Random polygons, Grassmannians, and a problem of Lewis Carroll.

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joint work with Clayton Shonkwiler, Tetsuo Deguchi,
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University of Georgia

MAA Invited Address
Joint Math Meetings (Atlanta)
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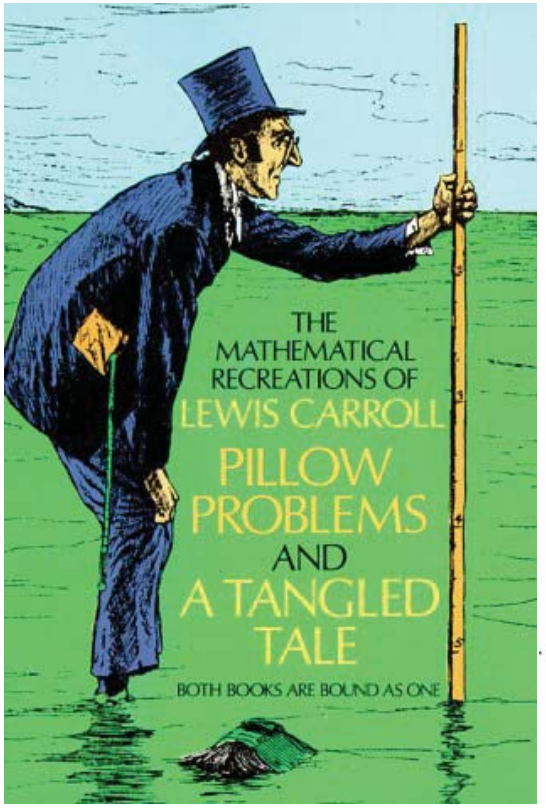
The Pillow Problem

In 1895, Charles Dodgson (better known by his pen name Lewis Carroll) published a book of 72 mathematical problems designed to be solved “while lying in bed”



57. (25, 80)

In a given Triangle describe three Squares, whose bases shall lie along the sides of the Triangle, and whose upper edges shall form a Triangle;

(1) geometrically; (2) trigonometrically. [27/1/91

58. (25, 83)

Three Points are taken at random on an infinite Plane. Find the chance of their being the vertices of an obtuse-angled Triangle. [20/1/84

Carroll's problem is ill-posed

Question

What does it mean to choose a random triangle?

> *The issue of choosing a "random triangle" is indeed problematic. I believe the difficulty is explained in large measure by the fact that there seems to be no natural group of transitive transformations acting on the set of triangles.*
>
> *–Stephen Portnoy, 1994*
> *(Editor, J. American Statistical Association)*

There have been many approaches which solve the problem of defining a random triangle in different ways [Guy, Kendall, Portnoy, Edelman/Strang, ...].

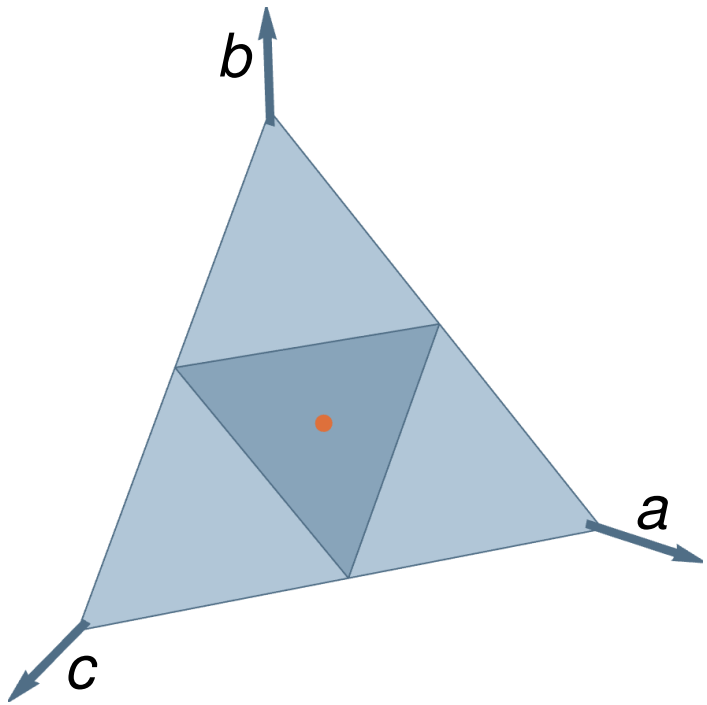
Choosing a random triangle

Let a, b, and c be the sidelengths of the triangle. The space of triangles is parametrized by choices of a, b and c satisfying a collection of conditions:

$$a \geq 0, b \geq 0, c \geq 0, \quad \text{and} \quad a + b + c = 2$$

and the triangle inequalities

$$b + c \geq a$$
$$a + c \geq b$$
$$a + b \geq c$$



Choosing a random triangle (2)

