials P

$$\tau(P\xi_i) = \tau \otimes \tau(\partial_i P)\,.$$

In particular we see that τ_V has $\mathcal{D}V$ has conjugate variables.

1.8 The isomorphism problem

One of the motivations of Voiculescu to construct free probability was in fact to disprove the isomorphism problem of free group factors. Let us consider the von Neumann algebra $L(F^{*m})$ generated by m free semi-circle variables. The isomorphism problem asks if these are isomorphic for different numbers of generators. It can be written in the free probability framework in terms of transport maps by wondering for $m \neq n$ whether there exists functions $G = (G_i)_{1\le i\le n}$ and $(F_j)_{1\le j\le m}$ so that

$$\begin{aligned}\sigma^m(P) &= \sigma^n\left(P(F_1(S_1,\dots,S_n),\dots,F_m(S_1,\dots,S_n))\right)\\ \sigma^n(Q) &= \sigma^n\left(Q(G_1(S_1,\dots,S_m),\dots,G_n(S_1,\dots,S_m))\right)\end{aligned}$$

In the commutative setting it is known since von Neumann (1932) that such a mapping exists ; as soon as one considers two probability measures which are for instance absolutely continuous with respect to Lebesgue measure, there exists measurable functions which map one on the other and vice-versa (in fact this is true in much greater generality). This is still an open question in the non-commutative setting. In the next sections we will describe a few attempts that were made to attack this question in free probability.

1.9 Transport theory

If the theory of existence of transport maps indeed goes back to von Neumann who was interested in more general isomorphisms results between von Neumann algebra, the theory of optimal transport is in fact more ancient and goes back to Monge. In the following, we will denote for two probability measures P, Q on $\mathbb{R}^d$ and a map $T : \mathbb{R}^d \to \mathbb{R}^d$ measurable, $T\#P = Q$ and say that T is a transport map from P to Q iff for all bounded continuous function f on $\mathbb{R}^d$ we have

$$\int f(T(x))dP(x) = \int f(x)dQ(x)\,.$$

Monge (1781), assuming the existence of a transport map T from P to Q wondered about how to characterize the transport map T which minimizes a cost such as

$$\int \|x - T(x)\|_1 dP(x)\,.$$

Monge was considering the ℓ^1 norm but in fact any norm on $\mathbb{R}^d$ could be used : we will hereafter consider the ℓ^2 norm $\|v\|_2 = (\sum v_i^2)^{1/2}$. Kantorovich in 1940 generalized this question and seeked for a probability measure $\pi(dx, dy)$ on $\mathbb{R}^{2d}$ whose restriction to the x variables (respectively the y variables) is given by P (resp. Q) and which minimizes

$$\int \|x - y\|_2^2 d\pi(x, y)\,.$$

These two questions were in fact shown to be equivalent by Brennier [Bre91]

Theorem 1.6 (Brenier) *Assume $P, Q \ll dx$. Then there exists a unique π so that $\pi|_x = P$ and $\pi|_y = Q$ which minimizes*

$$\int \|x-y\|_2^2 d\pi(x,y)\,.$$

Moreover, π is optimal iff

$$\int f(x,y)d\pi(x,y) = \int f(x,T(x))dP(x) \quad \forall f$$

where T is a transport map from P to Q which can be written as $T = \nabla\phi$ for some convex function ϕ. T is then said to be monotone.

The question of the smoothness of the optimal transport T was also considered. It turns out that if dP/dx and dQ/dx are smooth and positive, T is smooth. Hence, in this case, even the C^* algebras are isomorphic. Of course, generalizing optimal transport theory to free probability would allow to answer many isomorphisms questions. This program was undertaken in [GS12] where the laws introduced in section 1.6 were considered. For the time being the result is only perturbative but still yields some interesting isomorphisms results, for instance between q-Gaussian and free semicircle variables for q small enough. Namely, let us denote

$$\|P\|_A = \sum |\lambda_P(q)| A^{\deg(q)}$$

if P can be decomposed as $P = \sum \lambda_P(q)q$ where the sum runs over monomials. Then, it was proved [GS12] that

Theorem 1.7 *Fix $A > 4$. Then, there exists $\varepsilon(A) > 0$ so that if V is a self-adjoint polynomial so that $\|V - \frac{1}{2}\sum X_i^2\|_A \le \varepsilon(A)$ there exists functions F and G in the closure of polynomial functions by $\|.\|_A$ so that for all polynomial P*

$$\begin{aligned}
\sigma^d(P) &= \tau_V\left(P(\mathcal{D}_1 F(X_1,\ldots,X_d),\ldots,\mathcal{D}_m F(X_1,\ldots,X_n))\right)\\
\tau_V(Q) &= \sigma^d\left(Q(\mathcal{D}_1 G(X_1,\ldots,X_d),\ldots,\mathcal{D}_n G(X_1,\ldots,X_d))\right)
\end{aligned}$$

The proof is based on the Schwinger-Dyson equation (9), and more precisely a rewriting of this equation which resembles the Monge-Ampère equation. Let us remind the reader what is the latter. Consider again probability measures P, Q on $\mathbb{R}^d$ and assume they have smooth densities

$$P(dx) = e^{-V(x)}dx \qquad Q(dx) = e^{-W(x)}dx\,.$$

Then $T\#P = Q$ is equivalent, if T is increasing, to

$$\begin{aligned}\int f(T(x))e^{-V(x)}dx &= \int f(x)e^{-W(x)}dx \\ &= \int f(T(y))e^{-W(T(y))}JT(y)dy\end{aligned}$$

with JT the Jacobian of T. Hence, this equation is equivalent to the Monge-Ampère equation

$$V(x) = W(T(x)) - \log JT(x)\,.$$

Monge-Ampère equation is a good tool to study the transport map. In particular, when one knows that JT is bounded below by cI (for instance in convex situations, see [Caf00]) the above is an implicit smooth equation for T; it can be solved by the implicit function theorem. The strategy of [GS12] is to show that (9) is equivalent to an equation which in fact is the analogue of Monge-Ampère equation, and then show that in perturbative situations it has a unique solution. This strategy could generalize to locally strictly convex potentials but there is no angle of attack for anything more general since in general the conjugate variable does not define the law uniquely (take V to have two deep double wells and find out that the support of the limiting measure is disconnected and that the mass of each connected component is not specified by the equation). Moreover, we know that transport can not be generalized as much as in the commutative setting in view of a result of Ozawa [Oza04] which shows that there is no separable universal II_1-factor. Understanding transport in a non-commutative setting is therefore a challenging question.

1.10 The entropy dimension

The initial strategy of Voiculescu to attack the isomorphism problem of free group factors was in fact to try to disprove it. To this end he proposed a candidate for an invariant of von Neumann algebra that takes different values on free group factors according to the number of generators. His construction is based on a notion of entropy which generalizes the usual Boltzmann entropy as follows. One defines neighborhoods of a non-commutative law τ as the set of Hermitian $N \times N$ matrices whose non-commutative distribution approximates τ (the topology is the weak topology so one considers matrices whose empirical distribution evaluated at words of degree smaller than some k are at distance less than ϵ from those of τ. These neighborhood get finer as k goes to infinity and ϵ to zero). One then evaluates the probability of such an event when the matrices are independent and follow the GUE. As the dimension goes to infinity and then the neighborhood shrinks to the singleton $\{\tau\}$, this probability decays exponentially fast and the rate of this decay should be given by the so-called non-commutative entropy. This entropy hence appears as a rate function for the large deviations of the empirical distribution of independent GUE matrices. Sadly enough, there are still technical difficulties so that it is not even known if the limsup and the liminf defining the rate function match. This is why the entropy is still defined by taking the limsup ; it is then called the micro states entropy and denoted $\chi(\tau)$. The free entropy dimension $\delta(X_1, \ldots, X_d)$ is given by the formula

$$\delta(X_1, \ldots, X_d) = d + \limsup_{\epsilon \downarrow 0} \frac{\chi(\tau_\epsilon)}{|\log \epsilon|}$$

where τ_ϵ is the distribution of $(X_1 + \epsilon S_1, \ldots, X_d + \epsilon S_d)$ if $(S_1, \ldots, S_d)$ are free semi-circle variables, free from $(X_1, \ldots, X_d)$.

It is easy to see that $\delta(S_1, \ldots, S_d) = d$ but not clear at all why δ should be an invariant of the von Neumann algebra a priori nor how Voiculescu had such an intuition. In fact, if one performs the analogous construction in the classical setting [GS07], that is take the usual entropy

$$S(p) = -\int \log \frac{dp(x)}{dx} dp(x)$$

and put

$$\delta_c(p) = d + \limsup_{\epsilon \downarrow 0} \frac{S(p * \gamma_\epsilon)}{|\log \epsilon|}$$

where γ_ϵ is the centered Gaussian law with covariance ϵ^2, then δ_c is only invariant under Lipschitz transport (that is $\delta_c(f\#p) = \delta_c(p)$ only if f is Lipschitz rather than measurable).

However, the entropy and the entropy dimension (or its slight generalization) appeared as very interesting mathematical objects which allowed to solve other important problems such as the absence of Cartan subalgebras in free group factors ; we refer to [Voi02b] for such applications. It is still open whether this approach can allow to construct invariants for von Neumann algebra.