We can rewrite the triangle inequalities as

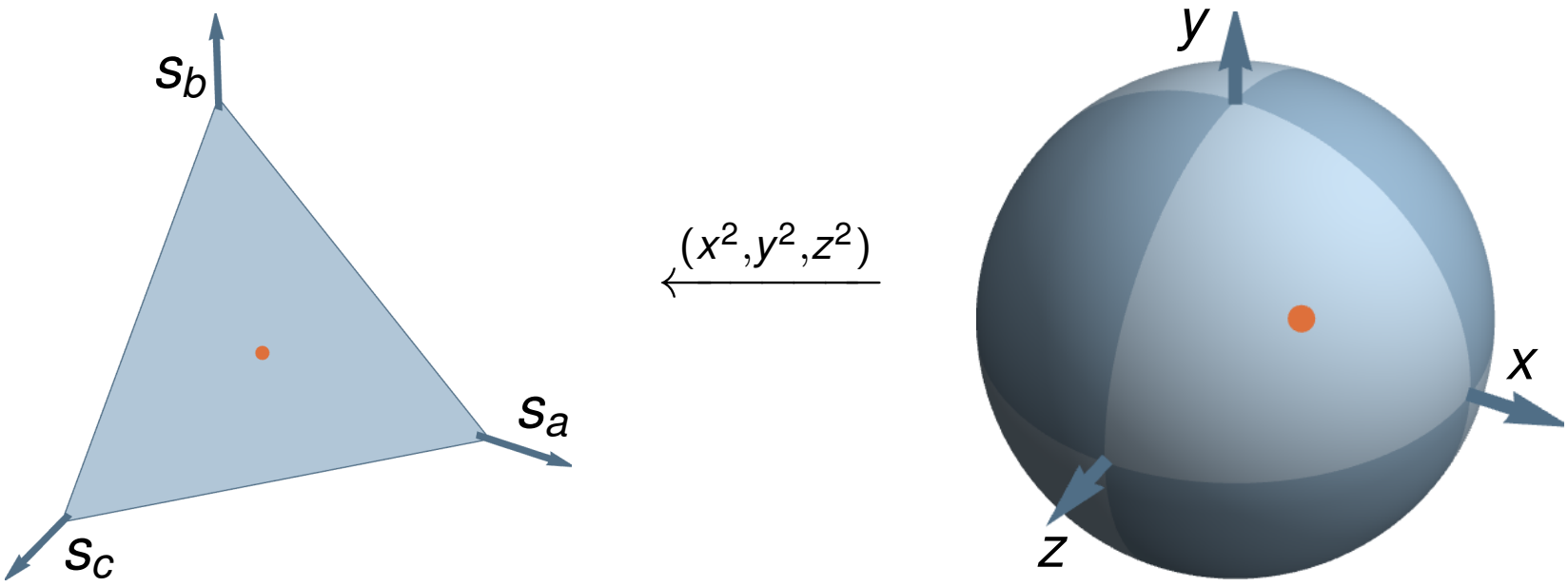
$$\begin{aligned} -a+b+c &\geq 0 \\ a-b+c &\geq 0 \\ a+b-c &\geq 0 \end{aligned}$$

If $s = \frac{a+b+c}{2}$ (the semiperimeter) this suggests new variables:

$$\begin{aligned} s_a := s-a &= \frac{-a+b+c}{2} \geq 0 \\ s_b := s-b &= \frac{a-b+c}{2} \geq 0 \\ s_c := s-c &= \frac{a+b-c}{2} \geq 0 \end{aligned}$$

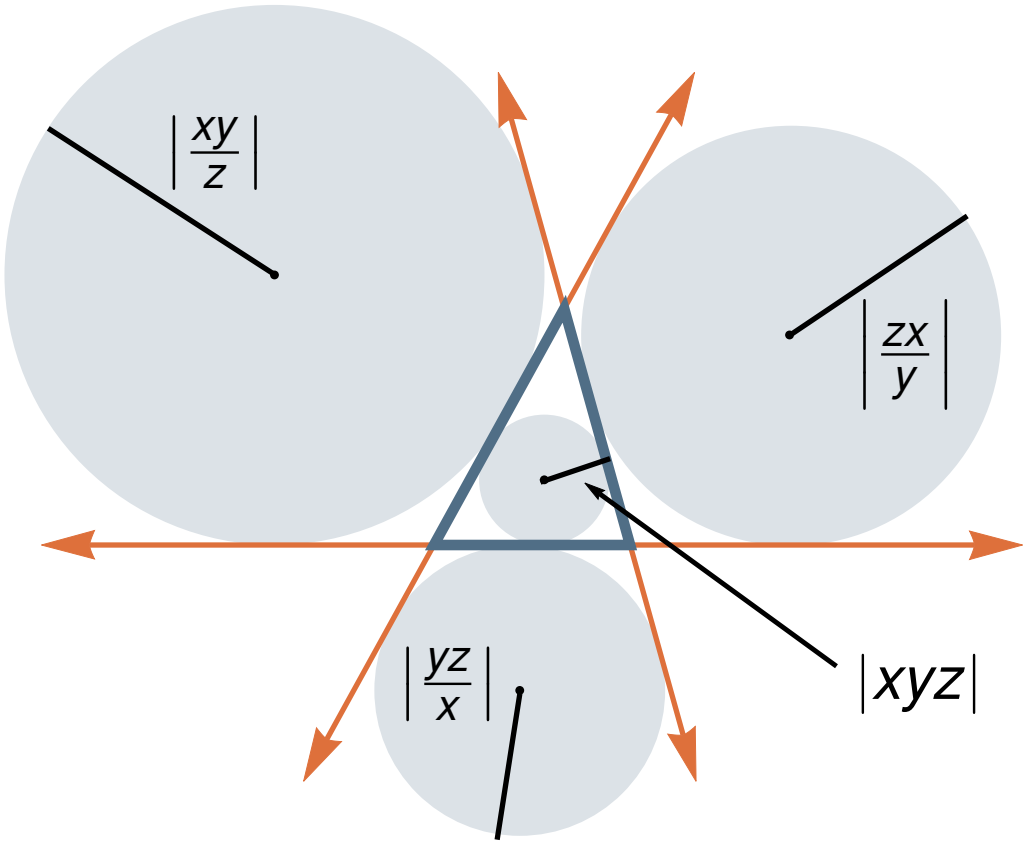
Note that $s_a + s_b + s_c = s = 1$.

This triangle of triangles is covered 8-fold by the sphere.



$$x^2 = s_a = 1 - a, \quad y^2 = s_b = 1 - b, \quad z^2 = s_c = 1 - c$$

We will use x, y, and z as coordinates on triangle space. The measure will be surface area on the sphere.

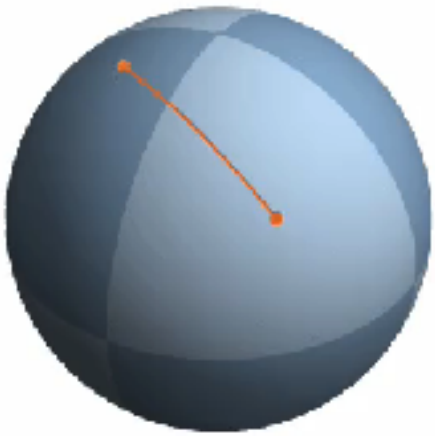


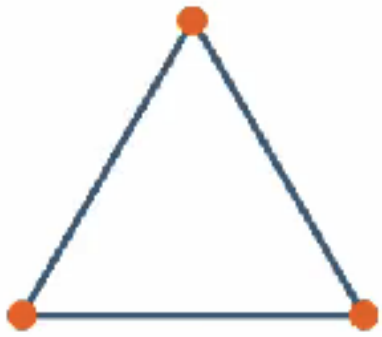
Proposition

$|x|$, $|y|$, and $|z|$ are (pairwise) geometric means of the exradii.

Transitive transformations on the sphere

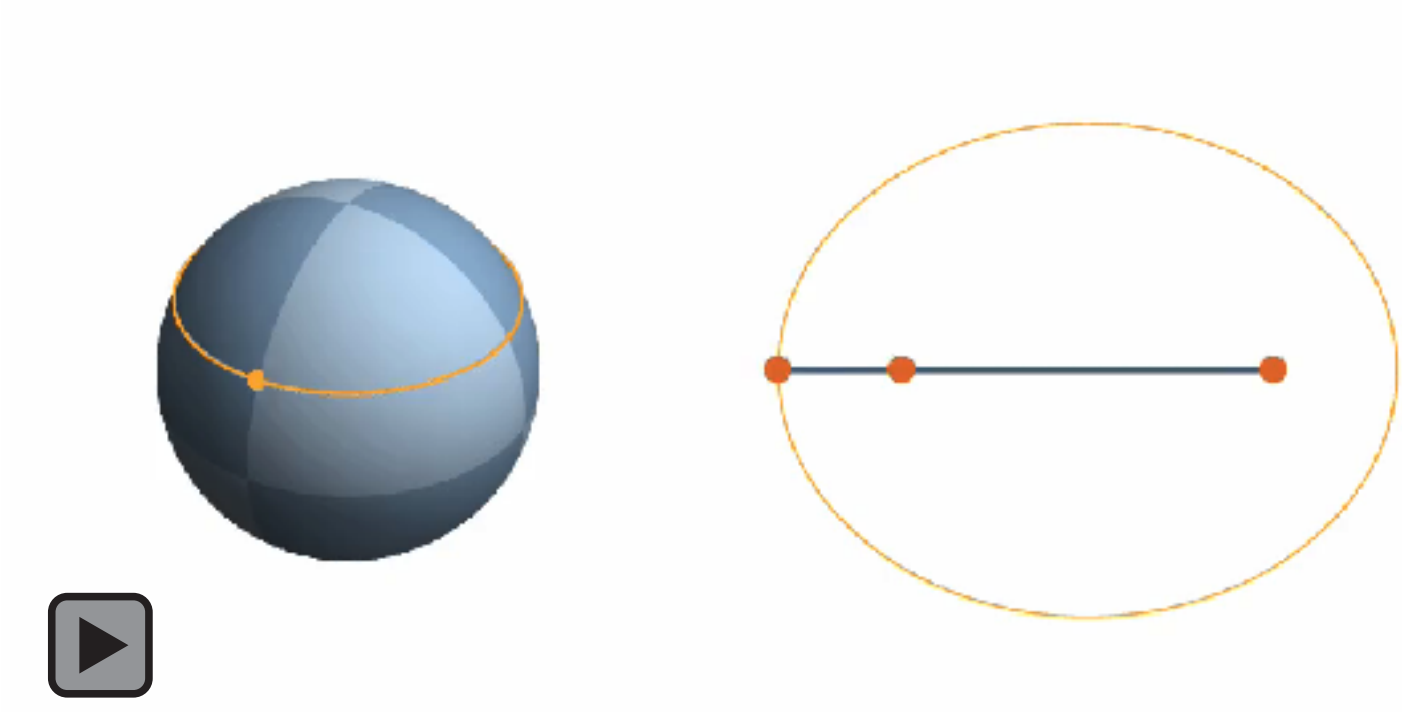
Rotations of S^2 can exchange any two triangles.





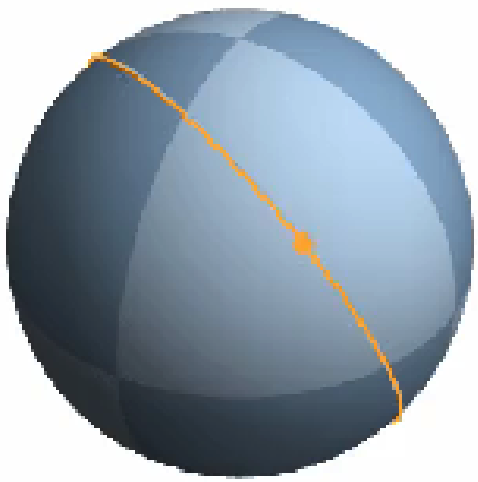
Rotation around an axis

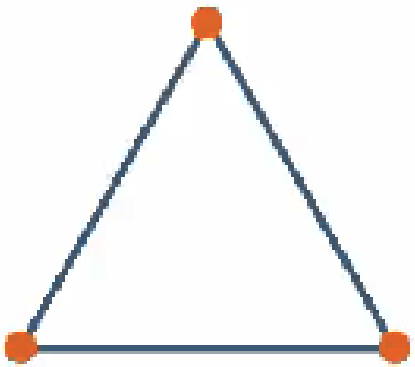
Rotations around an axis fix one edge and move the opposing vertex around an ellipse.