A point which may be a positive sign that entropy dimension is a good object is its relation with L^2 Betti numbers [CS05]. Given a measure-preserving action of a free group F_n on a probability space (X, μ), there is a classical construction (going back to von Neumann) which associates to this data a von Neumann algebra denoted $L^\infty(X) \rtimes_\alpha F_n$. In the case of a trivial action on a one-point space, this von Neumann algebra is precisely the free group factor $L(F^{*n})$. For non-trivial actions and continuous spaces, one obtains other von Neumann algebras $M(X, \alpha, F_n) = L^\infty(X) \rtimes_\alpha F_n$. One can ask more generally when $M(X, \alpha, F_n)$ are isomorphic ; in the case $X = \{\text{point}\}$ this is the isomorphism question for free group factors. Remarkably, it was recently shown by Popa and Vaes [1] that for *free* actions α on non-atomic spaces (X, μ), $M(X, \alpha, F_n) \cong M(X, \alpha', F_{n'})$ entails in particular $n = n'$. The key step in their proof involves showing that any isomorphism of $M(X, \alpha, F_n)$ with $M(X, \alpha', F_n)$ must give rise to an orbit equivalence between the actions α and α'. It then follows from Gaboriau's ℓ^2 cohomology theory (and its L^2 Betti numbers) for orbit equivalence relations that $n = n'$.

2 Topological expansions

Topological expansions build upon the formula for moments of Gaussian (GUE) matrices' which are given by

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}((X^N)^p)] = \sum_{\substack{\text{graph 1 vertex} \\ \text{degree p}}} N^{-2g} = \sum_{g\geq 0} N^{-2g} M(g,p) \qquad (13)$$

where $M(g,p)$ is the number of maps with genus g build over a vertex of degree p, that is the number of graphs build over one vertex of degree p which can be properly embedded onto a surface of genus g (but not in a smaller genus surface). This formula was proved in (8) based on Wick calculus. 't Hooft [Hoo74] and Brézin-Itzykson-Parisi-Zuber [BIPZ78] had the idea in the seventies to use further this remarkable relation between matrix moments and the enumeration of graphs to enumerate maps with several vertices. Topological expansions were since then used in many diverse context in physics or mathematics ; after the enumeration of triangulations following Brézin, Itzykson, Parisi and Zuber, it was used to study the enumeration of meanders (Di Francesco ...), the enumeration of loop configurations and the O(n) model (Eynard, Kostov or Guionnet-Jones-Shlyakhtenko-Zinn Justin), and its application to knot theory (Zinn-Justin, Zuber)... The full topological expansions were used in mathematics since the work of Harer and Zagier (1986) in their article on the Euler characteristics of the moduli space of curves, and the famous work of Kontsevich. It was also seen as a tool to construct invariants based on its relation with algebraic geometry and topological string theory (the famous Dijkgraaf-Vafa conjecture states that Gromov-Witten invariants generating functions should be matrix integrals).

It turns out that such topological expansions are closely related with the so-called loop (or Schwinger-Dyson) equations which are satisfied by matrix models but also can be seen as topological recursion relations. At the first order, these equations are just given by the type of non-commutative integration by parts formula that we described in (12).

The next orders equations appear as derivatives of the first loop equation taken at finite dimension, and allow to describe the full topological expansion. Hence, as put forward by B. Eynard, the loop equations can be used as the key to construct topological expansions and therefore interesting geometric quantities and invariants.

In this lecture we describe more precisely the relation between matrix integrals, topological expansions and loop equations. Based on this relation, we show that topological expansions can be derived in much greater generality than those related with matrices with Gaussian entries and Feynmann diagrams, namely in models for which loop equations given by a non-commutative derivative are valid. We detail the case of β ensembles and integrals over the unitary group. This lecture is related with the previous but hopefully can be followed independently.

2.1 Topological expansions and Wick formula

Expansion (13) can be generalized to monomials in several matrices and then count maps with one color vertex. The first natural idea to count maps with several vertices is to consider the expectation of a product of traces of words as follows. Let $q_1, \cdots, q_n$ be monomials in d non-commutative variables. Then, applying Gaussian calculus (that is Wick formula), we find that if $\mathbb{P}$ is the law of the GUE (see section 1.5)

$$\int \prod_{i=1}^{n} (N\mathrm{Tr}(q_i(X_1, \cdots, X_m)))d\mathbb{P}_N(X_1)\cdots d\mathbb{P}_N(X_m)$$

$$= \sum_{g\in\mathbb{N}} \sum_{c\geq 1} \frac{1}{N^{2g-2c}} \sharp\{G_{g,c}((q_i, 1), 1 \leq i \leq n)\}$$

Here $\sharp\{G_{g,c}((q_i, 1), 1 \leq i \leq n\}$ is the number of graphs (up to homeomorphism) that can be build on stars of type $q_i, 1 \leq i \leq n$ with exactly c connected components so that the sum of their genera is equal to g.

Hence, such expectations are related with the enumeration of graphs with several vertices but unfortunately do not sort the connected graphs. We next how this can be done.

2.2 Matrix models and topological expansions

To enumerate connected graphs, and more precisely maps, the idea [BIPZ78] is to consider instead of moments partition functions, that is logarithm of Laplace transforms of traces of monomials.

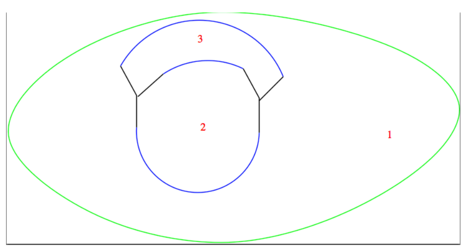
Before going any further let us define more precisely maps.

2.2.1 Maps

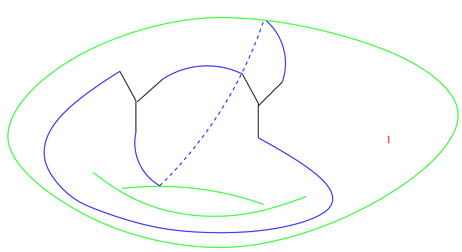
Maps are connected graphs which are properly embedded into a surface, that is embedded in such a way that its edges do not cross and the faces (obtained by cutting the surface along the edges of the graph) are homeomorphic to disks. The genus of a map is the smallest genus of a surface so that this can be done. By Euler formula, we have

$$2-2g = \#\{vertices\} + \#\{faces\} - \#\{edges\}\,.$$

Be careful that in the definition above the external face is counted. Here is a genus zero (or planar) map with two vertices, 3 edges and 3 faces



and here is a genus one map with two vertices, 3 edges and one face :



As surfaces come with an orientation, a fact is that any given cyclic order at the ends of edges of a graph around each vertex uniquely determines the imbedding of the graph into a surface.

Hence, to enumerate maps, we shall be given vertices equipped with "half-edges" and a cyclic order at the ends of their edges. Edges will just be created by matching the end points of the half-edges. As we shall count labeled maps, we shall assume also that each vertex is given a root, that is a marked edge. In fact, these vertices will be given by stars as defined in section 1.5. Such vertices can be equipped with colored half-edges and therefore will be bijectively associated with monomials simply by assigning to each letter a color, see section 1.5.

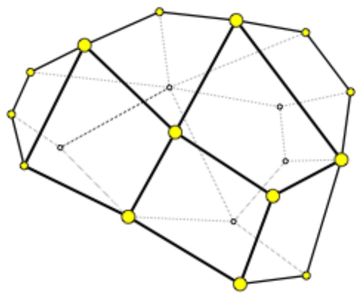
We will denote for $\mathbf{k} = (k_1, \cdots, k_n)$ and monomials $q_1, \ldots, q_n$,

$$\mathcal{M}_g((q_i, k_i), 1 \le i \le n) = \#\{ \text{ maps with genus } g$$

$$\text{and } k_i \text{ stars of type } q_i, 1 \le i \le n\}.$$

the number of maps with genus g build on k_i stars of type q_i, $1 \le i \le n$, by matching the half-edges of the stars which have the same color. The enumeration is done up to homeomorphisms.By convention, we will denote $\mathcal{M}_0(1) = 1$.

Note that stars can also be seen by duality as polygons with colored sides and one mark side, where each end point of the half-edge is replaced by a perpendicular segment of the same color. Maps are then "polygonizations" of a surface with given genus by polygons of prescribed nature. For instance, for the matrix model with $q(X) = X^4$, the stars are vertices with valence four, which in the dual picture are just square. We are thus counting quadrangulations of a surface with given genus and a given number of squares. The counting is done with labeled sides.