Off-axis rotations

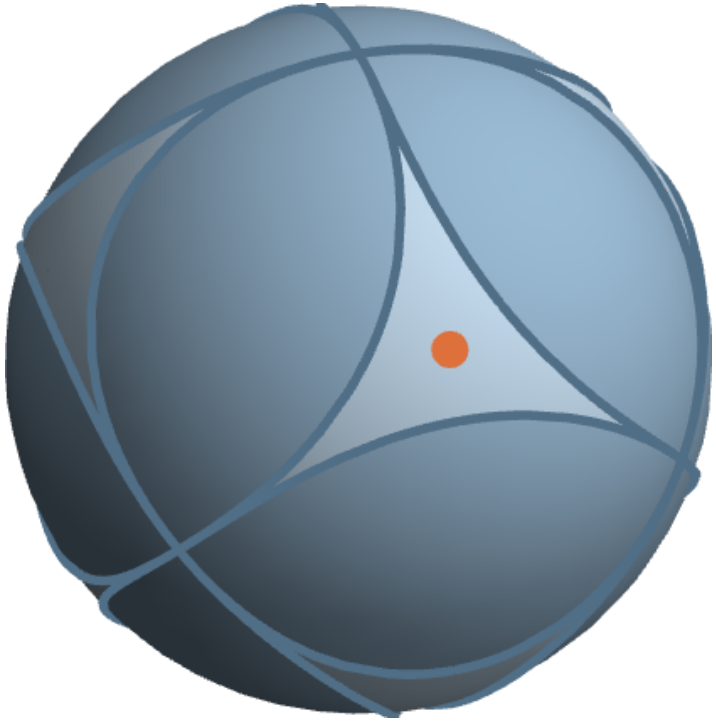
Open Question: is there an elegant triangle-theoretic description of this motion?





The Pythagorean theorem implies that right triangles have

$$x^2y^2 = z^2, \quad y^2z^2 = x^2, \quad \text{or} \quad z^2x^2 = y^2.$$



Theorem (with Needham, Shonkwiler, Stewart)

The fraction of obtuse triangles is $\frac{3}{2} - \frac{3\ln 2}{\pi} \approx 0.838093$

Generalizing to n-gons: start over

Suppose the edges are complex numbers $e_1, \ldots, e_n \in \mathbb{C}$.
The polygon is closed, so

$$e_1 + \cdots + e_n = 0$$

Let $e_i = z_i^2$ and $z_i = u_i + \mathbf{i}v_i$.

$$\begin{aligned} 0 = e_1 + \cdots + e_n &= z_1^2 + \cdots + z_n^2 \\ &= (u_1 + \mathbf{i}v_1)^2 + \cdots + (u_n + \mathbf{i}v_n)^2 \\ &= (u_1^2 - v_1^2) + \mathbf{i}(2u_1v_1) + \cdots + (u_n^2 - v_n^2) + \mathbf{i}(2u_nv_n) \\ &= (u_1^2 + \cdots + u_n^2 - v_1^2 - \cdots - v_n^2) + 2\mathbf{i}(u_1v_1 + \cdots + u_nv_n). \end{aligned}$$

or if $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$

$$\left|\vec{u}\right|^2 = \left|\vec{v}\right|^2 \quad \text{and} \quad \left\langle \vec{u}, \vec{v} \right\rangle = 0$$

Generalizing to n-gons: start over (2)

If we fix the total polygon length to be 2, we have

$$\begin{aligned} 2 &= |e_1| + \cdots + |e_n| = |z_1|^2 + \cdots + |z_n|^2 \\ &= u_1^2 + v_1^2 + \cdots + u_n^2 + v_n^2 \\ &= \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 . \end{aligned}$$

This gives us:

Theorem (Knutson/Hausmann 1997)

If the edges of an n-gon are $e_i = z_i^2$ and each $z_i = u_i + \mathbf{i}v_i$, then the polygon is closed and length 2 $\iff$ $\vec{u}$ and $\vec{v}$ are orthonormal.

Generalizing to n-gons: start over (2)

If we fix the total polygon length to be 2, we have

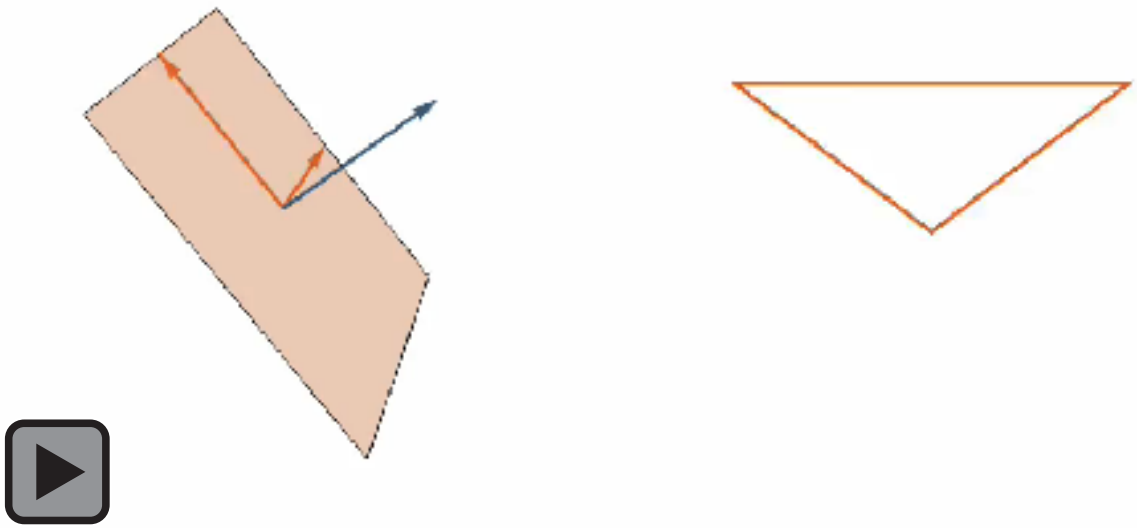
$$\begin{aligned} 2 &= |e_1| + \cdots + |e_n| = |z_1|^2 + \cdots + |z_n|^2 \\ &= u_1^2 + v_1^2 + \cdots + u_n^2 + v_n^2 \\ &= \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 . \end{aligned}$$