2.2.2 Random matrices and the enumeration of maps

Consider $q_1, \cdots, q_n$ monomials. Then, [BIPZ78] shows that

$$\log\left(\int e^{\sum_{i=1}^{n} t_i N \mathrm{Tr}(q_i(X_1,\cdots,X_m))} d\mathbb{P}_N(X_1)\cdots d\mathbb{P}_N^V(X_m)\right)$$

$$= \sum_{g\geq 0} \frac{1}{N^{2g-2}} \sum_{k_1,\cdots,k_n\in\mathbb{N}} \prod_{i=1}^{n} \frac{(t_i)^{k_i}}{k_i!} \mathcal{M}_g((q_i,k_i), 1\leq i\leq n) \tag{14}$$

where the equality means that derivatives of all orders at $t_i = 0, 1 \leq i \leq n$, match. The proof of this formula is simply done by developing the exponential and recalling that the logarithmic function will yield connected graphs. Adding in the potential a term tq, taking a formal derivative in t at the origin shows that if $V = \frac{1}{2}\sum X_i^2 - \sum t_i q_i(X_1, \ldots, X_m)$ then for any monomial q

$$\int \frac{1}{N} \mathrm{Tr}(q(X_1^N, \ldots, X_d^N)) d\mathbb{P}_N^V(X_1^N, \ldots, X_d^N)$$

$$= \sum_{g\geq 0} \frac{1}{N^{2g-2}} \sum_{k_1,\cdots,k_n\in\mathbb{N}} \prod_{i=1}^{n} \frac{(t_i)^{k_i}}{k_i!} \mathcal{M}_g((q,1); (q_i,k_i), 1\leq i\leq n) \tag{15}$$

where $\mathcal{M}_g((q,1); (q_i,k_i), 1 \leq i \leq n)$ is the number of maps of genus g build on one star of type q and k_i stars of type k_i, $1 \leq i \leq n$.

At this point the equality is formal but it can in fact be made asymptotic as soon as reasonable assumptions are made to insure that the integral converges and that the t_i are small enough to guarantee the convergence of the series. Equality (15) given asymptotically up to any order of correction N^{-k} is called an asymptotic topological expansion. We next discuss this issue, and how the loop equations can play a key role in deriving the topological expansions.

2.3 Loop equations and asymptotic expansions

It is possible to prove topological expansions by using functional calculus (namely integration by parts) rather than Wick formula and Gaussian calculus. This approach turns out to allow the proof of asymptotic topological expansions but also to generalize to different setting which are not related with any Gaussian variables, such as the integration over the unitary group or under the so-called β ensemble. We first describe the strategy for the law $\mathbb{P}_V^N$. As we already pointed out, by a simple integration by part, we can prove, see (10), that

$$\int \frac{1}{N}\mathrm{Tr}(P\mathcal{D}_i V)d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N) \tag{16}$$

$$= \int \frac{1}{N}\mathrm{Tr}\otimes\frac{1}{N}\mathrm{Tr}(\partial_i P)d\mathbb{P}_V^N(X_1^N,\ldots,X_d^N)\,.$$

In the case where V is strictly convex (in the sense that $\mathrm{Hess}(\mathrm{Tr}V) \geq cI$ for some $c > 0$), we can argue by standard concentration of measure (see (11)) and Brascamp Lieb inequality (see [AGZ10] and [Gu09]) that there exists a finite constant C(which only depends on c) so that for any monomial q of degree less than $\sqrt{N}$

$$\int |\frac{1}{N}\mathrm{Tr}(q(X_1^N,\ldots,X_d^N))|d\mathbb{P}_N^V(X_1^N,\ldots,X_d^N) \leq C^{\deg q}\,. \tag{17}$$

As a consequence, the family $\{\int \frac{1}{N}\mathrm{Tr}(q(X_1^N,\ldots,X_d^N))d\mathbb{P}_N^V(X_1^N,\ldots,X_d^N), q\}$ is tight. Any limit point $\{\tau(q), q\}$ satisfies the Schwinger-Dyson equation

$$\tau_V(P\mathcal{D}_i V) = \tau_V\otimes\tau_V(\partial_i P) \tag{18}$$

with $\tau_V(I) = 1$. Here τ_V is extended linearly to polynomials. Moreover, for any monomial q, we deduce from (17) that

$$|\tau_V(q)| \leq C^{\deg(q)} . \tag{19}$$

As a consequence, when $V - \frac{1}{2}\sum X_i^2 = \sum t_i q_i$ with the t_i small enough, there exists a unique solution to (18). Indeed, when $t_i = 0$, the moments are just defined inductively by (18). When the t_i are small enough, the equation still has a unique solution. Indeed, taking two solutions $\tau, \tilde{\tau}$ and denoting

$$\Delta_k := \sup_{q:\deg(q)\leq k} |\tau(q) - \tilde{\tau}(q)|$$

where the supremum is taken on monomials of degree smaller or equal to k, we have by using (18), (21) and $\Delta_0 = 0$, if $D + 1 = \max \deg(q_i)$

$$\begin{aligned}
\Delta_{k+1} &= \max_i \sup_{q:\deg \text{ q}\leq k} |\tau(X_i q) - \tilde{\tau}(X_i q)| \\
|\tau(X_i q) - \tilde{\tau}(X_i q)| &\leq |\tau \otimes \tau(\partial_i q) - \tilde{\tau} \otimes \tilde{\tau}(\partial_i q)| + D \sum t_j \Delta_{k+D} \\
&\leq \sum_{l=1}^{k} \Delta_l C^{k-l} + D \sum t_j \Delta_{k+D}
\end{aligned}$$

Hence,

$$\Delta_{k+1} \leq \sum_{l=1}^{k} \Delta_l C^{k-l} + D \sum t_j \Delta_{k+D}, \Delta_k \leq 2C^k$$

so that for $\gamma < 1/C$,

$$\Delta_\gamma = \sum_{k\geq 1} \gamma^k \Delta_k \leq \frac{\gamma}{1 - C\gamma} \Delta_\gamma + \frac{D \sum |t_j|}{\gamma^D} \Delta_\gamma \tag{20}$$

which entails $\Delta_\gamma = 0$ for $\gamma < 1/C \wedge$ so that

$$\frac{\gamma}{1 - C\gamma} + \frac{D \sum |t_j|}{\gamma^D} < 1 ,$$

which implies $\tau = \tilde{\tau}$.

We shall see that

Theorem 2.1 *For t_i small enough,*

$$\mathcal{M}_{\mathbf{t}}(q) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_0((q,1);(q_i,k_i), 1 \le i \le n)$$

is solution of equation (18) *and therefore*

$$\tau_V(q) = \mathcal{M}_{\mathbf{t}}(q)$$

Let us remark that by definition of τ_V, for all polynomials P, Q,

$$\tau_V(PP^*) \ge 0 \quad \tau_V(PQ) = \tau_V(QP).$$

As a consequence, $\mathcal{M}_{\mathbf{t}}$ also satisfy these equations : for all P, Q

$$\mathcal{M}_{\mathbf{t}}(PP^*) \ge 0 \quad \mathcal{M}_{\mathbf{t}}(PQ) = \mathcal{M}_{\mathbf{t}}(QP) \quad \mathcal{M}_{\mathbf{t}}(1) = 1.$$

This means that $\mathcal{M}_{\mathbf{t}}$ is a tracial state. The traciality property can easily be derived by symmetry properties of the maps. However, the positivity property $\mathcal{M}_{\mathbf{t}}(PP^*) \ge 0$ is not easy to prove combinatorial, and hence matrix models are an easy way to derive it. This property may be seen to be useful to actually solve the combinatorial problem (i.e. find an explicit formula for $\mathcal{M}_{\mathbf{t}}$).

Proof. Let us denote in short, for $\mathbf{k} = (k_1, \ldots, k_n)$ and a monomial q by $\mathcal{M}_{\mathbf{k}}(q) = \mathcal{M}_0((q,1);(q_i,k_i), 1 \le i \le n)$ the number of planar maps with k_i stars of type q_i and one of type q. We generalize this definition to polynomials P by linearity. We let

$$\mathcal{M}_{\mathbf{t}} = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}(q)$$

This series is a priori formal but we shall see below that in fact there exists a finite constant C so that for any monomials q_i

$$\mathcal{M}_{\mathbf{k}}(q) \le \prod k_i! C^{\sum k_i \deg(q_i)} \tag{21}$$

converges for $|t_i| < 1/C$. $\mathcal{M}_{\mathbf{t}}$ satisfies (18) if and only if for every $\mathbf{k}$ and P

$$\mathcal{M}_{\mathbf{k}}(X_i P) = \sum_{\substack{0\le p_j\le k_j\\ 1\le j\le n}} \prod_{j=1}^{n} C_{k_j}^{p_j} \mathcal{M}_{\mathbf{p}} \otimes \mathcal{M}_{\mathbf{k}-\mathbf{p}}(\partial_i P) + \sum_{1\le j\le n} k_j \mathcal{M}_{\mathbf{k}-1_j}([D_i q_j]P) \tag{22}$$

where $1_j(i) = 1_{i=j}$ and $\mathcal{M}_{\mathbf{k}}(1) = 1_{\mathbf{k}=\mathbf{0}}$.