This gives us:

Theorem (Knutson/Hausmann 1997)

The space of closed, length 2 plane polygons is 2^n-fold covered by the "Stiefel manifold" $V_2(\mathbb{R}^n)$ of orthonormal 2-frames in $\mathbb{R}^n$.

Rotations in the plane of $\vec{u}$ and $\vec{v}$ rotate the z_i and rotate the edges $e_i = z_i^2$ twice as fast.



Theorem (Knutson/Hausmann 1997)

The plane spanned by $\vec{u}$ and $\vec{v}$ determines the polygon up to rotation.

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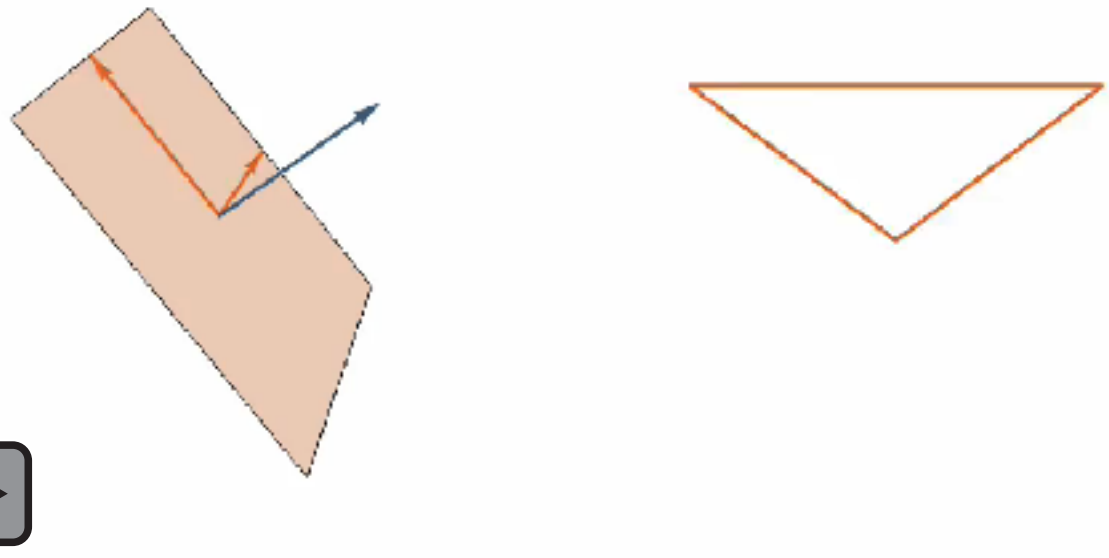


Theorem (Knutson/Hausmann 1997)

The space of closed, length-2 plane polygons up to translation and rotation is covered by the "Grassmann manifold" $G_2(\mathbb{R}^n)$ of 2-planes in $\mathbb{R}^n$.

Bringing the pictures together: $G_1(\mathbb{R}^3) = G_2(\mathbb{R}^3)$

If (x, y, z) is orthonormal to $\vec{u}$ and $\vec{v}$, then

$$\begin{pmatrix} x & u_1 & v_1 \\ y & u_2 & v_2 \\ z & u_3 & v_3 \end{pmatrix} \quad \text{is orthonormal.}$$

So

$$x^2 + u_1^2 + v_1^2 = 1, \quad \text{or} \quad x^2 = 1 - u_1^2 - v_1^2$$

but

$$|z_1|^2 = |e_1| = a$$

so this is our original equation

$$x^2 = 1 - a$$

Generalizing (x, y, z): Plücker Coordinates

Definition

Any 2-plane P in $\mathbb{R}^n$ spanned by $\vec{u}$, $\vec{v}$ is described by a skew-symmetric $n \times n$ matrix of Plücker coordinates

$$\Delta(P)_{ij} = \det \begin{pmatrix} u_i & v_i \\ u_j & v_j \end{pmatrix} = (u_i, v_i) \times (u_j, v_j)$$

defined up to multiplication by a common scalar. (Changing the basis for P only changes the scalar, so the Plücker coordinates depend only on the plane.)

Our coordinates (x, y, z) are the Plücker coordinates in the upper triangle of the 3x3 matrix for $G_2(\mathbb{R}^3)$.

Definition

The Positive Grassmannian is the portion of the Grassmannian where all Plücker coordinates in the upper triangle are positive

$$\Delta P_{ij} > 0 \iff i < j.$$

It has attracted a lot of interest in string theory and has a beautiful and somewhat mysterious structure.

Theorem (with Needham, Shonkwiler, Stewart)

The positive Grassmannian $G_2(\mathbb{R}^n)^+$ *consists of planes* P *where* (a_i, b_i) *lie in a common semicircle and the polygon is convex.* $G_2(\mathbb{R}^n)$ *is tiled by* $2^{n-2} \times (n-1)!$ *isometric copies of* $G_2(\mathbb{R}^n)^+$.

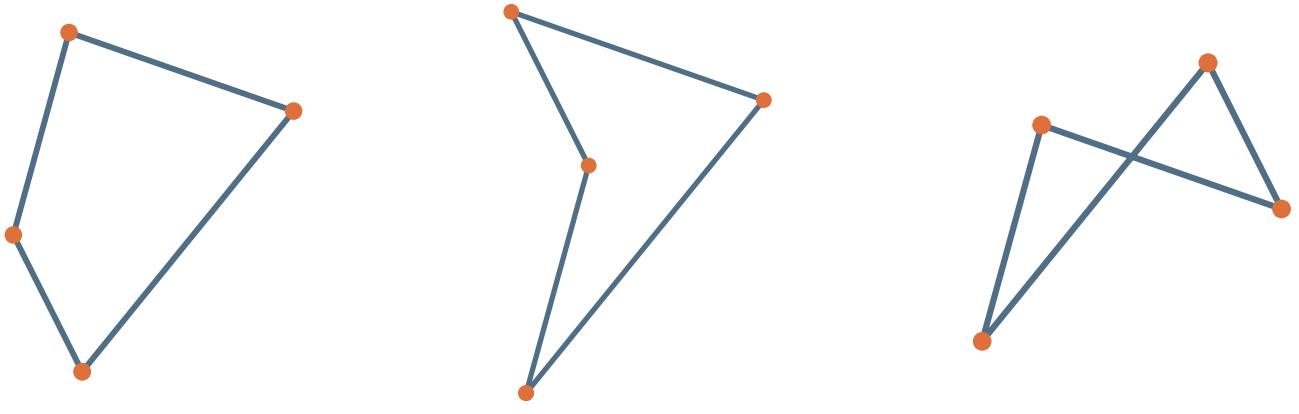
(A comparable interpretation appears in Section 5.3 of Arkani-Hamed, 2012.)

Random polygons

There is a natural way to measure volume in $G_2(\mathbb{R}^n)$ which is $O(n)$ invariant (Haar measure). Using this as a probability measure on polygons:

Theorem (with Needham, Shonkwiler, Stewart)

The probability that a random n-gon is convex is $2/(n-1)!$.

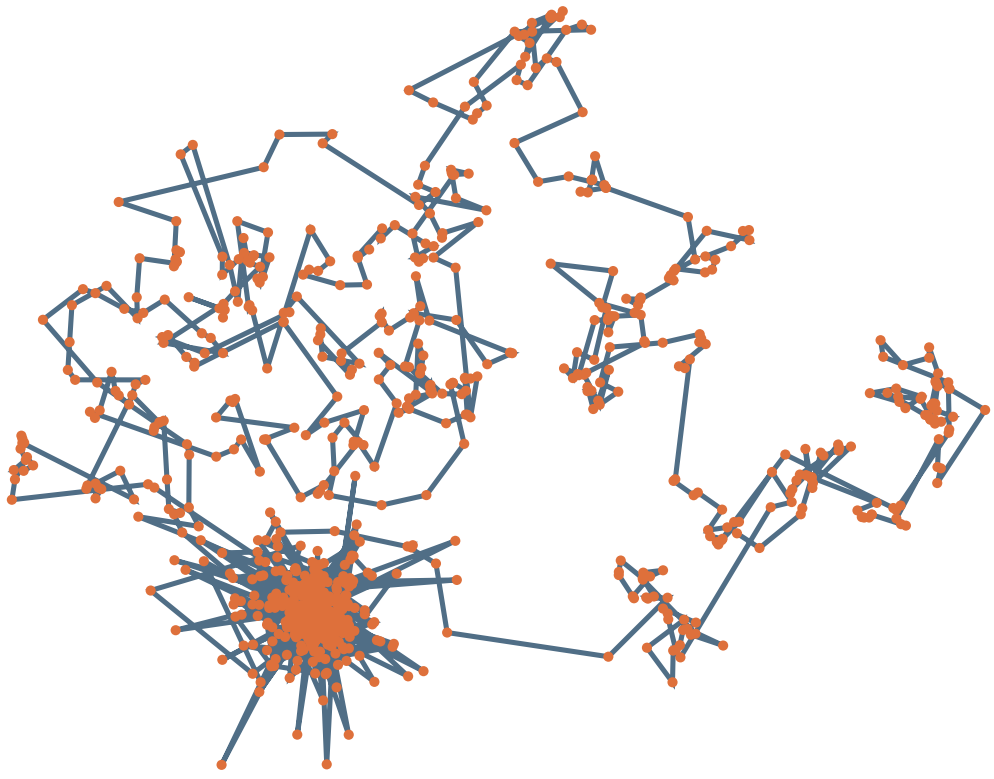


Theorem (with Needham, Shonkwiler, Stewart)

Among random quadrilaterals, 1/3 are convex, 1/3 are reflex, and 1/3 are self-intersecting.