- We first check (22) for $\mathbf{k} = \mathbf{0} = (0, \cdots, 0)$. By convention, $\mathcal{M}_{\mathbf{0}}(1) = 1$. We now check that

$$\mathcal{M}_{\mathbf{0}}(X_i P) = \mathcal{M}_{\mathbf{0}} \otimes \mathcal{M}_{\mathbf{0}}(\partial_i P) = \sum_{P=p_1 X_i p_2} \mathcal{M}_{\mathbf{0}}(p_1)\mathcal{M}_{\mathbf{0}}(p_2)$$

  But in any planar map with only one star of type $X_i P$, the half-edge corresponding to X_i has to be glued with another half-edge of P. If X_i is glued with the half-edge X_i coming from the decomposition $P = p_1 X_i p_2$, the map is then split into two (independent) planar maps with stars p_1 and p_2 respectively (note here that p_1 and p_2 inherite the structure of stars since they inherite the orientation from P as well as a marked half-edge corresponding to the first neighbour of the glued X_i.) Hence the relation is satisfied.
- We now proceed by induction over $\mathbf{k}$ and the degree of P ; we assume that (22) is true for $\sum k_i \le M$ and all monomials, and for $\sum k_i = M + 1$ when $\deg(P) \le L$. Note that $\mathcal{M}_{\mathbf{k}}(1) = 0$ for $|\mathbf{k}| \ge 1$ since we can not glue a vertex with no half-edges with any star. Hence, this induction can be started with $L = 0$. Now, consider $R = X_i P$ with P of degree less than L and the set of planar maps with a star of type $X_i P$ and k_j stars of type q_j, $1 \le j \le n$, with $|\mathbf{k}| = \sum k_i = M + 1$. Then,
  ⋄ either the half-edge corresponding to X_i is glued with an half-edge of P, say to the half-edge corresponding to the decomposition $P = p_1 X_i p_2$; we then can use the argument as above ; the map M is cut into two disjoint planar maps M_1 (containing the star p_1) and M_2 (resp. p_2), the stars of type q_i being distributed either in one or the other of these two planar maps ; there will be $r_i \le k_i$ stars of type q_i in M_1, the rest in M_2. Since all stars all labelled, there will be $\prod C_{k_i}^{r_i}$ ways to assign these stars in M_1 and M_2.

Hence, the total number of planar maps with a star of type X_iP and k_i stars of type q_i, such that the marked half-edge of X_iP is glued with an half-edge of P is

$$\sum_{P=p_1X_ip_2} \sum_{\substack{0\le r_i\le k_i\\1\le i\le n}} \prod_{i=1}^{n} C_{k_i}^{r_i}\mathcal{M}_{\mathbf{r}}(p_1)\mathcal{M}_{\mathbf{k}-\mathbf{r}}(p_2) \tag{23}$$

$\diamond$ Or the half-edge corresponding to X_i is glued with an half-edge of another star, say q_j ; let's say with the edge coming from the decomposition of q_j into $q_j = q_j^1 X_i q_j^2$. Then, once we are giving this gluing of the two edges, we can replace the two stars X_iP and $q_j^1 X_i q_j^2$ glued by their X_i by the star $q_j^2 q_j^1 P$.
We have k_j ways to choose the star of type q_j and the total number of such maps is

$$\sum_{q_j=q_j^1X_iq_j^2} k_j\mathcal{M}_{\mathbf{k}-1_j}(q_j^2q_j^1P)$$

Summing over j, we obtain by linearity of $\mathcal{M}_{\mathbf{k}}$

$$\sum_{j=1}^{n} k_j\mathcal{M}_{\mathbf{k}-1_j}([D_iq_j]P) \tag{24}$$

(23) and (24) give (22). Moreover, it is clear that (22) defines uniquely $\mathcal{M}_{\mathbf{k}}(P)$ by induction. In addition, we see that the solution to (22) satisfy (21) : Indeed this is true for $\mathbf{k} = 0$ as free semi-circle variables are bounded by 2 and then follows for large $\mathbf{k}$ by induction over $\sum k_i$.

It turns out that this strategy can be followed for each genera up to consider a family of loop equations which are obtained by differentiating the first one with respect to small additional potentials. The first point is to derive the second order Schwinger Dyson equation by varying V into $V + \varepsilon W$ and differentiating at $\varepsilon = 0$ the first order loop equation (16), hence getting equations for the cumulants. We refer the interested reader to [GMS07, Ma06] for full details, but outline the approach below. The first point is to prove an a priori rough estimate by showing that there exists a finite constant $C > 0$ so that for all t_i's small enough, all monomials q of degree less than $N^{1/2-\varepsilon}$ for $\varepsilon > 0$, we have

$$|\mathbb{E}[\tau_{X_N}[q]] - \tau_V(q)| \leq \frac{C^{\deg(q)}}{N^2}\,.$$

The proof elaborates on the ideas developed around (20) to prove uniqueness of the solution to Schwinger-Dyson equation and the concentration inequalities (11) which give a fine control on the error term in the loop equation satisfied by $\mathbb{E}[\tau_{X_N}]$ with respect to the Schwinger-Dyson equation. Once we have this a priori estimate, we write the second loop equation by making a small change in the potential $V \to V + \epsilon N^{-1} W$ and identifying the linear term in ε in the first order loop equation. We denote by

$$\begin{aligned} W_2^V(P,Q) &= \mathbb{E}[(\mathrm{Tr}P - \mathbb{E}\mathrm{Tr}P)(\mathrm{Tr}Q - \mathbb{E}[\mathrm{Tr}Q])] \\ &= \partial_\varepsilon \mathbb{P}^{V-\epsilon N^{-1}Q}(\mathrm{Tr}P)|_{\varepsilon=0} \\ W_3^V(P,Q,R) &= \partial_\varepsilon W_2^{V-\epsilon N^{-1}R}(P,Q) \end{aligned}$$

We denote by $\bar{\delta}^N(P) = \mathbb{E}[\mathrm{Tr}(P)] - N\tau_V(P)$. Note that equation (16) can be written as

$$\mathbb{E}[\mathrm{Tr}(\Xi_i P)] = \frac{1}{N} W_2(\partial_i P) + \frac{1}{N} \bar{\delta}^N \otimes \bar{\delta}^N(\partial_i P)\,, \tag{25}$$

where

$$\Xi_i P = \partial_i V \# P - (\tau_V \otimes I + I \otimes \tau_V)\partial_i P\,.$$

By our a priori estimate on $\bar{\delta}_N$ the last term is at most of order N^{-3}. Hence, to estimate the first order correction, we would like to estimate the asymptotics of W_2 as well as "invert" Ξ_i. It turns out that even though Ξ_i is hardly invertible, a combination of the Ξ_i is on polynomials so that $\tau_V(P) = 0$, namely

$$\Xi P = \sum_i \Xi_i \mathcal{D}_i P$$

Indeed, the latter can be seen as a small perturbation of the infinitesimal generator of the free Brownian motion, see [GMS07].

To estimate W_2, we obtain the second loop equation by changing $V \to V - \epsilon N^{-1} W$ in (16) and identifying the linear terms in ϵ; we find if

$$\begin{aligned} W_2(\Xi_i P, W) &= \mathbb{E}[\frac{1}{N}\mathrm{Tr}(P\mathcal{D}_i W)] \\ &\quad + N^{-1} W_3(\partial_i P, W) + (W_2 \otimes \bar{\delta}^N + \bar{\delta}^N \otimes W_2)(\partial_i P, W) \end{aligned}$$

It turns out that the term in W_3 is bounded by concentration inequalities whereas $\bar{\delta}^N$ is of order N^{-1} by our previous rough estimate. Hence we see that

$$\lim_{N\to\infty} W_2(\Xi_i P, W) = \tau_V(P\mathcal{D}_i W)$$

for all i and P. Applying this with $P = \mathcal{D}_i Q$ and summing we conclude that

$$\lim_{N\to\infty} W_2(P, W) = \tau_V(\sum_i \mathcal{D}_i \Xi^{-1} P \times \mathcal{D}_i W) =: w_2(P, W)$$

and therefore plugging this back into (25) we deduce the first order correction

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}(P)] = \tau_V(P) + \frac{1}{N^2} w_2[\sum_i \partial_i \mathcal{D}_i \Xi^{-1} P] + o(N^{-2}) \,.$$

The next orders of the asymptotic expansion can be found similarly.

It turns out that loop equations appear for many other models which are not directly related with Gaussian random matrices. It seems that a large family of loop equations give rise to topological expansions. We describe below the case of the β-ensembles and the integration over the unitary group.

2.4 Topological expansion for β-matrix models

The law of the eigenvalues of the GUE follows the distribution on $\mathbb{R}^N$

$$dP_N(\lambda) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum \lambda_i^2} \prod d\lambda_i$$

as can be checked by doing the change of variables associating to X its ordered eigenvalues and a parametrization of its eigenvectors. β-ensembles are the following generalization of this distribution :

$$dP^V_{N,\beta}(\lambda) = \frac{1}{Z^V_{N,\beta}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N\sum V(\lambda_i)} \prod d\lambda_i$$

It is related with invariant matrix ensembles only in the cases $\beta = 1, 2, 4$ and a priori has no relations with Gaussian entries otherwise. However, it was proved in [BG12], see [CE06] for a formal proof, that β-ensembles have a to pological expansion. More precisely, assume that V is analytic in a neighborhood of the real line and such that the unique probability measure μ_V which minimizes

$$\int V(x)d\mu(x) - \frac{\beta}{2} \int\int \log|x-y| d\mu(x)d\mu(y) \tag{26}$$

has a connected support, and V is off critical in the sense that $V'(x) - \beta \int (x-y)^{-1} d\mu_V(x)$ does not vanish in an open neighborhood of the support of μ_V, then for any $z \in \mathbb{C}\backslash\mathbb{R}$, and $K \geq 0$

$$\int \frac{1}{N} \sum_{i=1}^N \frac{1}{z-\lambda_i} dP^V_N(d\lambda) = \sum_{k=0}^K N^{-k}\, W^{V,k}(z) + o(N^{-K}) \tag{27}$$

where $o(N^{-K})$ is uniform on compacts. Moreover, we have

$$W^{V,k}(z) = \sum_{g=0}^{\lfloor k/2 \rfloor} \left(\frac{\beta}{2}\right)^{-g} \left(1 - \frac{2}{\beta}\right)^{k-2g+1} \mathcal{W}^{V;(g;k-2g+1)}$$

and if V is a small perturbation of the quadratic potential, $\mathcal{W}^{V;(g;k-2g+1)}$ expands as a generating function of maps of genus g when ribbons are twisted $k-2g+1$ times.