Random *n*-gons

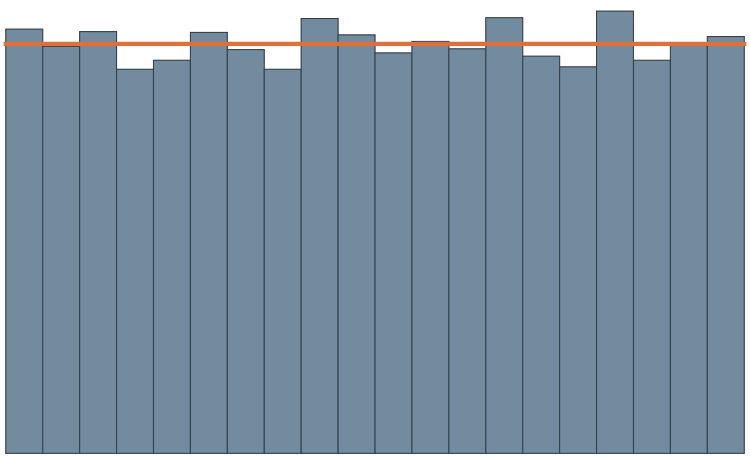
It is easy to sample a random 2-plane in $\mathbb{R}^n$ uniformly: just pick two vectors of n independent Gaussians and take the plane they span. Here's a random 500-gon:



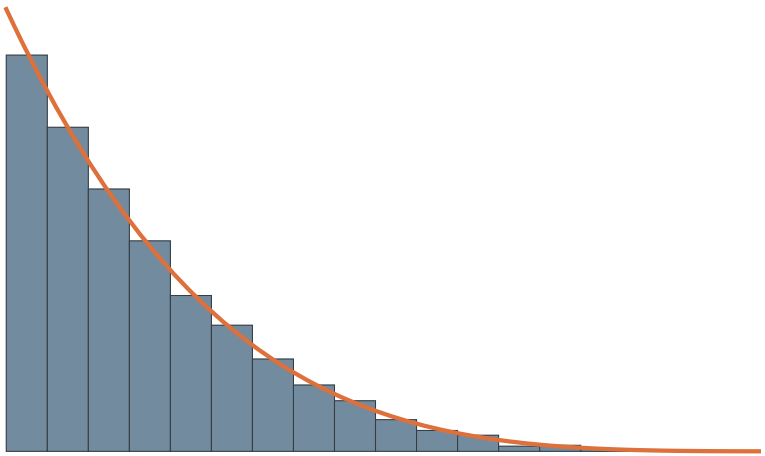
Theorem (with Deguchi, Shonkwiler)

The edgelength of a random quadrilateral is uniformly distributed on $[0, 1]$*. The edgelength of a random n-gon is sampled from a Beta distribution with probability density*

$$\phi(y) = \left(\frac{n}{2} - 1\right)(1 - y)^{\frac{n}{2}-2}$$



Quadrilaterals



12-gons

Definition

The radius of gyration of an n-gon $v_1, \ldots, v_n$ is the average (squared) distance between vertices:

$$\frac{1}{n^2} \sum_{i,j \in 1}^{n} |v_i - v_j|^2$$

Theorem (with Deguchi, Shonkwiler)

The expected radius of gyration of a random planar n-gon is

$$\frac{2}{3} \frac{n+1}{n(n+2)}$$

We can use the geometry of the Grassmannian to easily build geodesic (shortest) paths between (lifts of) polygons.

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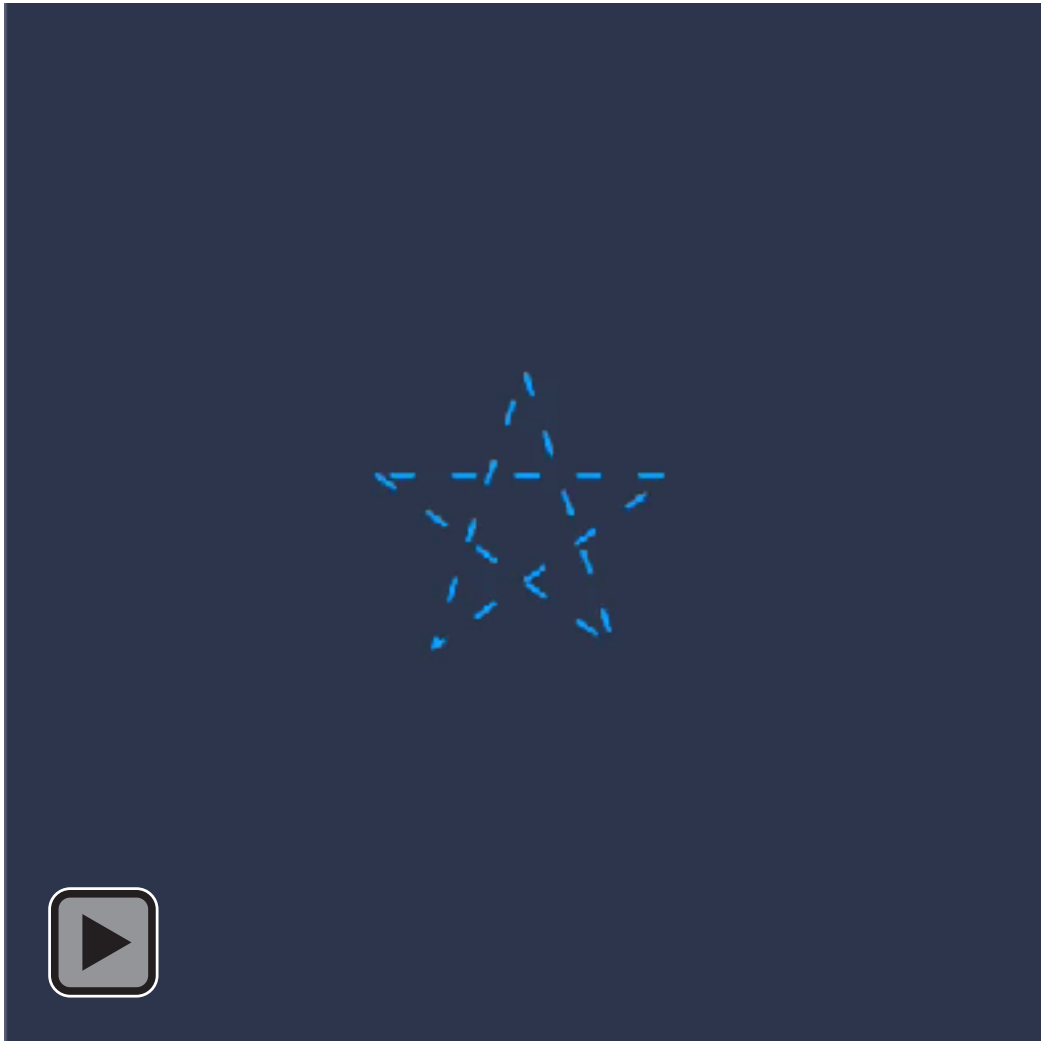