Note that the hypothesis that the support is connected is important since otherwise the result is not true in general. The proof of such expansion relies as well on the loop equations

$$\int \frac{\beta}{N^2} \sum_{i \neq j} \frac{f(x_i) - f(x_j)}{x_i - x_j} dP^V_{N,\beta} = \int [\frac{1}{N} \sum f(\lambda_i) V'(\lambda_i)] dP^V_{N,\beta} \tag{28}$$

which can be proved by integration by parts. As a consequence, one sees that the equilibrium measure μ_V satisfies the limiting equation

$$\int\int \frac{f(x)-f(y)}{x-y} d\mu_V(x)d\mu_V(y) = \int f(x)V'(x)d\mu_V(x) \tag{29}$$

If V is a small perturbation of the quadratic one can develop arguments similar to those of the previous section to check that moments under μ_V are generating functions for planar maps.

In fact, the limiting equation (29) does not always have a unique solution as it is a weak characterization of the minimizers of (26), but it does as soon as V is strictly convex for instance. In any case, μ_V governs the first order of the expansion. To get the higher order terms in the expansion the idea is, as in the previous section, to write equations for all the cumulants

$$W_n^V(x_1,\ldots,x_n) = \partial_{\epsilon_1}\cdots\partial_{\epsilon_n}\Big(\ln Z_{N,\beta}^{V-\frac{2}{\beta N}\sum_i \frac{\epsilon_i}{x_i-\bullet}}\Big)\Big|_{\epsilon_i=0}$$

by differentiating the loop equation (28) with respect to the potential.

2.5 Loop equations for the unitary group

In this section we shall consider the Haar measure dU on the unitary group, that is the unique measure on $U(N)$ which is invariant under left multiplication by unitary matrices. We consider matrix integrals given by

$$I_N(V,A_i) = \int e^{N\mathrm{Tr}(V(A_i,U_i,U_i^*,1\le i\le m))} dU_1\cdots dU_m$$

A well-known example is the Harich-Chandra-Itzykson-Zuber integral

$$HCIZ(A_1,A_2) = \int e^{N\mathrm{Tr}(A_1UA_2U^*)} dU$$

where $(A_i, 1\le i\le m)$ are $N\times N$ deterministic uniformly bounded matrices, dU denotes the Haar measure on the unitary group $U(N)$

(normalized so that $\int_{U(N)} dU = 1$) and V is a polynomial function in the non-commutative variables $(U_i, U_i^*, A_i, 1 \le i \le m)$. We assume that the joint distribution of the $(A_i, 1^2 i^2 m)$ converges ; namely for all polynomial function P in m non-commutative indeterminates

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}(P(A_i, 1 \le i \le m)) = \tau(P)$$

for some linear functional τ on the set of polynomials. For technical reasons, we assume that the polynomial V satisfies $\mathrm{Tr}(V(U_i, U_i^*, A_i, 1 \le i \le m)) \in \mathbb{R}$ for all $U_i \in U(N)$ and all Hermitian matrices $A_i, 1 \le i \le m$ and $N \in \mathbb{N}$. Under those very general assumptions, the formal convergence of the integrals could already be deduced from [C03]. The following Theorem is a precise description of the results from [CGMS09] which gives an asymptotic convergence :

Theorem 2.2 *Under the above hypotheses and if we further assume that the spectral radius of the matrices $(A_i, 1 \le i \le m, N \times N)$ is uniformly bounded (by say M), there exists $\epsilon = \epsilon(M, V) > 0$ so that for $z \in [-\epsilon, \epsilon]$,the limit*

$$F_{V,\tau}(z) = \lim_{N\to\infty} \frac{1}{N^2} \log I_N(zV, A_i)$$

exists. Moreover, $F_{V,\tau}(z)$ is an analytic function of $z \in \{z \in \mathbb{C} : |z| \le \epsilon\}$. Furthermore, if we let

$$P_N(dU, \ldots, dU) = \frac{1}{I_N(V, A_i)} e^{NTr(V(U_i, U_i^*, A_i, 1 \le i \le m))} dU_1 \cdots dU_m$$

for all polynomial P in $(U_i, U_i^*, A_i)_{1 \le i \le m}$ we have the convergence

$$\tau_{V,\tau}(P) = \lim_{N\to\infty} \int \frac{1}{N} \mathrm{Tr}(P((U_i, U_i^*, A_i)_{1 \le i \le m})) dP_N$$

Note that $\frac{1}{N^2} \log I_N(zV, A_i)$ is a uniformly bounded analytic function, hence its limit points are analytic and therefore uniquely determined by their values on a set with an accumulation point. Hence, the convergence above yields the convergence on the whole complex plane. Obtaining the next order of the expansion is actually a work in progress which should follow the same ideas that for integration over Gaussian matrices.

The strategy is again to find and study the Schwinger-Dyson (or loop) equations under the associated Gibbs measure P_N. This equation is based on the invariance of the Haar measure, which somehow generalize the Gaussian case where the loop equation was based on integration by parts, which can be seen as a consequence of the invariance by translation of Lebesgue measure.

To define this equation let us first define derivatives on polynomials in these matrices by the linear form such that for all $i, j \in \{1, \ldots, m\}$

$$\partial_j A_i = 0 \qquad \partial_j U_i = 1_{i=j} U_j \otimes 1 \qquad \partial_j U_i^* = -1_{i=j} 1 \otimes U_j^*$$

and satisfying the Leibnitz rule, namely, for monomials P, Q,

$$\partial_j(PQ) = \partial_j P \times (1 \otimes Q) + (P \otimes 1) \times \partial_j Q. \tag{30}$$

Here $\times$ denotes the product $P_1 \otimes Q_1 \times P_2 \otimes Q_2 = P_1 P_2 \otimes Q_1 Q_2$. We also let D_i be the corresponding cyclic derivatives such that if $m(A \otimes B) = BA$, then $D_j = m \circ \partial_j$. If q is a monomial, we more specifically have

$$\partial_j q = \sum_{q = q_1 U_j q_2} q_1 U_i \otimes q_2 - \sum_{q = q_1 U_j^* q_2} q_1 \otimes U_j^* q_2 \tag{31}$$

$$D_j q = \sum_{q = q_1 U_j q_2} q_2 q_1 U_j - \sum_{q = q_1 U_j^* q_2} U_j^* q_2 q_1 \tag{32}$$

Using the invariance by multiplication of the Haar measure one can prove the asymptotic Schwinger-Dyson equation :

$$P_N\left(\frac{1}{N}\mathrm{Tr} \otimes \frac{1}{N}\mathrm{Tr}(\partial_j P)\right) + P_N\left(\frac{1}{N}\mathrm{Tr}(P D_j V)\right) = 0$$

This is proved by noticing that if we set $U_j(t) = U_j e^{itB}$ and leave the other $U_k(t) = U_k$ unchanged for a Hermitian matrix B then for all

$k, l \in \{1, \ldots, N\}$

$$\partial_t \int P(U_p(t), 1 \le p \le m, A_i)(k,l) e^{NTr(V(U_p(t), U_p(t)^*, A_i)} dU_1 \cdots dU_m = 0$$

Taking $B = 1_{kl} + 1_{lk}$ and $B = i1_{kl} - i1_{lk}$ shows that we can by linearity choose $B = 1_{kl}$ even though this is not self-adjoint which yields the result after summation over k and l. By using concentration of measure, we know that for all polynomial P $N^{-1}\mathrm{Tr}(P(U_i, U_i^*, A_i))$ is not far from its expectation and therefore we deduce that the limit points of these (bounded) quantities $\tau(P)$ satisfy the Schwinger-Dyson equation

$$\tau \otimes \tau(\partial_j P) + \tau(D_j V P) = 0$$

Uniqueness of the solution to such an equation in the perturbative regime is done as in the Gaussian case ; when $V = 0$ it is clear as it defines all moments recursively from the knowledge of τ restricted to the A_i and a perturbation argument shows this is still true for small parameters. The uniqueness provides the convergence whereas the study of this solution shows that it expands as a generating series in the enumeration of some planar maps.

3 Loop models

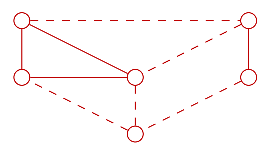
In this last lecture we show how matrix integrals can also be used to enumerate loop models, which are in some cases equivalent to the famous Potts model (on random planar maps). We first discuss the relation with the Potts model and then show how to construct a matrix model to solve the related enumeration question.