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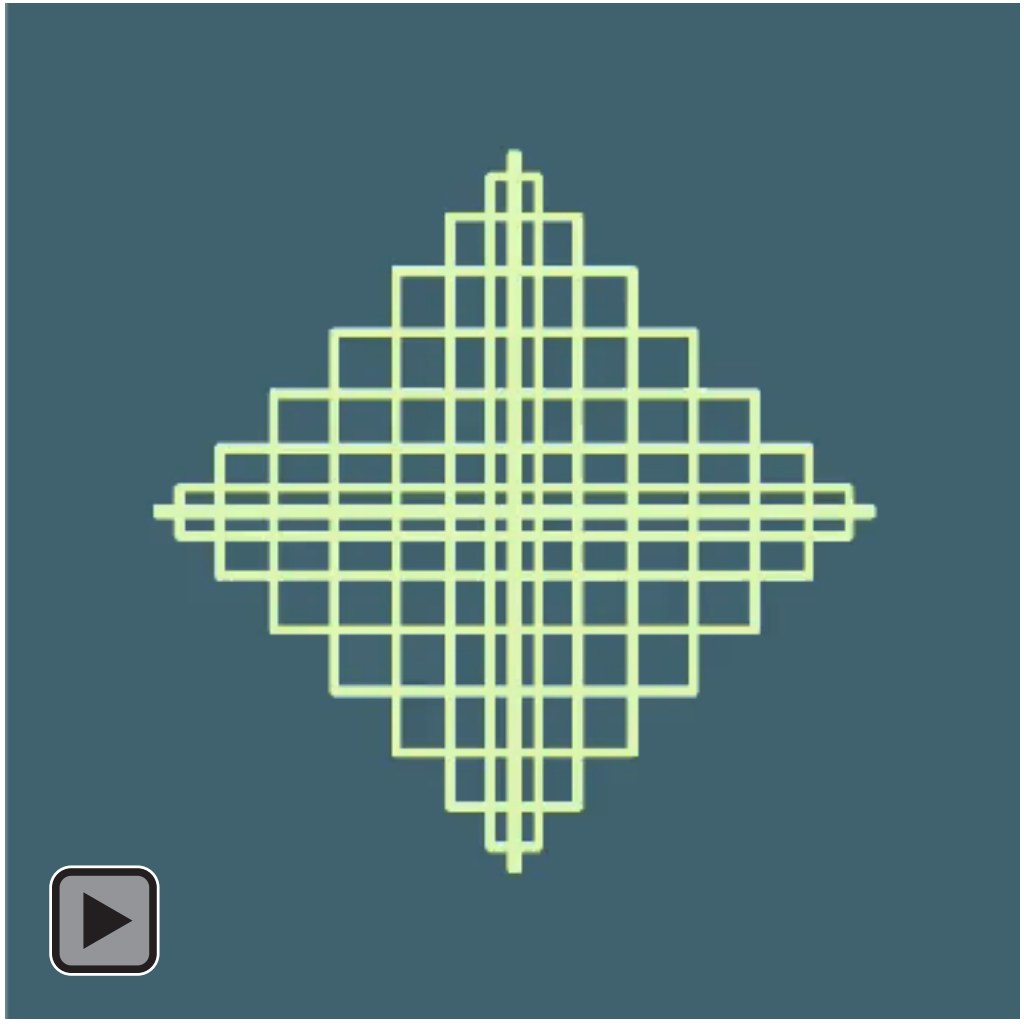

Space Polygons

Space polygons have a similar structure, if we replace $G_2(\mathbb{R}^n)$ with $G_2(\mathbb{C}^n)$, view edges as quaternions rather than complex numbers, and replace squaring with the Hopf map.

All this structure lets you compute some exact probabilities:

Theorem (with Deguchi, Shonkwiler)

The expected radius of gyration of a random n-gon in $\mathbb{R}^3$ *sampled from* $G_2(\mathbb{C}^n)$ *is*

$$\frac{1}{2n}$$

Theorem (with Grosberg, Kusner, Shonkwiler)

The expected total curvature of a random n-gon in $\mathbb{R}^3$ *sampled from* $G_2(\mathbb{C}^n)$ *is*

$$\frac{\pi}{2}n + \frac{\pi}{4}\frac{2n}{2n-3}$$

...and beyond!

From here, you can go in various directions:

- polygons of fixed edgelength (e.g. equilateral polygons) (with Shonkwiler-Duplantier-Uehara)
- polygons of fixed thickness (Chapman, Plunkett)
- linkages and computational geometry
- polygons of fixed bending angle (e.g. molecular models)
- curves instead of polygons (Needham)
- different topologies, such as θ-curves (Deguchi, Uehara)
- shape recognition (Needham, Mumford-Shah-Younes)
- random knots and links (Chapman, Hass, Millett, Rawdon)

. . . and we invite you to our paper session tomorrow morning!

8:30-11:50am, A602, Atrium Level, Marriott Marquis