3.1 The Potts model on random maps

The partition function of the Potts model on a graph $G = (V, E)$ is given by

$$\begin{aligned} Z_G &= \sum_{\sigma: V \to \{1,\ldots,Q\}} \exp(K \sum_{\{i,j\} \in E} \delta_{\sigma_i,\sigma_j}) \\ &= \sum_{\sigma: V \to \{1,\ldots,Q\}} \prod_{\{i,j\} \in E} (1 + v\delta_{\sigma_i,\sigma_j}) \\ &= \sum_{E' \subset E} v^{\sharp \text{ bonds}} \, Q^{\sharp \text{ clusters}} \end{aligned}$$

where $v = e^K - 1$, bonds are the edges in E', a subset of E, the clusters are the connected components of the subgraph (V, E'). For instance the following graph where the edges in E' are bold whereas those in $E \backslash E'$ are dashed,



has weight $v^4 Q^3$.

We shall consider the Potts model on random planar maps. Recall (see section 2.2.1) that a map is a connected graph which is embedded into a surface in such a way that edges do not cross and faces (obtained by cutting the surface along the edges) are homeomorphic to a disk.

The genus of the map is the minimal genus of a surface in which it can be properly embedded.

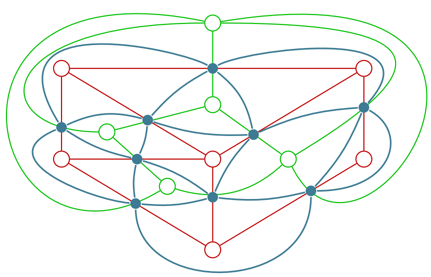
By Euler formula :

$$2 - 2g = \sharp \text{ vertices} + \sharp \text{ faces } - \sharp \text{ edges}.$$

We shall consider the Potts model on random planar maps. We assume these graphs are rooted, that is are given a distinguished oriented edge. It is given by the partition function

$$\begin{aligned} Z &= \sum_{G=(V,E)} x^{\#E} y^{\#V} Z_G \\ &= \sum_{G=(V,E)} x^{\#E} y^{\#V} \sum_{\sigma:V\to\{1,\ldots,Q\}} \exp(K \sum_{\{i,j\}\in E} \delta_{\sigma_i,\sigma_j}) \\ &= \sum_{G=(V,E)} x^{\#E} y^{\#V} \sum_{E'\subset E} v^{\sharp\text{ bonds}} Q^{\sharp\text{ clusters}} \end{aligned} \tag{33}$$

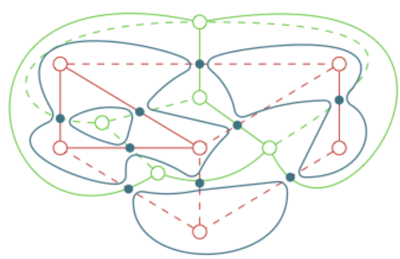
If G is a planar map, there is a dual (green) and a medial (blue) planar graph G_m. The vertices of the dual graph are given by a point in each of the faces of the original graph and each of the edges of the dual graph intersect one (and only one) edge of the original graph. The vertices of the medial graph are the intersection of the edges of the dual graph and the original graph. The medial graph has an edge in each face of the graph obtained by taking both edges and vertices of the original and dual graph. The edges of the medial graph do not cut the edges of the original or the medial graph. Hence,the medial graph is four-valent.



By construction, the original graph and the dual graph are in bijection. Note that in each face of the medial graph there is either a vertex from the dual or the original graph and that this choice is given by a checkerboard coloring of the medial graph corresponding to fix which type of vertex is inside a face. Hence, knowing the medial graph and the nature of the vertex in one of its face allows to reconstruct both the original and the dual graph.

We next describe the bijection between the configurations on the original, dual and medial graph. A configuration on the original graph just consists in coloring some edges and dashing the others. The dual configuration on the medial graph is given by splitting the vertex so that it does not intersect a colored edge. Hence, there are two sorts of vertices according to the nature of the colored edge it does not intersect. The two sorts of vertices on the medial graph are as follows :

Configuration are therefore described bijectively by the collection of loops of the medial graph as well as as a checkerboard coloring.



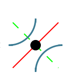
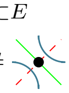
If G is a planar map, there is a bijection between the configuration on G and the set of loops and shaded vertices on the medial graph.

Moreover, writing Euler formula in each cluster gives the relation

$$\sharp\text{loops} = 2\sharp\text{clusters} + \sharp\text{bonds} - \sharp V$$

The equivalence to the loop model allows to state that if we take $y = Q^{-\frac{1}{2}}$ in (33)

$$Z = \sum_{G=(V,E)} x^{\#E} Q^{-\frac{1}{2}\#V} \sum_{E' \subset E} v^{\sharp\text{ bonds}} Q^{\sharp\text{ clusters}}$$

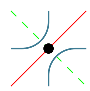
$$= \sum_{\Gamma} \delta^{\#\text{ loops}} \alpha^{\#}$$  $\beta^{\#}$ 

where the summation is restricted to 4-valent rooted planar maps, and

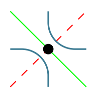
$$\delta = \sqrt{Q} \qquad \frac{\alpha}{\beta} = \frac{v}{\sqrt{Q}} \qquad \beta = x$$

δ is called the fugacity.

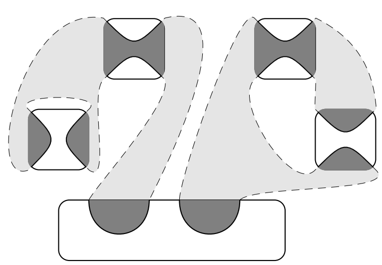
Hence, when $y = Q^{-1/2}$, the partition function Z of the Potts model on planar maps is a generating function for the number of possible matchings of the end points of n copies of the vertex



and m copies of the vertex



so that the resulting graph is planar, connected, has p loops, and is checkerboard shaded.



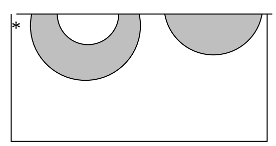
We shall consider generalizations of such enumeration questions in the following.

3.2 Loop models and Random matrices

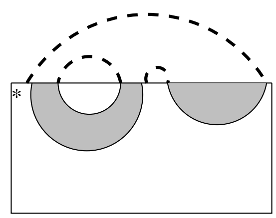
We have already seen in the previous lecture that random matrices could be used to enumerate planar graphs. In this section we show how this point can be specified to enumerate loop models.

In the following we shall consider loop models with vertices given by Temperley-Lieb elements.

The Temperley-Lieb elements are boxes with boundary points connected by non-intersecting strings, equipped with a shading and a marked boundary point.

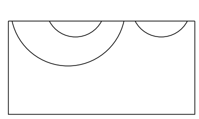


The easier loop models are those with only one vertex and the question one may ask is, being given a Temperley-Lieb element, to count the number of planar matching of the end points of the Temperley-Lieb element so that there are exactly n loops. The picture below shows the case of 2 loops :



This question was related with random matrices for a long time in the physics literature, see e.g. [EB99, EK95, KS92]. For a Temperley-Lieb element B, we denote $p \overset{B}{\sim} \ell$ if a string joins the pth boundary point with the ℓth boundary point in B, then we associate to B with k strings the polynomial

$$q_B(X) = \sum_{\substack{i_j = i_p \text{ if } j \overset{B}{\sim} p \\ 1 \le i_j \le n}} X_{i_1} \cdots X_{i_{2k}}.$$

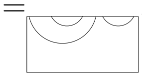
For instance, if B is given by  we have

$$q_B(X) = \sum_{i,j,k=1}^{n} X_i X_j X_j X_i X_k X_k \,.$$

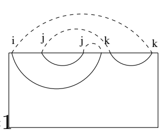
Theorem 3.1 *If ν^M denotes the law of n independent GUE matrices,*

$$\lim_{M\to\infty} \int \frac{1}{M} \mathrm{Tr}\,(q_B(X))\,\nu^M(dX) = \sum n^{\sharp\text{loops}}$$

where we sum over all planar maps that can be built on B.

Proof By Voiculescu's theorem, if $B =$ ,

$$\begin{aligned} &\lim_{M\to\infty} \int \frac{1}{M} \mathrm{Tr}\,(q_B(X))\,\nu^M(dX) \\ = &\sum_{i,j,k=1}^{n} \lim_{M\to\infty} \int \frac{1}{M} \mathrm{Tr}\,(X_i X_j X_j X_i X_k X_k)\,\nu^M(dX) \end{aligned}$$

$$= \sum \sum_{i,j,k=1}^{n}$$ 

$$= \sum n^{\sharp\text{loops}}$$

because the indices have to be constant along loops.

The problem with the previous theorem is that moments of random matrices can only be used so far as generating function for the enumeration of loop configurations taken at integer values of the fugacity. This is enough to characterize polynomials but not the series we shall consider later.

In [J98], V. Jones proposed a construction of a planar algebra associated with a bipartite graph. It was used in [GJS10] to overcome this point. The idea is to take random matrices which are indexed by the edges of a bipartite graph instead of the integer number and to modify the polynomial q_B in such a way that the fugacity is the Perron-Frobenius eigenvalue of the adjacency matrix of the graph.

To be more precise, let $\Gamma = (V = V_+ \cup V_-, E)$ be a bipartite graph with oriented edges so that if $e \in E$, its opposite e^o is also in E. Assume that the adjacency matrix of Γ has Perron-Frobenius eigenvalue. Note that this restrict the possible values of δ to $\{2\cos(\frac{\pi}{n}), n \geq 3\} \cup [2, +\infty[$ which is however a set which contains limit points.

Now, let us define for a Temperley-Lieb element B the polynomial

$$q_B^v(X) = \sum_{e_j = e_p^o \text{ if } j \overset{B}{\sim} p} \sigma_B(w) X_{e_1} \cdots X_{e_{2k}}$$

where we recall that $p \overset{B}{\sim} j$ if a string joins the pth boundary point with the jth boundary point in the TL element B. The sum runs over loops $w = e_1 \cdots e_{2k}$ in Γ which starts at $v \in V$. $v \in V_+$ iff $*$ is in a white region. σ_B is defined as follows. Denote $(\mu_v)_{v \in V}$ with $\mu_v \geq 0$ the eigenvector of Γ for the Perron-Frobenius eigenvalue δ and set, if $\sigma(e) := \sqrt{\frac{\mu_{t(e)}}{\mu_{s(e)}}}$, $e = (s(e), t(e))$,

$$\sigma_B(e_1 \cdots e_{2p}) = \prod_{\substack{i \overset{B}{\sim} j \\ i < j}} \sigma(e_i)$$

to be the sum over products of $\sigma(e)$ so that each string of B brings $\sigma(e)$ with e the edge which labels the start of the string.

For $e \in E$, $e = (s(e), t(e))$, let X_e^M be independent (except $X_{e^o} = X_e^*$) $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with i.i.d centered Gaussian entries with variance $1/(M\sqrt{\mu_{s(e)}\mu_{t(e)}})$.

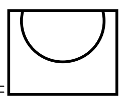
Theorem 3.2 (G-Jones-Shlyakhtenko [GJS10]) *Let Γ be a bipartite graph whose adjacency matrix has δ as Perron-Frobenius eigenvalue. Let B be Temperley-Lieb element so that $*$ is in an unshaded region.*

Then, for all $v \in V^+$

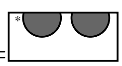
$$\tau_\delta(B) := \lim_{M\to\infty} E[\frac{1}{M\mu_v}\mathrm{Tr}(q_B^v(X^M))] = \sum \delta^{\sharp\mathrm{loops}}$$

where the sum runs above all planar maps built on B*.*

Maybe the best proof is by trying examples.

If $B =$ , for all $v \in V^+$

$$\begin{aligned}\mathbb{E}[\frac{1}{M\mu_v}\mathrm{Tr}(\sum_{e:s(e)=v}\sigma(e)X_eX_{e^0}))] &= \frac{1}{M\mu_v}\sum_{e:s(e)=v}\sqrt{\frac{\mu_{t(e)}}{\mu_v}}\frac{M\mu_v M\mu_{t(e)}}{M\sqrt{\mu_{t(e)}\mu_{s(e)}}} \\ &= \frac{1}{\mu_v}\sum_{e:s(e)=v}\mu_{t(e)} = \delta\end{aligned}$$

If $B =$ , for all $v \in V^+$

$$\lim_{M\to\infty}\mathbb{E}[\frac{1}{M\mu_v}\mathrm{Tr}(\sum_{\substack{e:s(e)=v\\ s(f)=v}}\sigma(e)\sigma(f)X_eX_{e^0}X_fX_{f^0})]$$

$$= \delta^2 + \frac{1}{\mu_v}\sum_{e=f}\frac{\mu_{t(e)}}{\mu_v}\frac{\mu_v^2\mu_{t(e)}}{\mu_{t(e)}\mu_v} = \delta^2 + \delta$$

More generally, the edges are constant along the loops and brings the contribution $\mu_{t(e)}/\mu_v$ hence leading after summation to δ.

As in the previous section we can make these enumeration questions more interesting by adding a potential, and in turn enumerating loop models with several Temperley-Lieb vertices. Let B_i be Temperley Lieb elements with $*$ with color $\sigma_i \in \{+,-\}$, $1 \le i \le p$. Let Γ be a bipartite graph whose adjacency matrix has eigenvalue δ as before. Let ν^M be the law of the previous independent rectangular Gaussian matrices and set

$$d\nu^M_{(B_i)_i}(X_e) = \frac{1_{\|X_e\|_\infty\le L}}{Z_B^M}e^{M\mathrm{Tr}(\sum_{i=1}^p\beta_i\sum_{v\in V_{\sigma_i}}\mu_v q^v_{B_i}(X))}d\nu^M(X_e).$$

Theorem 3.3 (G-Jones-Shlyakhtenko-Zinn Justin [GJSZ12]) *For any $L > 2$, for β_i small enough real numbers, for any Temperley-Lieb element B with color σ, any $v \in V_\sigma$,*

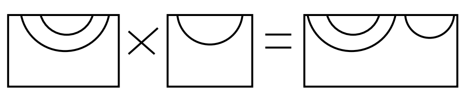
$$\tau_{\delta,\beta}(B) := \lim_{M\to\infty} \int \frac{1}{M\mu_v} \mathrm{Tr}(q_B^v(X)) d\nu^M_{(B_i)_i}(X) = \sum_{n_i \geq 0} \sum \delta^{\sharp \mathrm{loops}} \prod_{i=1}^p \frac{\beta_i^{n_i}}{n_i!}$$

where we sum over the planar maps build on n_i TL elements B_i and one B.

The proof is based, as in the previous section, on Schwinger-Dyson's equation and concentration of measure.

3.3 Loop models and subfactors

Another point of view on the previous section is subfactor theory. In fact, Temperley-Lieb algebra can be viewed as a special case of planar algebras and $\tau_{\delta,\beta}$ are tracial states on this planar algebra if they are equipped with the multiplication



and the involution which is given by taking the symmetric picture of the element.

Theorem 3.4 (G-Jones-Shlyakhtenko [GJS10]) .

Take $\delta \in \{2\cos(\pi/n), n \geq 3\} \cup [2, +\infty[$. Then

- *$\tau_{\delta,0}$ is a tracial state on the Temperley Lieb algebra.*
- *The von Neumann algebra associated by the GNS construction* (5) *is a factor, namely its center is trivial. A tower of sub factors with index δ^2 can be built.*

The tower is build by changing the multiplication so that the nearest boundary points of both Temperley-Lieb elements are capped. The construction presented here can be generalized to any planar algebra. Hence, it shows that there is a canonical way to construct a tower of sub factors from any subfactor planar algebra. It is still unknown whether the von Neumann algebras associated to $\tau_{\delta,\beta}$ are factors for $\beta \neq 0$.

3.4 Matrix model for the Potts model

Let $\delta \in \{2\cos(\frac{\pi}{n})\}_{n\geq 3} \cup [2,\infty[$ and $\Gamma = (V_+ \cup V_-, E)$ be a bipartite graph with eigenvalue δ and Perron-Frobenius eigenvector μ. Assume that Γ is finite, as otherwise the construction requires to define the joint law of an infinite number of random matrices, which need an additional Gibbs measure type approach. This includes $\delta = 2\cos(\pi/n), n \geq 3$, and therefore a set with an accumulation point. We set

$$\nu^M_{\beta_\pm}(dX_e) = \frac{1_{\|X_e\|_\infty \leq L}}{Z^M_{\beta_\pm}} e^{M\mathrm{Tr}\left(\sum_{v\in V}\mu_v \sum_{\sigma=\pm}\beta_\sigma 1_{v\in V_\sigma}\left(\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)^2\right)}$$

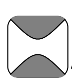
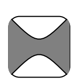
$$\prod_e e^{-\frac{M}{2}(\mu_{s(e)}\mu_{t(e)})^{\frac{1}{2}}\mathrm{Tr}(X_eX_e^*)} dX_e dX_e^*$$

Theorem 3.5 (G-Jones-Shlyakhtenko-Zinn Justin [GJSZ12]) *Then, for L large enough, $\beta_\pm$ small enough, for all TL B, $v \in V_{\sigma_B}$,*

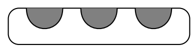
$$\lim_{M\to\infty} \frac{1}{M\mu_v} \int \mathrm{Tr}(q^v_B(X))\nu^M_{\beta_\pm}(dX)$$

$$= \sum_{n_+,n_-\geq 0} \sum \prod_{i=\pm} \frac{\beta_i^{n_i}}{n_i!}\delta^{\sharp\mathrm{loops}} =: \mathrm{Tr}_{\beta_\pm,\delta}(B)$$

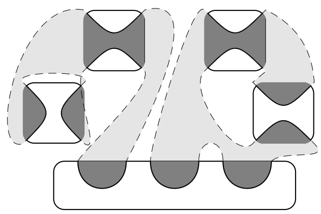
where we sum over all planar maps build by matching the end points of n_- (resp. n_+) vertices of type

 *resp.* 

and one of type B. If B is given by

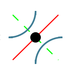
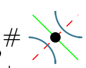


we count the number of matchings of the following type :



3.5 Solving the Potts matrix model

By construction

$$\sum_{\Gamma} \delta^{\#\text{ loops}} \beta_-^{\#}$$  $$\beta_+^{\#}$$ 

equals

$$\lim_{M\to\infty} \frac{1}{M^2 \sum \mu(v)^2} \log Z^M_{\beta_\pm}$$

with for L large enough $Z^M_{\beta_\pm}$ equals to

$$\int_{\|X_e\|\le L} e^{M\mathrm{Tr}\left(\sum_{v\in V} \mu_v(\beta_+ 1_{v\in V_+} + \beta_- 1_{v\in V_-})\left(\sum_{e:s(e)=v} \sqrt{\mu_v \mu_{t(e)}} X_e X_e^*\right)^2\right)} \nu^M(dX)$$

if ν^M is the law of X_e, $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with iid centered Gaussian entries with covariance $(M^2\mu_{s(e)}\mu_{t(e)})^{-\frac{1}{2}}$

We can compute the matrix model and therefore solve the original combinatorial problem by using Hubbard-Stratonovich transformation. Namely, let G_v $[M\mu_v] \times [M\mu_v]$ independent matrices from the GUE and X_e be $[M\mu_{s(e)}] \times [M\mu_{t(e)}]$ matrices with covariance $(M^2\mu_{s(e)}\mu_{t(e)})^{-\frac{1}{2}}$ under ν^M, with $\alpha_\pm = \sqrt{2\beta_\pm}$, $\Delta(\lambda) = \prod_{i\neq j}(\lambda_i - \lambda_j)$,

$$\begin{aligned} Z^M_{\beta_\pm} &= \int_{\|X_e\|\le L} e^{M\mathrm{Tr}\left(\sum_{v\in V}\mu_v(\beta_+ 1_{v\in V_+}+\beta_- 1_{v\in V_-})\left(\sum_{e:s(e)=v}\sigma(e)X_eX_e^*\right)^2\right)}\nu^M(dX) \\ &= \int_{\|X_e\|\le L} e^{M\mathrm{Tr}\left(\sum_{\sigma=\pm}\sum_{v\in V_\sigma}\alpha_\sigma G_v\left(\sum_{e:s(e)=v}\sqrt{\mu_v\mu_{t(e)}}X_eX_e^*\right)\right)}\nu^M(dX,dG) \\ &\approx \int_{\|G_v\|\le L'} \prod_{e\in E_+} e^{-\mathrm{Tr}\otimes\mathrm{Tr}(\log(I+\alpha_+ I\otimes G_{s(e)}+\alpha_- G_{t(e)}\otimes I))}\nu^M(dG) \\ &= \int_{|\lambda_i^e|\le L'} \prod_{e\in E_+} e^{-\sum_{i=1}^{[M\mu_{s(e)}]}\sum_{j=1}^{[M\mu_{t(e)}]}\log(1+\alpha_+\lambda_i^e+\alpha_-\eta_j^e)} \\ &\qquad \Delta(\eta^e)\Delta(\lambda^e)e^{-\frac{[M\mu_{s(e)}]}{2}\sum_{i=1}^{[M\mu_{s(e)}]}(\lambda_i^e)^2-\frac{[M\mu_{t(e)}]}{2}\sum_{j=1}^{[M\mu_{t(e)}]}(\eta_j^e)^2}d\lambda^e d\eta^e. \end{aligned}$$

where in the second line we used Hubbard-Stratonovich transformation and in the third took the expectation over the X_e (note here that the bound on the $\|X_e\|$ insured that the log-density of the joint law in X, G was strictly concave; it thus keep the G_v bounded with overwhelming probability by Brascamp-Lieb inequality which is the reason why the bound on $\|X_e\|$ transferred into a bound on $\|G\|_v$).

Hence, Hubbard-Stratonovich led us to define an auxiliary measure $P^{M,L}_\alpha$, absolutely continuous with respect to Lebesgue measure and with density

$$\frac{1_{|\lambda^v|\le L}}{Z^{M,L}_\alpha}\prod_{e\in E_+}\prod_{\substack{1\le i\le[M\mu_{s(e)}]\\1\le j\le[M\mu_{t(e)}]}}\frac{1}{1+\alpha_+\lambda_i^{s(e)}+\alpha_-\lambda_j^{t(e)}}\prod_{v\in V}\Delta(\lambda^v)e^{-\frac{M\mu_v}{2}\sum(\lambda_i^v)^2}.$$

By large deviation analysis, see [BaG97, AGZ10], the asymptotics of this model can easily be studied and one finds that

$$P^{M,L}_\alpha\left(d(\frac{1}{M\mu_v}\sum\delta_{\lambda_i^v},\nu_v)<\epsilon\,\forall\, v\right)\approx e^{-M^2[I(\nu_v,v\in V)-\inf I]}$$

with if $\Sigma(\nu)=\int\log|x-y|d\nu(x)d\nu(y)$

$$I(\nu_v,v\in V)=\sum_v\frac{\mu_v^2}{2}(\int x^2d\nu_v(x)-2\Sigma(\nu_v))$$

$$-\sum_{e\in E_+}\mu_v\mu_{t(e)}\int\log|1+\alpha_+x+\alpha_-y|d\nu_v(x)d\nu_{t(e)}(y)\,.$$

Moreover $\lim_{M\to\infty}M^{-2}\log Z_\alpha^{M,L}=-\inf I$

Theorem 3.6 ([GJSZ12]) • *I achieves its minimal value at a unique set of probability measures $\nu_v, v\in V$.*

• *$\exists\,\nu_+,\nu_-\in P(\mathbb{R})$, so that $\nu_v=\nu_\pm$ if $v\in V_\pm$. (ν_-,ν_+) are the unique minimizers of*

$$I_{\delta,\alpha_+,\alpha_-}(\nu_+,\nu_-)=\sum_{\epsilon=\pm}\left(\frac{1}{2}\int x^2d\nu_\epsilon(x)-\int\log|x-y|d\nu_\epsilon(x)d\nu_\epsilon(y)\right)$$

$$+\delta\int\log|1+\alpha_+x+\alpha_-y|d\nu_+(x)d\nu_-(y).$$

• *For all $L>2$ not too large,*

$$\lim_{M\to\infty}E[\frac{1}{M\mu_v}\sum_{i=1}^{[M\mu_v]}(\lambda_i^v)^p]=\int x^pd\nu_v(x)\qquad\forall\quad p\in\mathbb{N},\quad v\in V.$$

We next relate the asymptotic measures ν_+,ν_- with the original combinatorial problem we wished to solve, that is the enumeration for the loop model. Let $M(z)=\int\sum_{n\geq 0}z^nx^nd\nu_+(x)$ On the other hnd, consider the generating function we are interested in, namely put

$$\gamma(z)=\alpha_+z/(1-z^2M(z))$$

and

$$C(z,\alpha_+,\alpha_-)=\sum_{n\geq 0}z^n\sum\delta^\ell\frac{\alpha_+^{n_+}}{n_+!}\frac{\alpha_-^{n_-}}{n_-!}$$

where we sum over the planar maps build over n_+ (resp. n_-) vertices with two strings and two black (resp. white) regions and one vertex with n strings with black inside so that there are exactly ℓ loops. Then, for small z, we have

$$C(z,\alpha_+,\alpha_-) = \frac{\alpha_+}{z}[1 - \frac{\alpha_+\gamma^{-1}(z)}{z}].$$

Hence, we see that the original combinatorial question encapsulated in $C(z,\alpha_+,\alpha_-)$ can be reduced to a variational problem, namely minimizing $I_{\delta,\alpha_+,\alpha_-}$. It turns out that this minimizing problem can be solved explicitly as follows. Let p_+ (resp. p_-) be the law of $1+\alpha_+ x$ and $-\alpha_- y$ under ν_+ (resp. ν_-).

• For $\alpha_\pm$ small enough, $p_\pm$ has a connected support $[a_\pm, b_\pm]$ around 1 (resp. 0) and $a_- < b_- < a_+ < b_+$.

• Set $G_\pm(z) = \int (z-x)^{-1} dp_\pm(x)$. Then

$$G_\pm(z+i0) + G_\pm(z-i0) = P_\pm(z) + \delta G_\mp(z) \qquad z \in [a_\pm, b_\pm]$$

with $P_-(z) = z/\alpha_-$, $P_+(z) = (1-z)/\alpha_+$.

Introduce

$$u(z) = \int_{b_-}^{z} \frac{1}{\sqrt{(v-a_+)(v-a_-)(v-b_+)(v-b_-)}} dv,$$

with inverse $z(u)$. With $\delta = q + q^{-1}$ set

$$\omega_\pm(u) = q^{\pm 1} G_+(z(u)) - G_-(z(u)) \pm \frac{1}{q-q^{-1}}(P_+ + q^{\pm 1} P_-) z(u).$$

Then, we have

$$\omega_\pm(u+2K) = \omega_\pm(u) \qquad \omega_\pm(u+2iK') = q^{\pm 2}\omega_\pm(u)$$

$\omega_\pm$ are meromorphic with only poles at $\pm u_\infty$. If we set

$$\Theta(u) = 2\sum_{k=0}^{\infty} e^{i\pi\frac{\omega_2}{\omega_1}(k+1/2)^2} \sin(2k+1)\frac{\pi u}{\omega_1}$$

Then we have

$$\omega_+(u) = c_+ \frac{\Theta(u-u_\infty-\nu\omega_1)}{\Theta(u-u_\infty)} + c_- \frac{\Theta(u+u_\infty-\nu\omega_1)}{\Theta(u+u_\infty)}$$

All the parameters can be computed as solutions of fixed point equations.

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